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Positive entire solutions of higher order semilinear elliptic equations

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1. Introduction

This paper is concerned with the existence of positive entire solutions of semilinear elliptic equations of the type

$$(1.1) \qquad \Delta^N u + a_1 \Delta^{N-1} u + \cdots + a_{N-1} \Delta u + a_N u = f(|x|, u), \quad x \in R^n,$$

where $n \geqq 3$, $N \geqq 1$, a_j, $1 \leqq j \leqq N$, are real constants, Δ^k, $1 \leqq k \leqq N$, are iterates of the Laplacian $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$, and $f(t, u)$ is a real-valued continuous function defined in $[0, \infty) \times (0, \infty)$. By an entire solution of (1.1) we mean a function $u \in C^{2N}(R^n)$ which satisfies equation (1.1) at every point of R^n.

The problem of existence (and nonexistence) of entire solutions for higher order nonlinear elliptic equations was first investigated by Walter [9, 10] in the late fifties; see also Walter and Rhee [11]. However, a systematic study of this problem has recently been initiated by Kusano and Swanson [7], and Kusano, Naito and Swanson [4, 5, 6]. See Usami [8] for further study in this direction. In particular, it is shown [5] that the particular case of (1.1)

$$(1.2) \qquad \Delta^N u = f(|x|, u), \quad x \in R^n, \quad n \geqq 3$$

possesses a variety of entire solutions with different asymptotic behavior at infinity.

The purpose of this paper is to extend the existence theory of [5] to a more general equation (1.1) in which the differential operator

$$L = \Delta^N + a_1 \Delta^{N-1} + \cdots + a_{N-1} \Delta + a_N$$

has a decomposition of the form

$$L = (\Delta - \alpha_1^2)^{p_1} \cdots (\Delta - \alpha_M^2)^{p_M},$$

where α_m, $1 \leqq m \leqq M$, are nonnegative constants with $\alpha_1 < \cdots < \alpha_M$ and p_m, $1 \leqq m \leqq M$, are positive integers. The unperturbed equation

$$(1.3) \qquad (\Delta - \alpha_1^2)^{p_1} \cdots (\Delta - \alpha_M^2)^{p_M} u = 0$$

has a set of radial entire solutions

$$\zeta^i_{\alpha_m}(|x|),\quad 0 \leqq i \leqq p_m - 1,\quad 1 \leqq m \leqq M,$$

where $\zeta^i_\alpha(|x|)$ behaves as $|x| \to \infty$ like a positive constant multiple of the function

$$|x|^{i-\frac{n-1}{2}} \exp(\alpha|x|) \qquad \text{for} \quad \alpha > 0; \quad |x|^{2i} \qquad \text{for} \quad \alpha = 0.$$

We first give conditions under which equation (1.1) has positive radial entire solutions $u(x)$ which are asymptotic to $\zeta^i_{\alpha_m}(|x|)$ at infinity in the sense that the limit

$$\lim_{|x|\to\infty} \frac{u(x)}{\zeta^i_{\alpha_m}(|x|)}$$

exists and is positive.

An interesting problem is to find entire solutions which are asymptotic to none of the $\zeta^i_{\alpha_m}(|x|)$ at infinity. We also study this problem and establish the existence of four kinds of radial entire solutions $u_1(x)$, $u_2(x)$, $u_3(x)$ and $u_4(x)$ for (1.1) with asymptotic properties

$$\lim_{|x|\to\infty} \frac{u_1(x)}{\zeta^{i-1}_{\alpha_m}(|x|)} = \infty, \quad \lim_{|x|\to\infty} \frac{u_1(x)}{\zeta^i_{\alpha_m}(|x|)} = 0, \quad 1 \leqq i \leqq p_m - 1;$$

$$\lim_{|x|\to\infty} \frac{u_2(x)}{\zeta^{p_m-1}_{\alpha_m}(|x|)} = \infty, \quad \lim_{|x|\to\infty} \frac{u_2(x)}{\zeta_{\alpha_{m+1}}(|x|)} = 0;$$

$$\lim_{|x|\to\infty} \frac{u_3(x)}{\zeta_{\alpha_1}(|x|)} = 0; \quad \lim_{|x|\to\infty} \frac{u_4(x)}{\zeta^{p_M-1}_{\alpha_M}(|x|)} = \infty.$$

It is known that equation (1.3) has a set of radial solutions

$$\eta^i_{\alpha_m}(|x|),\quad 0 \leqq i \leqq p_m - 1,\quad 1 \leqq m \leqq M,$$

where $\eta^i_\alpha(|x|)$ is defined in $R^n \setminus \{0\}$ and behaves as $|x| \to \infty$ like a constant multiple of

$$|x|^{i-\frac{n-1}{2}} \exp(-\alpha|x|) \qquad \text{for} \quad \alpha > 0; \quad |x|^{2i+2-n} \qquad \text{for} \quad \alpha = 0.$$

We show that, under certain conditions, equation (1.1) possesses a decaying radial entire solution $u(x)$ such that the limit

$$\lim_{|x|\to\infty} \frac{u(x)}{\eta^{p_1-1}_{\alpha_1}(|x|)}$$

exists and is positive.

All the existence theorems are proved in Section 3. In each of the theorems the desired entire solution is obtained, via the Schauder-Tychonoff fixed point theorem, as a solution of a suitable integral equation, whose integral operator is composed of a finite number of integral operators of the forms G_α and H_α:

$$(G_\alpha g)(t) = \zeta_\alpha(t) \int_0^t \frac{dr}{r^{n-1}\zeta_\alpha(r)^2} \int_0^r s^{n-1}\zeta_\alpha(s)g(s)ds,$$

$$(H_\alpha g)(t) = \zeta_\alpha(t) \int_t^\infty \frac{dr}{r^{n-1}\zeta_\alpha(r)^2} \int_0^r s^{n-1}\zeta_\alpha(s)g(s)ds,$$

where $\zeta_\alpha(t)$ is defined by

$$\zeta_\alpha(t) = \sum_{k=0}^\infty \frac{(\alpha t/2)^{2k}}{2^\nu k!\Gamma(\nu+k+1)}, \quad \nu = \frac{n}{2} - 1, \quad \alpha \geqq 0.$$

Note that the operators G_α and H_α were used by the present author [2] to construct entire solutions of second order elliptic equations of the type $\Delta u - \alpha^2 u = f(x, u)$, $x \in R^n$, $n \geqq 3$. Basic properties of G_α and H_α needed in the proofs of our results are collected in Section 2. An example illustrating the main results is given in Section 4.

2. Fundamental integral operators

In our existence theory to be developed in Section 3 a crucial role will be played by the integral operators G_α and H_α (see (2.10) and (2.11) below) giving rise to radial entire solutions of the linear elliptic equation of the form

$$\Delta u - \alpha^2 u = g(|x|), \quad x \in R^n, \quad n \geqq 3.$$

The purpose of this preparatory section is to collect basic properties of these integral operators.

We begin by considering, for $n \geqq 3$ and $\alpha > 0$, the functions

$$\zeta_\alpha(t) = (\alpha t)^{-\nu} I_\nu(\alpha t), \quad \nu = \frac{n}{2} - 1, \tag{2.1}$$

$$\eta_\alpha(t) = \zeta_\alpha(t) \int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2}, \tag{2.2}$$

where $I_\nu(t)$ is the modified Bessel function of order ν:

$$I_\nu(t) = \sum_{k=1}^\infty \frac{(t/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)}.$$

These functions constitute linearly independent solutions of the differential equation

$$y'' + \frac{n-1}{t} y' - \alpha^2 y = 0, \quad t > 0.$$

Using the facts [12, pp. 77–80]

$$I_\nu(t) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{t}{2}\right)^\nu \quad \text{as} \quad t \to +0,$$

$$I_\nu(t) \sim (2\pi t)^{-1/2} e^t \quad \text{as} \quad t \to \infty,$$

we see that

$$\zeta_\alpha(t) \sim 2^{-\nu}\Gamma(\nu+1)^{-1} \quad \text{as} \quad t \to +0, \tag{2.3}$$

$$\eta_\alpha(t) \sim (n-2)^{-1} 2^\nu \Gamma(\nu+1) t^{2-n} \quad \text{as} \quad t \to +0, \tag{2.4}$$

$$\zeta_\alpha(t) \sim (2\pi)^{-1/2}(\alpha t)^{(1-n)/2} e^{\alpha t} \quad \text{as} \quad t \to \infty, \tag{2.5}$$

$$\eta_\alpha(t) \sim (2\pi)^{1/2}(2\alpha)^{-1}(t/\alpha)^{(1-n)/2} e^{-\alpha t} \quad \text{as} \quad t \to \infty, \tag{2.6}$$

$$\zeta_\alpha(t)\eta_\alpha(t) \sim (2\alpha)^{-1} t^{1-n} \quad \text{as} \quad t \to \infty. \tag{2.7}$$

It follows that $\zeta_\alpha(|x|)$ is a positive entire solution of the equation $\Delta u - \alpha^2 u = 0$ which increases exponentially to ∞ as $|x| \to \infty$, and that $\eta_\alpha(|x|)$ is a solution of the same equation which is defined in $R^n \setminus \{0\}$ and decreases exponentially to zero as $|x| \to \infty$. Put

$$\zeta_0(t) = 2^{-\nu}\Gamma(\nu+1)^{-1}, \quad \eta_0(t) = (n-2)^{-1} 2^\nu \Gamma(\nu+1) t^{2-n}; \tag{2.8}$$

then $\zeta_0(t) = \lim_{\alpha \to +0} \zeta_\alpha(t)$, $\eta_0(t) = \lim_{\alpha \to +0} \eta_\alpha(t)$, and $\zeta_0(|x|)$ and $\eta_0(|x|)$ are solutions of the Laplace equation $\Delta u = 0$ in R^n and $R^n \setminus \{0\}$, respectively.

Let $\Lambda_\alpha(0, \infty)$, $\alpha \geqq 0$, denote the set of all real-valued continuous functions $g(t)$ in $(0, \infty)$ such that

$$\int_0^\delta t^{n-1}\zeta_\alpha(t)|g(t)|dt < \infty, \quad \int_\delta^\infty t^{n-1}\eta_\alpha(t)|g(t)|dt < \infty \tag{2.9}$$

for any $\delta > 0$. We define the integral operators G_α: $C[0, \infty) \to C^2[0, \infty)$ and H_α: $\Lambda_\alpha(0, \infty) \to C^2(0, \infty)$ by the formulas

$$\begin{aligned}(G_\alpha g)(t) &= \zeta_\alpha(t)\int_0^t \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)g(s)ds \\ &= -\eta_\alpha(t)\int_0^t s^{n-1}\zeta_\alpha(s)g(s)ds + \zeta_\alpha(t)\int_0^t s^{n-1}\eta_\alpha(s)g(s)ds, \quad t \geqq 0,\end{aligned} \tag{2.10}$$

for $g \in C[0, \infty)$,

$$\begin{aligned}(H_\alpha g)(t) &= \zeta_\alpha(t)\int_t^\infty \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)g(s)ds \\ &= \eta_\alpha(t)\int_0^t s^{n-1}\zeta_\alpha(s)g(s)ds + \zeta_\alpha(t)\int_t^\infty s^{n-1}\eta_\alpha(s)g(s)ds, \quad t > 0,\end{aligned} \tag{2.11}$$

for $g \in \Lambda_\alpha(0, \infty)$.

It is obvious that $G_\alpha g \geqq 0$ and $H_\alpha g \geqq 0$ for $g \geqq 0$ and that the image of $C[0, \infty) \cap \Lambda_\alpha(0, \infty)$ under H_α is contained in $C^2[0, \infty)$. Note that if in particular $\alpha = 0$, then (2.10) and (2.11) reduce, respectively, to

$$(G_0 g)(t) = \int_0^t r^{1-n} dr \int_0^r s^{n-1} g(s) ds$$
$$= \frac{1}{n-2} \int_0^t \left(1 - \left(\frac{s}{t}\right)^{n-2}\right) s g(s) ds, \quad t \geq 0, \quad g \in C[0, \infty),$$

and

$$(H_0 g)(t) = \int_t^\infty r^{1-n} dr \int_0^r s^{n-1} g(s) ds$$
$$= \frac{1}{n-2} \left(\int_0^t \left(\frac{s}{t}\right)^{n-2} s g(s) ds + \int_t^\infty s g(s) ds \right), \quad t > 0, \quad g \in \Lambda_\alpha(0, \infty);$$

see Kawano [3] and the present author [1].

The following result is an easy consequence of (2.10), (2.11) and the polar form of $\Delta - \alpha^2$:

$$\Delta - \alpha^2 = \frac{1}{t^{n-1} \zeta_\alpha(t)} \frac{d}{dt} t^{n-1} \zeta_\alpha(t)^2 \frac{d}{dt} \frac{\cdot}{\zeta_\alpha(t)}, \quad t = |x|.$$

LEMMA 2.1. *G_α and H_α, $\alpha \geqq 0$, have the following properties*:

(i) $[(\Delta - \alpha^2) G_\alpha g](|x|) = g(|x|)$, $x \in R^n$ *for all* $g \in C[0, \infty)$.

(ii) $[(\Delta - \alpha^2) H_\alpha g](|x|) = -g(|x|)$, $x \in R^n \setminus \{0\}$ [resp. $x \in R^n$] *for all* $g \in \Lambda_\alpha(0, \infty)$ [*resp.* $g \in C[0, \infty) \cap \Lambda_\alpha(0, \infty)$].

For $\alpha \geqq 0$ and $i \geqq 0$ we define

$$\zeta_\alpha^i(t) = (G_\alpha^i \zeta_\alpha)(t), \quad \eta_\alpha^i(t) = (H_\alpha^i \eta_\alpha)(t), \tag{2.12}$$

where G_α^i and H_α^i denote the i-th iterates of the operators G_α and H_α.

LEMMA 2.2. *For $\alpha > 0$ and $i \geqq 0$ the functions $\zeta_\alpha^i(t)$ and $\eta_\alpha^i(t)$ have the following properties*:

(i) $\zeta_\alpha^i \in C[0, \infty)$ *and*

$$\zeta_\alpha^i(t) \sim \frac{1}{i!} \left(\frac{t}{2\alpha}\right)^i \zeta_\alpha(t) \quad as \quad t \to \infty. \tag{2.13}$$

(ii) (a) *If* $0 \leqq i \leqq [(n-3)/2]$, *then* $\eta_\alpha^i \in \Lambda_\alpha(0, \infty)$ *and*

$$\eta_\alpha^i(t) \sim \frac{1}{i!} \left(\frac{t}{2\alpha}\right)^i \eta_\alpha(t) \quad as \quad t \to \infty, \tag{2.14}$$

$$\eta_\alpha^i(t) \sim \frac{2^\nu \Gamma(\nu+1)}{2^i i!(n-2)\cdots(n-2i-2)} t^{2i+2-n} \quad as \quad t \to +0. \tag{2.15}$$

(b) *If n is even and $i=(n/2)-1$, then $\eta_\alpha^i \in \Lambda_\alpha(0, \infty)$ and*

$$\eta_\alpha^i(t) \sim \frac{1}{((n/2)-1)!}\left(\frac{t}{2\alpha}\right)^{(n/2)-1} \eta_\alpha(t) \quad as \quad t \to \infty, \tag{2.16}$$

$$\eta_\alpha^i(t) \sim \frac{2^\nu \Gamma(\nu+1)}{2^{n-2}[((n/2)-1)!]^2} \log(1/t) \quad as \quad t \to +0. \tag{2.17}$$

(c) *If $i \geqq [n/2]$, then $\eta_\alpha^i \in C[0, \infty) \cap \Lambda_\alpha(0, \infty)$ and* (2.14) *is satisfied.*

PROOF. (i) It is clear that $\zeta_\alpha^i \in C[0, \infty)$ for all $i \geqq 0$. Since $\zeta_\alpha^0 = \zeta_\alpha$, (2.13) holds for $i=0$. Suppose that (2.13) is true for some $i \geqq 0$. Then, by (2.10),

$$\begin{aligned}\zeta_\alpha^{i+1}(t) &= -\eta_\alpha(t)\int_0^t s^{n-1}\zeta_\alpha(s)\zeta_\alpha^i(s)ds + \zeta_\alpha(t)\int_0^t s^{n-1}\eta_\alpha(s)\zeta_\alpha^i(s)ds \\ &= J_1(t) + J_2(t).\end{aligned}$$

Using l'Hospital's rule, (2.2), (2.7) and (2.13), we obtain

$$\begin{aligned}\lim_{t\to\infty}\frac{J_1(t)}{t^{i+1}\zeta_\alpha(t)} &= \lim_{t\to\infty}\frac{-\int_0^t s^{n-1}\zeta_\alpha(s)\zeta_\alpha^i(s)ds}{t^{i+1}\left(\int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2}\right)^{-1}} \\ &= \lim_{t\to\infty}\frac{-t^{n-1}\zeta_\alpha(t)\zeta_\alpha^i(t)}{(i+1)t^i\left(\int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2}\right)^{-1} + \frac{t^{i+1}}{t^{n-1}\zeta_\alpha(t)^2}\left(\int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2}\right)^{-2}} \\ &= \frac{1}{i!}\left(\frac{1}{2\alpha}\right)^i \lim_{t\to\infty}\frac{-[t^{n-1}\zeta_\alpha(t)\eta_\alpha(t)]^2}{(i+1)t^{n-1}\zeta_\alpha(t)\eta_\alpha(t)+t} = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\frac{J_2(t)}{t^{i+1}\zeta_\alpha(t)} &= \lim_{t\to\infty}\frac{\int_0^t s^{n-1}\eta_\alpha(s)\zeta_\alpha^i(s)ds}{t^{i+1}} \\ &= \lim_{t\to\infty}\frac{t^{n-1}\eta_\alpha(t)\zeta_\alpha^i(t)}{(i+1)t^i} = \frac{1}{(i+1)!}\left(\frac{1}{2\alpha}\right)^{i+1},\end{aligned}$$

which shows that (2.13) with i replaced by $i+1$ is true. Therefore (2.13) must hold for all $i \geqq 0$.

(ii) (a) Suppose that $\eta_\alpha^i \in \Lambda_\alpha(0, \infty)$ and (2.14)–(2.15) hold for some i ($0 \leqq i \leqq [(n-3)/2]-1$). Then $\eta_\alpha^{i+1} \in C(0, \infty)$ is obvious. In view of (2.11) we get

(2.18) $$\eta_\alpha^{i+1}(t) = \eta_\alpha(t)\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^i(s)ds + \zeta_\alpha(t)\int_t^\infty s^{n-1}\eta_\alpha(s)\eta_\alpha^i(s)ds$$

$$= K_1(t) + K_2(t).$$

L'Hospital's rule, together with (2.5), (2.6) and (2.14), implies that

$$\lim_{t\to\infty}\frac{K_1(t)}{t^{i+1}\eta_\alpha(t)} = \lim_{t\to\infty}\frac{\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^i(s)\,ds}{t^{i+1}}$$

$$= \lim_{t\to\infty}\frac{t^{n-1}\zeta_\alpha(t)\eta_\alpha^i(t)}{(i+1)t^i} = \frac{1}{(i+1)!}\left(\frac{1}{2\alpha}\right)^{i+1}$$

and

$$\lim_{t\to\infty}\frac{K_2(t)}{t^{i+1}\eta_\alpha(t)} = \lim_{t\to\infty}\frac{\int_t^\infty s^{n-1}\eta_\alpha(s)\eta_\alpha^i(s)\,ds}{t^{i+1}\int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2}}$$

$$= \lim_{t\to\infty}\frac{-t^{n-1}\eta_\alpha(t)\eta_\alpha^i(t)}{(i+1)t^i\int_t^\infty \frac{ds}{s^{n-1}\zeta_\alpha(s)^2} - \frac{t^{i+1}}{t^{n-1}\zeta_\alpha(t)^2}}$$

$$= \frac{1}{i!}\left(\frac{1}{2\alpha}\right)^i \lim_{t\to\infty}\frac{-[t^{n-1}\zeta_\alpha(t)\eta_\alpha(t)]^2}{(i+1)t^{n-1}\zeta_\alpha(t)\eta_\alpha(t)-t} = 0,$$

proving the truth of (2.14) with i replaced by $i+1$.

Noting that $t^{2i+4-n}\to\infty$ as $t\to+0$ and $\eta_\alpha^i(t)\sim c_i t^{2i+2-n}$ as $t\to+0$, where $c_i = 2^\nu\Gamma(\nu+1)/2^i i!(n-2)\cdots(n-2i-2)$, and using (2.3), (2.4) and (2.18) we have

$$\lim_{t\to+0}\frac{K_1(t)}{t^{2i+4-n}} = \lim_{t\to+0}\frac{2^\nu\Gamma(\nu+1)\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^i(s)\,ds}{(n-2)t^{2i+2}}$$

$$= \lim_{t\to+0}\frac{2^\nu\Gamma(\nu+1)t^{n-1}\zeta_\alpha(t)\eta_\alpha^i(t)}{(n-2)(2i+2)t^{2i+1}} = \frac{c_i}{(n-2)(2i+2)},$$

$$\lim_{t\to+0}\frac{K_2(t)}{t^{2i+4-n}} = \lim_{t\to+0}\frac{\int_t^\infty s^{n-1}\eta_\alpha(s)\,\eta_\alpha^i(s)\,ds}{2^\nu\Gamma(\nu+1)t^{2i+4-n}}$$

$$= \lim_{t\to+0}\frac{-t^{n-1}\eta_\alpha(t)\eta_\alpha^i(t)}{2^\nu\Gamma(\nu+1)(2i+4-n)t^{2i+3-n}} = \frac{c_i}{(n-2)(n-4-2i)},$$

and consequently

$$\lim_{t\to+0}\frac{\eta_\alpha^{i+1}(t)}{t^{2i+4-n}} = \frac{c_i}{(2i+2)(n-4-2i)},$$

proving that (2.15) is true for i replaced by $i+1$. Thus, we have $\eta_\alpha^{i+1}\in\Lambda_\alpha(0,\infty)$.

Since $\eta_\alpha \in \Lambda_\alpha(0, \infty)$ and (2.14)–(2.15) are trivial for $i=0$, it follows that $\eta_\alpha^i \in \Lambda_\alpha(0, \infty)$ and (2.14)–(2.15) are true for $0 \leqq i \leqq [(n-3)/2]$.

(b) Let n be even and $i=(n/2)-1$. Then, (2.16) is obtained by applying H_α to (2.14) with $i=[(n-3)/2]$ and proceeding exactly as in the derivation of (2.14) in (a). The functions $K_1(t)$ and $K_2(t)$ in (2.18) with i replaced by $i-1$ satisfy in view of (2.3), (2.4) and (2.15)

$$\begin{aligned}\lim_{t\to+0} \frac{K_1(t)}{\log(1/t)} &= \lim_{t\to+0} \frac{2^\nu\Gamma(\nu+1)\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^{i-1}(s)ds}{(n-2)t^{n-2}\log(1/t)} \\ &= \lim_{t\to+0} \frac{2^\nu\Gamma(\nu+1)t^{n-1}\zeta_\alpha(t)\eta_\alpha^{i-1}(t)}{(n-2)[(n-2)t^{n-3}\log(1/t)-t^{n-3}]} \\ &= \lim_{t\to+0} \frac{c_{i-1}}{(n-2)[(n-2)\log(1/t)-1]} = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to+0} \frac{K_2(t)}{\log(1/t)} &= \lim_{t\to+0} \frac{\int_t^\infty s^{n-1}\eta_\alpha(s)\eta_\alpha^{i-1}(s)ds}{2^\nu\Gamma(\nu+1)\log(1/t)} \\ &= \lim_{t\to+0} \frac{t^{n-1}\eta_\alpha(t)\eta_\alpha^{i-1}(t)}{2^\nu\Gamma(\nu+1)t^{-1}} = \frac{c_{i-1}}{n-2} = \frac{2^\nu\Gamma(\nu+1)}{2^{n-2}[((n/2)-1)!]^2},\end{aligned}$$

which shows (2.17). Therefore, $\eta_\alpha^i \in \Lambda_\alpha(0, \infty)$.

(c) By the definition of $\eta_\alpha^{[n/2]}(t)$,

$$\begin{aligned}\eta_\alpha^{[n/2]}(t) = \eta_\alpha(t)&\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^{(n-3)/2}(s)ds \\ &+ \zeta_\alpha(t)\int_t^\infty s^{n-1}\eta_\alpha(s)\eta_\alpha^{(n-3)/2}(s)ds\end{aligned}$$

if n is odd, and

$$\begin{aligned}\eta_\alpha^{[n/2]}(t) = \eta_\alpha(t)&\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^{(n/2)-1}(s)ds \\ &+ \zeta_\alpha(t)\int_t^\infty s^{n-1}\eta_\alpha(s)\eta_\alpha^{(n/2)-1}(s)ds\end{aligned}$$

if n is even. This, combined with the relations (see (a) and (b))

$$\eta_\alpha^{(n-3)/2}(t) \sim \frac{2^\nu\Gamma(\nu+1)}{(n-2)!}t^{-1} \quad \text{as} \quad t\to+0 \quad (\text{for } n \text{ odd}),$$

$$\eta_\alpha^{(n/2)-1}(t) \sim \frac{2^\nu\Gamma(\nu+1)}{2^{n-2}[((n/2)-1)!]^2}\log(1/t) \quad \text{as} \quad t\to+0 \quad (\text{for } n \text{ even}),$$

implies that $\eta_\alpha^{[n/2]} \in C[0, \infty)$. The proof of (2.14) for $i \geqq [n/2]$ is similar to that of part (a), and hence $\eta_\alpha^i \in C[0, \infty) \cap \Lambda_\alpha(0, \infty)$ for $i \geqq [n/2]$. This completes the proof of Lemma 2.2.

Repeated application of G_0 and H_0 starting from (2.8) yields the explicit expressions for $\zeta_0^i(t)$ and $\eta_0^i(t)$.

LEMMA 2.3. *$\zeta_0^i(t)$ and $\eta_0^i(t)$ are given by*

$$\zeta_0^i(t) = \frac{2^{-\nu}\Gamma(\nu+1)^{-1}}{2^i i!\, n \cdots (n+2i-2)} t^{2i}, \quad i \geqq 1, \tag{2.19}$$

$$\eta_0^i(t) = \frac{2^{\nu}\Gamma(\nu+1)}{2^i i!(n-2)\cdots(n-2i-2)} t^{2i+2-n}, \quad 1 \leqq i \leqq [(n-3)/2]. \tag{2.20}$$

It should be noticed that $\eta_0^i(t)$ cannot be defined for $i \geqq [(n-1)/2]$, since $\eta_0^{[(n-3)/2]} \notin \Lambda_0(0, \infty)$ by (2.20).

We employ the notation

$$L_\alpha^i[0, \infty) = \left\{ g \in C[0, \infty) \colon \int_0^\infty t^{n-1} \eta_\alpha^i(t) |g(t)| dt < \infty \right\} \tag{2.21}$$

for $\alpha \geqq 0$, $i \geqq 0$. Obviously $L_\alpha^0[0, \infty) = C[0, \infty) \cap \Lambda_\alpha(0, \infty)$.

LEMMA 2.4. *If $g(t)$ and $h(t)$ are nonnegative functions in $\Lambda_\alpha(0, \infty)$, $\alpha \geqq 0$, then*

$$\int_0^\infty t^{n-1} h(t)(H_\alpha g)(t) dt = \int_0^\infty t^{n-1}(H_\alpha h)(t) g(t) dt. \tag{2.22}$$

The verification of this lemma is straightforward on the basis of the second expression for H_α in (2.11). Note that the integrals in (2.22) may converge or diverge.

LEMMA 2.5. *Let $\alpha \geqq 0$ and $j \geqq 1$. If $g \in L_\alpha^{j-1}[0, \infty)$, then for any $i \geqq 0$*

$$|(G_\alpha^i H_\alpha^j g)(t)| \leqq \zeta_\alpha^i(t) \int_0^\infty s^{n-1} \eta_\alpha^{j-1}(s) |g(s)| ds, \quad t \geqq 0, \tag{2.23}$$

and

$$\lim_{t \to \infty} \frac{(G_\alpha^i H_\alpha^j g)(t)}{\zeta_\alpha^i(t)} = 0. \tag{2.24}$$

PROOF. We first consider the case where $i=0$. Since $\zeta_\alpha(t)$ is increasing and $\eta_\alpha(t)$ is decreasing, we have from (2.11)

$$|(H_\alpha g)(t)| \leqq (H_\alpha |g|)(t) \leqq \zeta_\alpha(t) \int_t^\infty s^{n-1} \eta_\alpha(s) |g(s)| ds, \quad t \geqq 0.$$

From the relation

$$(2.25)\qquad \frac{(H_\alpha g)(t)}{\zeta_\alpha(t)} = \int_0^t \frac{\eta_\alpha(t)}{\zeta_\alpha(t)} s^{n-1}\zeta_\alpha(s)g(s)ds + \int_t^\infty s^{n-1}\eta_\alpha(s)g(s)ds,$$

we see, via the Lebesgue dominated convergence theorem applied to the first integral in (2.25), that $\lim_{t\to\infty}(H_\alpha g)(t)/\zeta_\alpha(t)=0$. Thus (2.23) and (2.24) are true for $i=0$ and $j=1$. Assume that truth of (2.23) and (2.24) for $i=0$ and some $j\geqq 1$. Then, if $g\in L_\alpha^j[0,\infty)$, using Lemma 2.4 we have

$$\begin{aligned}|(H_\alpha^{j+1}g)(t)| &\leqq \zeta_\alpha(t)\int_0^\infty s^{n-1}\eta_\alpha^{j-1}(s)(H_\alpha|g|)(s)ds\\ &= \zeta_\alpha(t)\int_0^\infty s^{n-1}\eta_\alpha^j(s)\,|g(s)|ds, \quad t\geqq 0,\end{aligned}$$

and

$$\lim_{t\to\infty}\frac{(H_\alpha^{j+1}g)(t)}{\zeta_\alpha(t)} = \lim_{t\to\infty}\frac{(H_\alpha^j H_\alpha g)(t)}{\zeta_\alpha(t)} = 0.$$

Thus induction shows that (2.23) and (2.24) hold for $i=0$ and all $j\geqq 1$.

Now let $j\geqq 1$ be fixed, and assume the truth of (2.23) and (2.24) for some $i\geqq 0$. That (2.23) and (2.24) with i replaced by $i+1$ hold is seen as follows:

$$\begin{aligned}|(G_\alpha^{i+1}H_\alpha^j g)(t)| &= |(G_\alpha G_\alpha^i H_\alpha^j g)(t)| \leqq (G_\alpha|G_\alpha^i H_\alpha^j g|)(t)\\ &\leqq (G_\alpha\zeta_\alpha^i)(t)\int_0^\infty s^{n-1}\eta_\alpha^{j-1}(s)\,|g(s)|ds\\ &= \zeta_\alpha^{i+1}(t)\int_0^\infty s^{n-1}\eta_\alpha^{j-1}(s)\,|g(s)|ds, \quad t\geqq 0,\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}\frac{(G_\alpha^{i+1}H_\alpha^j g)(t)}{\zeta_\alpha^{i+1}(t)} &= \lim_{t\to\infty}\frac{\displaystyle\int_0^t \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)(G_\alpha^i H_\alpha^j g)(s)ds}{\displaystyle\int_0^t \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)\zeta_\alpha^i(s)ds}\\ &= \lim_{t\to\infty}\frac{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)(G_\alpha^i H_\alpha^j g)(s)ds}{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)\,\zeta_\alpha^i(s)ds} = \lim_{t\to\infty}\frac{(G_\alpha^i H_\alpha^j g)(t)}{\zeta_\alpha^i(t)} = 0.\end{aligned}$$

It follows that (2.23) and (2.24) hold for all $i\geqq 0$ and $j\geqq 1$.

REMARK. In view of Lemma 2.3, if $\alpha=0$ in Lemma 2.5, then the integer j must not exceed $[(n-1)/2]$. A similar remark applies to the subsequent lemmas in which the function $\eta_0^j(t)$ appears.

LEMMA 2.6. (i) *If $g(t)$ is a nonnegative function in $C[0,\infty)$, then for $\alpha\geqq 0$ and $i\geqq 1$*

(2.26) $$\lim_{t\to\infty}\frac{(G_\alpha^i g)(t)}{\zeta_\alpha^{i-1}(t)}=\int_0^\infty s^{n-1}\eta_\alpha(s)g(s)ds.$$

(ii) *If $g(t)$ is a nonnegative function in $L_\alpha^{j-1}[0,\infty)$, $\alpha\geqq 0$, $j\geqq 1$, then for $i\geqq 1$*

(2.27) $$\lim_{t\to\infty}\frac{(G_\alpha^i H_\alpha^j g)(t)}{\zeta_\alpha^{i-1}(t)}=\int_0^\infty s^{n-1}\eta_\alpha^j(s)g(s)ds.$$

Note that the integrals in (2.26) and (2.27) may converge or diverge.

Proof. (i) By (2.10), $(G_\alpha g)(t)/\zeta_\alpha(t)$ is nondecreasing and

$$\begin{aligned}\lim_{t\to\infty}\frac{(G_\alpha g)(t)}{\zeta_\alpha(t)}&=\int_0^\infty\frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)g(s)ds\\&=\int_0^\infty s^{n-1}\eta_\alpha(s)g(s)ds,\end{aligned}$$

proving (2.26) for $i=1$. If we suppose that (2.26) holds for some $i\geqq 1$, then we have via l'Hospital's rule

$$\begin{aligned}\lim_{t\to\infty}\frac{(G_\alpha^{i+1}g)(t)}{\zeta_\alpha^i(t)}&=\lim_{t\to\infty}\frac{\displaystyle\int_0^t\frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)(G_\alpha^i g)(s)ds}{\displaystyle\int_0^t\frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)\zeta_\alpha^{i-1}(s)ds}\\&=\lim_{t\to\infty}\frac{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)(G_\alpha^i g)(s)ds}{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)\zeta_\alpha^{i-1}(s)ds}=\lim_{t\to\infty}\frac{(G_\alpha^i g)(t)}{\zeta_\alpha^{i-1}(t)},\end{aligned}$$

which shows that (2.26) is true for all $i\geqq 1$.

(ii) If $g\in L_\alpha^{j-1}[0,\infty)$, then $H_\alpha^j g$ is well defined (see Lemma 2.5), and from (2.26) (with g replaced by $H_\alpha^j g$) and Lemma 2.4 it follows that

$$\begin{aligned}\lim_{t\to\infty}\frac{(G_\alpha^i H_\alpha^j g)(t)}{\zeta_\alpha^{i-1}(t)}&=\int_0^\infty s^{n-1}\eta_\alpha(s)(H_\alpha^j g)(s)ds\\&=\int_0^\infty s^{n-1}\eta_\alpha^j(s)g(s)ds.\end{aligned}$$

This completes the proof.

Lemma 2.7. *Let $\alpha>\beta\geqq 0$, $i\geqq 1$ and $j\geqq 0$.*

(i) *If $g\in C[0,\infty)$ and $\lim_{t\to\infty}g(t)/\zeta_\alpha^j(t)$ exists in the extended real line $R\cup\{\pm\infty\}$, then*

(2.28) $$\lim_{t\to\infty}\frac{(G_\beta^i g)(t)}{\zeta_\alpha^j(t)}=\frac{1}{(\alpha^2-\beta^2)^i}\lim_{t\to\infty}\frac{g(t)}{\zeta_\alpha^j(t)}.$$

(ii) *If* $g \in \Lambda_\alpha(0, \infty)$ *and* $\lim_{t\to\infty} g(t)/\zeta^j_\beta(t)$ *exists in* R, *then*

$$\lim_{t\to\infty} \frac{(H^i_\alpha g)(t)}{\zeta^j_\beta(t)} = \frac{1}{(\alpha^2-\beta^2)^i} \lim_{t\to\infty} \frac{g(t)}{\zeta^j_\beta(t)}. \tag{2.29}$$

(iii) *If* $g \in \Lambda_\alpha(0, \infty)$ *and* $\lim_{t\to\infty} g(t)/\eta^j_\beta(t)$ *exists in* R, *then*

$$\lim_{t\to\infty} \frac{(H^i_\alpha g)(t)}{\eta^j_\beta(t)} = \frac{1}{(\alpha^2-\beta^2)^i} \lim_{t\to\infty} \frac{g(t)}{\eta^j_\beta(t)}. \tag{2.30}$$

PROOF. (i) Suppose that $\beta>0$. That (2.28) holds for $i=1$ is verified as follows:

$$\lim_{t\to\infty} \frac{(G_\beta g)(t)}{\zeta^j_\alpha(t)} = \lim_{t\to\infty} \frac{\displaystyle\int_0^t \frac{dr}{r^{n-1}\zeta_\beta(r)^2} \int_0^r s^{n-1}\zeta_\beta(s)g(s)ds}{\zeta^j_\alpha(t)/\zeta_\beta(t)}$$

$$= \lim_{t\to\infty} \frac{\displaystyle j! \int_0^t \frac{dr}{r^{n-1}\zeta_\beta(r)^2} \int_0^r s^{n-1}\zeta_\beta(s)g(s)ds}{(\alpha/\beta)^{(1-n)/2}(t/2\alpha)^j e^{(\alpha-\beta)t}}$$

$$= \lim_{t\to\infty} \frac{\displaystyle \frac{j!}{t^{n-1}\zeta_\beta(t)^2} \int_0^t s^{n-1}\zeta_\beta(s)g(s)ds}{(\alpha-\beta)(\alpha/\beta)^{(1-n)/2}(t/2\alpha)^j e^{(\alpha-\beta)t}}$$

$$= \lim_{t\to\infty} \frac{\displaystyle 2\pi j! \int_0^t s^{n-1}\zeta_\beta(s)g(s)ds}{(\alpha-\beta)(\alpha\beta)^{(1-n)/2}(t/2\alpha)^j e^{(\alpha+\beta)t}}$$

$$= \lim_{t\to\infty} \frac{2\pi j!\, t^{n-1}\zeta_\beta(t)g(t)}{(\alpha^2-\beta^2)(\alpha\beta)^{(1-n)/2}(t/2\alpha)^j e^{(\alpha+\beta)t}}$$

$$= \lim_{t\to\infty} \frac{2\pi j!\, t^{n-1}\zeta_\beta(t)\zeta^j_\alpha(t)}{(\alpha^2-\beta^2)(\alpha\beta)^{(1-n)/2}(t/2\alpha)^j e^{(\alpha+\beta)t}} \lim_{t\to\infty} \frac{g(t)}{\zeta^j_\alpha(t)}$$

$$= \frac{1}{\alpha^2-\beta^2} \lim_{t\to\infty} \frac{g(t)}{\zeta^j_\alpha(t)}.$$

Here l'Hospital's rule, (2.5) and (2.13) have been used. Using this result, we see that if (2.28) holds for some $i \geqq 1$, then

$$\lim_{t\to\infty} \frac{(G^{i+1}_\beta g)(t)}{\zeta^j_\alpha(t)} = \lim_{t\to\infty} \frac{(G_\beta G^i_\beta g)(t)}{\zeta^j_\alpha(t)}$$

$$= \frac{1}{\alpha^2-\beta^2} \lim_{t\to\infty} \frac{(G^i_\beta g)(t)}{\zeta^j_\alpha(t)} = \frac{1}{(\alpha^2-\beta^2)^{i+1}} \lim_{t\to\infty} \frac{g(t)}{\zeta^j_\alpha(t)},$$

showing the truth of (2.28) with i replaced by $i+1$. Therefore (2.28) holds for all $i \geqq 1$ if $\beta>0$. A similar computation with the use of (2.8), (2.19) and (2.20) shows that (2.28) also holds if $\beta=0$. This completes the proof of (i).

The statements (ii) and (iii) can be proved analogously.

We now introduce the notation:

$$\gamma_\alpha(t) = \min\{\zeta_\alpha(t), \eta_\alpha(t)\}, \quad t > 0, \tag{2.31}$$

$$\Gamma_\alpha(t) = \max\{\zeta_\alpha(t), \eta_\alpha(t)\}, \quad t > 0, \tag{2.32}$$

and

$$\gamma_\alpha^j(t) = (H_\alpha^j \gamma_\alpha)(t), \quad t > 0, \tag{2.33}$$

where $\alpha \geqq 0$ and $j \geqq 0$. Let $\tilde{L}_\alpha[0, \infty)$, $\alpha \geqq 0$, denote the set of all functions $g \in C[0, \infty)$ such that

$$\int_0^\infty s^{n-1}\Gamma_\alpha(s)|g(s)|ds < \infty. \tag{2.34}$$

It is clear that $\tilde{L}_\alpha[0, \infty) \subset L_\alpha[0, \infty)$.

LEMMA 2.8. *If $g(t)$ is a nonnegative function in $\tilde{L}_\alpha[0, \infty)$, $\alpha \geqq 0$, then for all $j \geqq 1$*

$$\begin{aligned} \gamma_\alpha^{j-1}(t)\int_0^\infty s^{n-1}\gamma_\alpha(s)g(s)ds &\leqq (H_\alpha^j g)(t) \\ &\leqq \gamma_\alpha^{j-1}(t)\int_0^\infty s^{n-1}\Gamma_\alpha(s)g(s)ds, \quad t \geqq 0, \end{aligned} \tag{2.35}$$

and

$$\lim_{t\to\infty}\frac{(H_\alpha^j g)(t)}{\eta_\alpha^{j-1}(t)} = \int_0^\infty s^{n-1}\zeta_\alpha(s)g(s)ds. \tag{2.36}$$

PROOF. We prove this lemma by induction on j. By (2.11) and (2.31) we have

$$(H_\alpha g)(t) \geqq \gamma_\alpha(t)\int_0^\infty s^{n-1}\gamma_\alpha(s)g(s)ds, \quad t \geqq 0, \tag{2.37}$$

for $g \in \tilde{L}_\alpha[0, \infty)$ with $g(t) \geqq 0$. On the other hand, (2.11) together with the monotonicity of $\zeta_\alpha(t)$ and $\eta_\alpha(t)$ implies that

$$\begin{aligned} (H_\alpha g)(t) &\leqq \zeta_\alpha(t)\int_0^\infty s^{n-1}\eta_\alpha(s)g(s)ds \\ &\leqq \zeta_\alpha(t)\int_0^\infty s^{n-1}\Gamma_\alpha(s)g(s)ds, \quad t \geqq 0, \\ (H_\alpha g)(t) &\leqq \eta_\alpha(t)\int_0^\infty s^{n-1}\zeta_\alpha(s)g(s)ds \\ &\leqq \eta_\alpha(t)\int_0^\infty s^{n-1}\Gamma_\alpha(s)g(s)ds, \quad t \geqq 0, \end{aligned}$$

which shows that

$$(2.38)\qquad (H_\alpha g)(t) \leqq \gamma_\alpha(t)\int_0^\infty s^{n-1}\Gamma_\alpha(s)g(s)ds, \quad t \geqq 0.$$

From (2.37) and (2.38) we see that (2.35) holds for $j=1$. That (2.36) holds for $j=1$ follows from the relations:

$$\frac{(H_\alpha g)(t)}{\eta_\alpha(t)} = \int_0^t s^{n-1}\zeta_\alpha(s)g(s)ds + \frac{\zeta_\alpha(t)}{\eta_\alpha(t)}\int_t^\infty s^{n-1}\eta_\alpha(s)g(s)ds$$

and

$$0 \leqq \frac{\zeta_\alpha(t)}{\eta_\alpha(t)}\int_t^\infty s^{n-1}\eta_\alpha(s)g(s)ds \leqq \int_t^\infty s^{n-1}\zeta_\alpha(s)g(s)ds.$$

Suppose that (2.35) and (2.36) are true for some $j \geqq 1$. Then, applying the operator H_α to (2.35), we obtain (2.35) with j replaced by $j+1$, and using (2.36) we find

$$\lim_{t\to\infty}\frac{(H_\alpha^{j+1}g)(t)}{\eta_\alpha^j(t)} = \lim_{t\to\infty}\frac{\displaystyle\int_t^\infty \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)(H_\alpha^j g)(s)ds}{\displaystyle\int_t^\infty \frac{dr}{r^{n-1}\zeta_\alpha(r)^2}\int_0^r s^{n-1}\zeta_\alpha(s)\eta_\alpha^{j-1}(s)\,ds}$$

$$= \lim_{t\to\infty}\frac{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)(H_\alpha^j g)(s)ds}{\displaystyle\int_0^t s^{n-1}\zeta_\alpha(s)\eta_\alpha^{j-1}(s)\,ds} = \lim_{t\to\infty}\frac{(H_\alpha^j g)(t)}{\eta_\alpha^{j-1}(t)},$$

proving (2.36) with j replaced by $j+1$. This completes the proof.

3. Existence of positive entire solutions

In this section the existence of positive radial entire solutions will be established for the elliptic equation

$$(3.1)\qquad (\Delta-\alpha_1^2)^{p_1}\cdots(\Delta-\alpha_M^2)^{p_M}u = f(|x|, u), \quad x \in R^n, \quad n \geqq 3,$$

where α_i, $1 \leqq i \leqq M$, are constants such that $0 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_M$, and p_i, $1 \leqq i \leqq M$, are positive integers. Hypotheses on $f(t, u)$ will be selected from the following list.

(f_1) f: $[0, \infty)\times(0, \infty)\to R$ is continuous.

(f_2) There exists a continuous function f^*: $[0, \infty)\times(0, \infty)\to[0, \infty)$ such that

$$|f(t, u)| \leqq f^*(t, u) \qquad \text{for} \quad (t, u) \in [0, \infty)\times(0, \infty).$$

(f_3) (Superlinearity) $u^{-1}f^*(t, u)$ is nondecreasing in $u \in (0, \infty)$ for each fixed $t \geqq 0$ and satisfies

$$\lim_{t \to +0} u^{-1}f^*(t, u) = 0, \quad t \geqq 0.$$

(f_4) (Sublinearity) $u^{-1}f^*(t, u)$ is nonincreasing in $u \in (0, \infty)$ for each fixed $t \geqq 0$ and satisfies

$$\lim_{t \to \infty} u^{-1}f^*(t, u) = 0, \quad t \geqq 0.$$

(f_5) $f^*(t, u)$ is nondecreasing in $u \in (0, \infty)$ for each fixed $t \geqq 0$.

(f_6) $f^*(t, u)$ is nonincreasing in $u \in (0, \infty)$ for each fixed $t \geqq 0$.

Noting that the functions $\{\zeta^i_{\alpha_m}(t)\colon 0 \leqq i \leqq p_m - 1,\ 1 \leqq m \leqq M\}$ defined by (2.1), (2.8) and (2.12) yield the positive entire solutions

$$\zeta^i_{\alpha_m}(|x|), \quad 0 \leqq i \leqq p_m - 1, \quad 1 \leqq m \leqq M \tag{3.2}$$

of the unperturbed elliptic equation

$$(\Delta - \alpha_1^2)^{p_1} \cdots (\Delta - \alpha_M^2)^{p_M} u = 0, \quad x \in R^n, \tag{3.3}$$

we first discuss the situation in which equation (3.1) possesses positive radial entire solutions $u(x)$ which are asymptotic to $\zeta^i_{\alpha_m}(|x|)$ as $|x| \to \infty$ in the sense that

$$\lim_{|x| \to \infty} \frac{u(x)}{\zeta^i_{\alpha_m}(|x|)} = \tau \tag{3.4}$$

for some positive value τ.

Let S denote the set of all positive radial entire solutions of (3.1), and define the subsets $S(\zeta^i_{\alpha_m})$ of S and the subsets $T(\zeta^i_{\alpha_m})$ of $(0, \infty)$ as follows:

$$S(\zeta^i_{\alpha_m}) = \{u \in S\colon u(x) \text{ satisfies (3.4) for some finite value } \tau > 0\},$$

$$T(\zeta^i_{\alpha_m}) = \{\tau \in (0, \infty)\colon \text{there exists a } u \in S \text{ such that (3.4) holds}\}.$$

In what follows we use the notation F and F^* to denote the Nemytskii operators corresponding to the functions $f(t, u)$ and $f^*(t, u)$ in (f_1)–(f_6):

$$(Fy)(t) = f(t, y(t)), \quad (F^*y)(t) = f^*(t, y(t)), \quad y \in C[0, \infty).$$

Our first results are the following two theorems.

THEOREM 3.1. *Suppose that* (f_1), (f_2) *and* (f_3) *are satisfied. Let* α_m *be one of the numbers* $\alpha_1, \ldots, \alpha_M$ *appearing in* (3.1) *and let* i *be an integer such that*

$$\tag{3.5} \begin{aligned} &0 \leqq i \leqq p_m - 1 \quad \text{if} \quad \alpha_m > 0, \\ &\max\left\{0,\ p_m - \left[\frac{n-1}{2}\right]\right\} \leqq i \leqq p_m - 1 \quad \text{if} \quad \alpha_m = 0. \end{aligned}$$

If there exists a positive constant λ *such that*

$$\tag{3.6} F^*(\lambda\zeta^i_{\alpha_m}) \in L^{p_m-i-1}_{\alpha_m}[0, \infty),$$

then $T(\zeta^i_{\alpha_m})$ *contains an interval of the form* $(0, \tau_0)$*, that is, there is a* $\tau_0 > 0$ *such that equation* (3.1) *has a positive radial entire solution* $u(x)$ *satisfying* (3.4) *for every* $\tau \in (0, \tau_0)$.

THEOREM 3.2. *Suppose that* (f_1), (f_2) *and* (f_4) *are satisfied. Let* α_m *be one of the numbers* $\alpha_1, \dots, \alpha_M$ *appearing in* (3.1) *and let* i *be an integer satisfying* (3.5). *If* (3.6) *holds for some* $\lambda > 0$*, then* $T(\zeta^i_{\alpha_m})$ *contains an interval of the form* (τ_0, ∞)*, that is, there is a* $\tau_0 > 0$ *such that equation* (3.1) *has a positive radial entire solution* $u(x)$ *satisfying* (3.4) *for every* $\tau \in (\tau_0, \infty)$.

PROOF OF THEOREM 3.1. Let $C[0, \infty)$ be the locally convex space of all continuous functions in $[0, \infty)$ with the topology of uniform convergence on every compact subinterval of $[0, \infty)$. Define

$$\tag{3.7} \xi^0_{\alpha_m}(t) = \zeta_{\alpha_m}(t), \quad \xi^i_{\alpha_m}(t) = \zeta_{\alpha_m}(t) + \zeta^i_{\alpha_m}(t) \qquad \text{for} \quad i \geqq 1.$$

Let $\tau \in (0, 2\lambda/3)$ and consider the closed convex subset Y_τ of $C[0, \infty)$ defined by

$$\tag{3.8} Y_\tau = \left\{y \in C[0, \infty) \colon \frac{1}{2}\tau\xi^i_{\alpha_m}(t) \leqq y(t) \leqq \frac{3}{2}\tau\xi^i_{\alpha_m}(t),\ t \geqq 0\right\}.$$

Condition (3.6), together with (f_2) and (f_3), implies that $Fy \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$ for every $y \in Y_\tau$. Let us define integral operators A, B and C by

$$\tag{3.9} A = G^{p_1}_{\alpha_1} \cdots G^{p_{m-1}}_{\alpha_{m-1}}, \quad B = G^i_{\alpha_m} H^{p_m-i}_{\alpha_m}, \quad C = H^{p_{m+1}}_{\alpha_{m+1}} \cdots H^{p_M}_{\alpha_M}.$$

Since, by Lemma 2.5,

$$|(Bg)(t)| \leqq \zeta^i_{\alpha_m}(t) \int_0^\infty s^{n-1} \eta^{p_m-i-1}_{\alpha_m}(s) |g(s)| ds, \quad t \geqq 0,$$

for $g \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$, using Lemma 2.4 we obtain for $y \in Y_\tau$

$$\begin{aligned} |(BCFy)(t)| &\leqq \zeta^i_{\alpha_m}(t) \int_0^\infty s^{n-1} \eta^{p_m-i-1}_{\alpha_m}(s) |(CFy)(s)| ds \\ &\leqq \zeta^i_{\alpha_m}(t) \int_0^\infty s^{n-1} (C^* \eta^{p_m-i-1}_{\alpha_m})(s) |(Fy)(s)| ds, \quad t \geqq 0, \end{aligned}$$

where $C^* = H^{p_M}_{\alpha_M} \cdots H^{p_{m+1}}_{\alpha_{m+1}}$, and hence

$$(3.10) \qquad |(ABCFy)(t)| \leqq (A\zeta^i_{\alpha_m})(t)\int_0^\infty s^{n-1}(C^*\eta^{p_m-i-1}_{\alpha_m})(s)(F^*y)(s)ds, \quad t \geqq 0.$$

Noting that

$$\lim_{t\to\infty}(A\zeta^i_{\alpha_m})(t)/\zeta^i_{\alpha_m}(t) = \prod_{k=1}^{m-1}(\alpha_m^2-\alpha_k^2)^{-p_k} > 0$$

and

$$\lim_{t\to\infty}(C^*\eta^{p_m-i-1}_{\alpha_m})(t)/\eta^{p_m-i-1}_{\alpha_m}(t) = \prod_{k=m+1}^{M}(\alpha_k^2-\alpha_m^2)^{-p_k} > 0$$

by Lemma 2.7, we see from (3.10) that there is a constant $c>0$ such that

$$(3.11) \qquad \begin{aligned} |(ABCFy)(t)| &\leqq c\zeta^i_{\alpha_m}(t)\int_0^\infty s^{n-1}\eta^{p_m-i-1}_{\alpha_m}(s)(F^*y)(s)ds \\ &\leqq c\zeta^i_{\alpha_m}(t)\int_0^\infty s^{n-1}\eta^{p_m-i-1}_{\alpha_m}(s)f^*\Big(s, \frac{3}{2}\tau\xi^i_{\alpha_m}(s)\Big)ds, \quad t \geqq 0, \end{aligned}$$

for $y \in Y_\tau$. In view of (3.6) and the fact that $\lim_{\tau\to+0}\tau^{-1}f^*(t, (3\tau/2)\xi^i_{\alpha_m}(t))=0$, $t \geqq 0$, by (f_3), we have

$$(3.12) \qquad \lim_{\tau\to+0}\frac{1}{\tau}\int_0^\infty s^{n-1}\eta^{p_m-i-1}_{\alpha_m}(s)f^*\Big(s, \frac{3}{2}\tau\xi^i_{\alpha_m}(s)\Big)ds = 0$$

with the aid of the Lebesgue dominated convergence theorem. From (3.11) and (3.12) it follows that there is a constant $\tau_0>0$ such that

$$(3.13) \qquad |(ABCFy)(t)| \leqq \frac{\tau}{2}\zeta^i_{\alpha_m}(t), \quad t \geqq 0,$$

for all $y \in Y_\tau$ and $0<\tau<\tau_0$.

Fix τ, $0<\tau<\tau_0$, and consider the mapping Φ_τ: $Y_\tau \to C[0, \infty)$ defined by

$$(3.14) \qquad (\Phi_\tau y)(t) = \tau\xi^i_{\alpha_m}(t) + (-1)^\rho(ABCFy)(t), \quad t \geqq 0,$$

where $\rho=\rho_m+\cdots+\rho_M-i$. By (3.13) it is clear that Φ_τ maps Y_τ into Y_τ. If $\{y_k\}$ is a sequence in Y_τ converging to $y \in Y_\tau$ in the $C[0, \infty)$ topology, use of (f_2), (f_3) and (3.6) together with the dominated convergence theorem shows that $\{(\Phi_\tau y_k)(t)\}$ converges to $(\Phi_\tau y)(t)$ uniformly on compact subintervals of $[0, \infty)$, implying the continuity of Φ_τ. Ascoli-Arzela's theorem can be used to show that $\Phi_\tau(Y_\tau)$ is relatively compact in $C[0, \infty)$. The Schauder-Tychonoff fixed point theorem then guarantees the existence of $y \in Y_\tau$ such that $y=\Phi_\tau y$. Put $u(x)=y(|x|)$, $x \in R^n$. Applying Lemma 2.1 repeatedly, we conclude that $u(x)$ is a positive entire solution of equation (3.1). Since $CFy \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$ for $y \in Y_\tau$, we have

$$\lim_{t\to\infty} \frac{(BCFy)(t)}{\zeta^i_{\alpha_m}(t)} = 0, \quad y \in Y_\tau,$$

by Lemma 2.5, and hence

$$\lim_{t\to\infty} \frac{(ABCFy)(t)}{\zeta^i_{\alpha_m}(t)} = 0, \quad y \in Y_\tau, \tag{3.15}$$

by (i) of Lemma 2.7. Thus the solution $u(x)=y(|x|)$ has the desired asymptotic property: $\lim_{|x|\to\infty} u(x)/\zeta^i_{\alpha_m}(|x|)=\tau$.

PROOF OF THEOREM 3.2. Let $\tau>2\lambda$, and define Y_τ and Φ_τ by (3.8) and (3.14), respectively. We observe that $Fy \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$ for $y \in Y_\tau$; in fact, in view of (f_2) and (f_4), we have

$$\begin{aligned}\frac{|f(t, y(t))|}{y(t)} &\leqq \frac{f^*(t, y(t))}{y(t)} \\ &\leqq \frac{f^*(t, (\tau/2)\xi^i_{\alpha_m}(t))}{(\tau/2)\xi^i_{\alpha_m}(t)} \leqq \frac{f^*(t, \lambda\xi^i_{\alpha_m}(t))}{\lambda\xi^i_{\alpha_m}(t)}, \quad t \geqq 0,\end{aligned}$$

and hence

$$|f(t, y(t))| \leqq 3f^*\Big(t, \frac{\tau}{2}\xi^i_{\alpha_m}(t)\Big) \leqq \frac{3\tau}{2\lambda} f^*(t, \lambda\xi^i_{\alpha_m}(t)), \quad t \geqq 0.$$

It follows that there is a constant $c>0$ such that

$$|(ABCFy)(t)| \leqq c\zeta^i_{\alpha_m}(t)\int_0^\infty s^{n-1}\eta^{p_m-i-1}_{\alpha_m}(s) f^*\Big(s, \frac{\tau}{2}\xi^i_{\alpha_m}(s)\Big)ds, \quad t \geqq 0, \tag{3.16}$$

for $y \in Y_\tau$, where A, B and C are defined by (3.9). Since $\lim_{\tau\to\infty} \tau^{-1} f^*(t, (\tau/2)\xi^i_{\alpha_m}(t))=0$, $t \geqq 0$, by (f_4), we have

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\infty s^{n-1}\eta^{p_m-i-1}_{\alpha_m}(s) f^*\Big(s, \frac{\tau}{2}\xi^i_{\alpha_m}(s)\Big)ds = 0,$$

which, combined with (3.16), implies the existence of a $\tau_0>2\lambda$ such that (3.13) holds for all $\tau>\tau_0$ and all $y \in Y_\tau$. We then conclude that, for any fixed $\tau>\tau_0$, the mapping Φ_τ has a fixed element $y \in Y_\tau$, which gives rise to a positive entire solution $u(x)=y(|x|)$ of equation (3.1). To verify the asymptotic property of $u(x)$ it suffices to observe that (3.15) also holds in this case. This completes the proof.

COROLLARY 3.1. *Suppose that* (f_1), (f_2) *and* (f_3) *hold. Let* α_m *be one of the numbers* $\alpha_1, \dots, \alpha_M$ *appearing in* (3.1), *and suppose that* $p_m \leqq [(n-1)/2]$ *if* $\alpha_m=0$. *Then,* $S(\zeta^i_{\alpha_m}) \neq \phi$ *for all* i, $0 \leqq i \leqq p_m-1$, *if there exists a constant* $\lambda>0$ *such that*

$$F^*(\lambda\zeta^{p_m-1}_{\alpha_m}) \in L^0_{\alpha_m}[0, \infty). \tag{3.17}$$

COROLLARY 3.2. *Suppose that* (f_1), (f_2) *and* (f_4) *hold. Let* α_m *be one of the numbers* $\alpha_1,\dots,\alpha_M$ *appearing in* (3.1), *and suppose that* $p_m \leqq [(n-1)/2]$ *if* $\alpha_m = 0$. *Then,* $S(\zeta^i_{\alpha_m}) \neq \phi$ *for all* i, $0 \leqq i \leqq p_m - 1$, *if there exists a constant* $\lambda > 0$ *such that*

$$F^*(\lambda\zeta_{\alpha_m}) \in L^{p_m-1}_{\alpha_m}[0, \infty). \tag{3.18}$$

PROOF OF COROLLARY 3.1. Let i, $0 \leqq i \leqq p_m - 1$, be fixed. Since, by Lemmas 2.2 and 2.3, $\zeta^i_{\alpha_m}(t) \leqq \zeta^{p_m-1}_{\alpha_m}(t)$ for sufficiently large t, say $t \geqq t_0$, (f_3) implies that

$$\zeta^{p_m-1}_{\alpha_m}(t) f^*(t, \lambda\zeta^i_{\alpha_m}(t)) \leqq \zeta^i_{\alpha_m}(t) f^*(t, \lambda\zeta^{p_m-1}_{\alpha_m}(t)), \quad t \geqq t_0.$$

Using this and the relation

$$\zeta^i_{\alpha_m}(t)\eta^{p_m-i-1}_{\alpha_m}(t) \sim c\zeta^{p_m-1}_{\alpha_m}(t)\eta_{\alpha_m}(t) \quad \text{as} \quad t \to \infty$$

for some constant $c > 0$, which also follows from Lemmas 2.2 and 2.3, we see from (3.17) that $F^*(\lambda\zeta^i_{\alpha_m}) \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$, so that $S(\zeta^i_{\alpha_m}) \neq \phi$ by Theorem 3.1. Since i is arbitrary, the conclusion follows.

PROOF OF COROLLARY 3.2. Let i be as above. Let t_0 be such that $\zeta^i_{\alpha_m}(t) \geqq \zeta_{\alpha_m}(t)$ for $t \geqq t_0$. We also have $F^*(\lambda\zeta^i_{\alpha_m}) \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$, because ($f_4$) implies

$$\zeta_{\alpha_m}(t) f^*(t, \lambda\zeta^i_{\alpha_m}(t)) \leqq \zeta^i_{\alpha_m}(t) f^*(t, \lambda\zeta_{\alpha_m}(t)), \quad t \geqq t_0,$$

and Lemmas 2.2 and 2.3 show

$$\zeta^i_{\alpha_m}(t)\eta^{p_m-i-1}_{\alpha_m}(t) \sim c\zeta_{\alpha_m}(t)\eta^{p_m-1}_{\alpha_m}(t) \quad \text{as} \quad t \to \infty$$

for some $c > 0$. Hence $S(\zeta^i_{\alpha_m}) \neq \phi$ by Theorem 3.2.

We will show that the conclusion of Theorem 3.2 can be strengthened if more restrictive conditions are placed on the nonlinearity of equation (3.1).

THEOREM 3.3. *Suppose that* (f_1), (f_2), (f_4) *and* (f_5) *are satisfied. Let* α_m *be one of the numbers* $\alpha_1,\dots,\alpha_M$ *appearing in* (3.1) *and let i be an integer satisfying* (3.5). *Suppose moreover that*

$$(-1)^\rho f(t, u) \geqq 0 \qquad \text{for} \quad (t, u) \in [0, \infty) \times (0, \infty), \tag{3.19}$$

where $\rho = p_m + \cdots + p_M - i$. *If* (3.6) *holds for some* $\lambda > 0$, *then* $T(\zeta^i_{\alpha_m}) = (0, \infty)$, *that is, for any given* $\tau > 0$ *there exists a positive radial entire solution* $u(x)$ *of equation* (3.1) *satisfying* (3.4).

PROOF. Fix $\tau > 0$ arbitrarily and define

$$Y_{\tau,\sigma} = \{y \in C[0, \infty): \tau\xi^i_{\alpha_m}(t) \leqq y(t) \leqq \sigma\xi^i_{\alpha_m}(t),\ t \geqq 0\},$$

where $\sigma \geqq \max\{\tau, \lambda\}$. If $y \in Y_{\tau,\sigma}$, then in view of (f_2), (f_4) and (f_5)

$$(-1)^{\rho} f(t, y(t)) \leqq f^*(t, y(t))$$
$$\leqq f^*(t, \sigma\xi^i_{\alpha_m}(t)) \leqq (\sigma/\lambda) f^*(t, \lambda\xi^i_{\alpha_m}(t)), \quad t \geqq 0,$$

which implies that $(-1)^{\rho} Fy \in L^{p_m-i-1}_{\alpha_m}[0, \infty)$. Letting A, B and C be as in (3.9) and arguing as in the proof of Theorem 3.2, we see that there is a $\sigma_0 \geqq \max\{\tau, \lambda\}$ such that

$$0 \leqq (-1)^{\rho}(ABCFy)(t) \leqq \frac{\sigma}{2}\zeta^i_{\alpha_m}(t), \quad t \geqq 0, \tag{3.20}$$

for all $\sigma \geqq \sigma_0$ and $y \in Y_{\tau,\sigma}$.

Put $\sigma^* = \max\{2\tau, \sigma_0\}$, and consider the mapping Φ_τ defined by (3.14). From (3.20) it follows that Φ_τ maps Y_{τ,σ^*} into itself. The continuity of Φ_τ and the relative compactness of $\Phi_\tau(Y_{\tau,\sigma^*})$ are verified without difficulty, and hence Φ_τ has a fixed element $y \in Y_{\tau,\sigma^*}$ by the Schauder-Tychonoff theorem. The function $u(x) = y(|x|)$ then gives a positive entire solution with the desired asymptotic behavior at infinity. This completes the proof.

The following theorem establishes a conclusion similar to Theorem 3.3 for the case where $f(t, u)$ is bounded by a function which is nonincreasing in u.

THEOREM 3.4. *Suppose that* (f_1), (f_2) *and* (f_6) *hold. Let* α_m, i *and* ρ *be as in Theorem* 3.3 *and suppose that* (3.19) *is satisfied. If* (3.6) *holds for all* $\lambda > 0$, *then* $T(\zeta^i_{\alpha_m}) = (0, \infty)$, *that is, for every* $\tau > 0$ *equation* (3.1) *has a positive radial entire solution* $u(x)$ *satisfying* (3.4).

PROOF. Let $\tau > 0$ be any fixed constant, and consider the set

$$Y_\tau = \{y \in C[0, \infty) \colon \tau\xi^i_{\alpha_m}(t) \leqq y(t) \leqq (\Phi_\tau(\tau\xi^i_{\alpha_m}))(t),\ t \geqq 0\},$$

where Φ_τ is given by (3.14); Y_τ is not empty because of (3.19). Hypotheses (f_2) and (f_4) imply that

$$(-1)^{\rho} f(t, y(t)) \leqq f^*(t, y(t)) \leqq f^*(t, \tau\xi^i_{\alpha_m}(t))), \quad t \geqq 0,$$

for $y \in Y_\tau$, and so Φ_τ obviously maps Y_τ into itself. Application of the Schauder-Tychonoff theorem completes the proof.

In what follows we are concerned with the problem of existence of positive entire solutions of equation (3.1) which are not asymptotic to any of the entire solution $\zeta^i_{\alpha_m}(|x|)$, $0 \leqq i \leqq p_m - 1$, $1 \leqq m \leqq M$, of the corresponding linear equation (3.3).

THEOREM 3.5. *Let* α_m *be one of the number* $\alpha_1, \ldots, \alpha_M$ *appearing in* (3.1) *and let* i *be an integer such that*

$$1 \leqq i \leqq p_m - 1 \quad \textit{if} \quad \alpha_m > 0,$$

$$\max\left\{1,\ p_m - \left[\frac{n-3}{2}\right]\right\} \leqq i \leqq p_m - 1 \quad \textit{if} \quad \alpha_m = 0.$$

In addition to (f_1) *suppose that*

(3.21) $f^*(t, u) \equiv (-1)^\rho f(t, u) \geqq 0$ *for* $(t, u) \in [0, \infty) \times (0, \infty)$,

where $\rho = p_m + \cdots + p_M - i$, *and* $f^*(t, u)$ *satisfies* (f_4) *and* (f_5). *If*

$$F^*(\zeta_{\alpha_m}^{i-1}) \notin L_{\alpha_m}^{p_m - i}[0, \infty) \quad \textit{and} \quad F^*(\zeta_{\alpha_m}^{i}) \in L_{\alpha_m}^{p_m - i - 1}[0, \infty), \tag{3.22}$$

then equation (3.1) *has a positive radial entire solution* $u(x)$ *such that*

$$\lim_{|x|\to\infty} \frac{u(x)}{\zeta_{\alpha_m}^{i-1}(|x|)} = \infty \quad \textit{and} \quad \lim_{|x|\to\infty} \frac{u(x)}{\zeta_{\alpha_m}^{i}(|x|)} = 0. \tag{3.23}$$

THEOREM 3.6. *Let m be an integer with* $1 \leqq m \leqq M-1$. *In addition to* (f_1) *suppose that* (3.21) *holds for* $\rho = p_{m+1} + \cdots + p_M$, *and* $f^*(t, u)$ *satisfies* (f_4) *and* (f_5). *If*

$$F^*(\zeta_{\alpha_m}^{p_m - 1}) \notin L_{\alpha_m}^{0}[0, \infty) \quad \textit{and} \quad F^*(\zeta_{\alpha_{m+1}}) \in L_{\alpha_{m+1}}^{p_{m+1}}[0, \infty), \tag{3.24}$$

then equation (3.1) *has a positive radial entire solution* $u(x)$ *such that*

$$\lim_{|x|\to\infty} \frac{u(x)}{\zeta_{\alpha_m}^{p_m - 1}(|x|)} = \infty \quad \textit{and} \quad \lim_{|x|\to\infty} \frac{u(x)}{\zeta_{\alpha_{m+1}}(|x|)} = 0. \tag{3.25}$$

PROOF OF THEOREM 3.5. For each $\tau \geqq 1$, define a set Y_τ by

$$Y_\tau = \{y \in C[0, \infty) \colon \tau \xi_{\alpha_m}^{i-1}(t) \leqq y(t) \leqq \tau(\xi_{\alpha_m}^{i-1}(t) + \zeta_{\alpha_m}^{i}(t)),\ t \geqq 0\}.$$

Since $\xi_{\alpha_m}^{i-1}(t) \leqq \zeta_{\alpha_m}^{i}(t)$ for sufficiently large t, (f_5) and the second condition of (3.22) imply that $(-1)^\rho F y \in L_{\alpha_m}^{p_m - i - 1}[0, \infty)$ for $y \in Y_\tau$, and it follows that

$$(-1)^\rho (ABCFy)(t) \leqq c\zeta_{\alpha_m}^{i}(t) \int_0^\infty s^{n-1} \eta_{\alpha_m}^{p_m - i - 1}(s) f^*(s, y(s))ds$$

$$\leqq c\zeta_{\alpha_m}^{i}(t) \int_0^\infty s^{n-1} \eta_{\alpha_m}^{p_m - i - 1}(s) f^*(s, \tau(\xi_{\alpha_m}^{i-1}(s) + \zeta_{\alpha_m}^{i}(s)))ds, \quad t \geqq 0,$$

for some constant $c > 0$, where A, B and C are given by (3.9) (see (3.11)). We now choose $\tau_0 \geqq 1$ so large that

$$(-1)^\rho (ABCFy)(t) \leqq \tau \zeta_{\alpha_m}^{i}(t), \quad t \geqq 0, \tag{3.26}$$

for all $\tau \geqq \tau_0$ and $y \in Y_\tau$. This is possible because $\lim_{\tau\to\infty} \tau^{-1} f^*(t, \tau(\xi_{\alpha_m}^{i-1}(t) + \zeta_{\alpha_m}^{i}(t))) = 0$, $t \geqq 0$, by (f_4), which implies

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\infty s^{n-1}\eta_{\alpha_m}^{p_m-i-1}(s)f^*(s,\ \tau(\xi_{\alpha_m}^{i-1}(s)+\zeta_{\alpha_m}^{i}(s)))ds = 0.$$

Fix $\tau\geqq\tau_0$ and define Φ_τ by

$$(\Phi_\tau y)(t) = \tau\xi_{\alpha_m}^{i-1}(t) + (-1)^\rho(ABCFy)(t), \quad t\geqq 0.$$

Then it follows from (3.26) that Φ_τ maps Y_τ into itself, and it is easy to show that Φ_τ is continuous and $\Phi_\tau(Y_\tau)$ is relatively compact. Consequently, there exists a fixed point $y\in Y_\tau$ of Φ_τ giving rise to entire solution $u(x)=y(|x|)$ of equation (3.1).

It remains to study the asymptotic behavior of $u(x)$ at infinity. Note first that the proof of the second relation in (3.23) is the same as in Theorem 3.2. Application of Lemma 2.6 shows that

$$\begin{aligned}\lim_{t\to\infty}\frac{(-1)^\rho(BCFy)(t)}{\zeta_{\alpha_m}^{i-1}(t)} &= (-1)^\rho\int_0^\infty s^{n-1}\eta_{\alpha_m}^{p_m-i}(s)(CFy)(s)ds\\ &= (-1)^\rho\int_0^\infty s^{n-1}(C^*\eta_{\alpha_m}^{p_m-i})(s)(Fy)(s)ds \qquad (3.27)\\ &\geqq (-1)^\rho c\int_0^\infty s^{n-1}\eta_{\alpha_m}^{p_m-i}(s)(Fy)(s)ds\end{aligned}$$

for some constant $c>0$, where $C^*=H_{\alpha_M}^{p_M}\cdots H_{\alpha_{m+1}}^{p_{m+1}}$, and Lemma 2.4 and (iii) of Lemma 2.7 have been used. Since

$$(-1)^\rho(Fy)(t)\geqq(-1)^\rho f(t,\ \tau\zeta_{\alpha_m}^{i-1}(t)), \quad t\geqq 0,$$

the last integral in (3.27) is equal to ∞, so that from (i) of Lemma 2.7 it follows that

$$\lim_{t\to\infty}\frac{(-1)^\rho(ABCFy)(t)}{\zeta_{\alpha_m}^{i-1}(t)} = \infty,$$

which establishes the first relation (3.23). This completes the proof.

PROOF OF THEOREM 3.6. We define Y_τ and Φ_τ, $\tau>0$, by

$$Y_\tau = \{y\in C[0,\infty):\ \tau\xi_{\alpha_m}^{p_m-1}(t)\leqq y(t)\leqq\tau(\xi_{\alpha_m}^{p_m-1}(t)+\zeta_{\alpha_{m+1}}(t)),\ t\geqq 0\}$$

and

$$(\Phi_\tau y)(t) = \tau\xi_{\alpha_m}^{p_m-1}(t) + (-1)^\rho(DEFy)(t), \quad t\geqq 0,$$

where $D=G_{\alpha_1}^{p_1}\cdots G_{\alpha_m}^{p_m}$ and $E=H_{\alpha_{m+1}}^{p_{m+1}}\cdots H_{\alpha_M}^{p_M}$. As in the proof of Theorem 3.5 it can be shown that there exists a $\tau_0\geqq 1$ such that

$$0\leqq(-1)^\rho(DEFy)(t)\leqq\tau\zeta_{\alpha_{m+1}}(t), \quad t\geqq 0,$$

for all $\tau \geqq \tau_0$ and $y \in Y_\tau$. With such a choice of τ, Φ_τ is shown to be continuous and to map Y_τ into a compact subset of Y_τ, and hence Φ_τ has a fixed element $y \in Y_\tau$.

Using Lemma 2.5, we see that $y(t)$ satisfies $\lim_{t\to\infty} (-1)^\rho (EFy)(t)/\zeta_{\alpha_{m+1}}(t)=0$, whence, applying (i) of Lemma 2.7, we find $\lim_{t\to\infty} (-1)^\rho (DEFy)(t)/\zeta_{\alpha_{m+1}}(t)=0$. This proves the second relation in (3.25). On the other hand, in view of Lemmas 2.6 and 2.4 and (iii) of Lemma 2.7 we have

$$
\begin{aligned}
\lim_{t\to\infty} \frac{(-1)^\rho (G_{\alpha_m}^{p_m} EFy)(t)}{\zeta_{\alpha_m}^{p_m-1}(t)} &= (-1)^\rho \int_0^\infty s^{n-1}\eta_{\alpha_m}(s)(EFy)(s)ds \\
&= \int_0^\infty s^{n-1}(E^*\eta_{\alpha_m})(s)(F^*y)(s)ds \\
&\geqq c\int_0^\infty s^{n-1}\eta_{\alpha_m}(s)(F^*y)(s)ds,
\end{aligned} \tag{3.28}
$$

where $E^* = H_{\alpha_M}^{p_M}\cdots H_{\alpha_{m+1}}^{p_{m+1}}$ and $c>0$ is a constant. This, combined with the first condition in (3.24), implies that the last integral in (3.28) diverges, and so repeated application of (i) of Lemma 2.7 shows that

$$\lim_{t\to\infty} \frac{y(t)}{\zeta_{\alpha_m}^{p_m-1}(t)} = \infty.$$

It follows that the function $u(x)=y(|x|)$ is an entire solution of (3.1) with the required asymptotic property. Thus the proof is complete.

THEOREM 3.7. *In addition to* (f_1) *suppose that* $f(t, u)\geqq 0$ *for* $(t, u)\in[0, \infty)\times(0, \infty)$ *and* $f(t, u)$ *is nonincreasing in* u. *If*

$$F(\lambda\zeta_{\alpha_M}^{p_M-1}) \notin L_{\alpha_M}^0[0, \infty) \tag{3.29}$$

for every constant $\lambda>0$, *then equation* (3.1) *possesses a positive radial entire solution* $u(x)$ *such that*

$$\lim_{|x|\to\infty} \frac{u(x)}{\zeta_{\alpha_M}^{p_M-1}(|x|)} = \infty. \tag{3.30}$$

PROOF. Fix $\tau>0$ arbitrarily and define

$$(\Phi_\tau y)(t) = \tau\zeta_{\alpha_M}^{p_M-1}(t) + (GFy)(t), \quad t\geqq 0,$$

where $G = G_{\alpha_1}^{p_1}\cdots G_{\alpha_M}^{p_M}$, and

$$Y_\tau = \{y\in C[0, \infty)\colon \tau\zeta_{\alpha_M}^{p_M-1}(t)\leqq y(t)\leqq(\Phi_\tau(\tau\zeta_{\alpha_M}^{p_M-1}))(t),\ t\geqq 0\}.$$

In view of the nonincreasing nature of f, Φ_τ maps Y_τ into itself. The continuity of Φ_τ and the relative compactness of $\Phi_\tau(Y_\tau)$ are easily verified. Therefore there exists $y\in Y_\tau$ such that $y=\Phi_\tau y$. Now, Lemma 2.6 shows that

$$\lim_{t\to\infty}\frac{(G_{\alpha_M}^{p_M}Fy)(t)}{\zeta_{\alpha_M}^{p_M-1}(t)}=\int_0^\infty s^{n-1}\eta_{\alpha_M}(s)(Fy)(s)ds,$$

whence, via (i) of Lemma 2.7, it follows that

$$\begin{aligned}\lim_{t\to\infty}\frac{y(t)}{\zeta_{\alpha_M}^{p_M-1}(t)}&=\tau+\lim_{t\to\infty}\frac{(GFy)(t)}{\zeta_{\alpha_M}^{p_M-1}(t)}\\&=\tau+\prod_{k=1}^{M-1}(\alpha_M^2-\alpha_k^2)^{-p_k}\int_0^\infty s^{n-1}\eta_{\alpha_M}(s)(Fy)(s)ds.\end{aligned}$$

If this limit is finite, then there exists a constant $c>0$ such that $y(t)\leqq c\zeta_{\alpha_M}^{p_M-1}(t)$ for $t\geqq 0$, so that $(Fy)(t)\geqq(F(c\zeta_{\alpha_M}^{p_M-1}))(t)$, $t\geqq 0$, by the nonincreasing nature of f. Therefore, by (3.29),

$$\int_0^\infty s^{n-1}\eta_{\alpha_M}(s)(Fy)(s)ds\geqq\int_0^\infty s^{n-1}\eta_{\alpha_M}(s)(F(c\zeta_{\alpha_M}^{p_M-1}))(s)ds=\infty,$$

which implies $\lim_{t\to\infty}y(t)/\zeta_{\alpha_M}^{p_M-1}(t)=\infty$, a contradiction. Thus, $u(x)=y(|x|)$ is an entire solution of (3.1) satisfying (3.30). This completes the proof.

The following theorem shows that a class of higher order sublinear elliptic equations may possess positive entire solutions which decay to zero at infinity.

THEOREM 3.8. *In addition to* (f_1) *suppose that*

(3.31) $$f^*(t,u)\equiv(-1)^\rho f(t,u)>0\quad \textit{for}\quad (t,u)\in[0,\infty)\times(0,\infty),$$

where $\rho=p_1+\cdots+p_M$. *Suppose that* $f^*(t,u)$ *satisfies* (f_4), (f_5) *and*

(3.32) $$\lim_{u\to+0}u^{-1}f^*(t,u)=\infty,\quad t\geqq 0.$$

Suppose moreover that $p_1\leqq[(n-1)/2]$ *if* $\alpha_1=0$.

(i) *If there is a constant* $\lambda>0$ *such that*

(3.33) $$F^*(\lambda\zeta_{\alpha_1})\in L_{\alpha_1}^{p_1-1}[0,\infty),$$

then equation (3.1) *has a positive radial entire solution* $u(x)$ *such that*

(3.34) $$\lim_{|x|\to\infty}\frac{u(x)}{\zeta_{\alpha_1}(|x|)}=0.$$

(ii) *If there is a constant* $\lambda>0$ *such that*

(3.35) $$F^*(\lambda\gamma_{\alpha_1}^{p_1-1})\in\tilde{L}_{\alpha_1}[0,\infty),$$

then equation (3.1) *has a positive radial entire solution* $u(x)$ *such that*

(3.36) $$\lim_{|x|\to\infty}\frac{u(x)}{\eta_{\alpha_1}^{p_1-1}(|x|)}=\mu$$

for some constant $\mu>0$.

PROOF. (i) We first note that the relations

$$\lim_{\sigma\to\infty}\sigma^{-1}f^*(t,\sigma\zeta_{\alpha_1}(t))=0 \quad\text{and}\quad \lim_{\tau\to+0}\tau^{-1}f^*(t,\tau\gamma_{\alpha_1}^{p_1-1}(t))=\infty$$

which follow from (f_4) and (3.32), respectively, imply that

$$\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\infty s^{n-1}\eta_{\alpha_1}^{p_1-1}(s)f^*(s,\sigma\zeta_{\alpha_1}(s))ds=0 \tag{3.37}$$

and

$$\lim_{\tau\to+0}\frac{1}{\tau}\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)f^*(s,\tau\gamma_{\alpha_1}^{p_1-1}(s))ds=\infty. \tag{3.38}$$

Let $\tau>0$ and $\sigma>0$ be constants such that

$$\tau\gamma_{\alpha_1}^{p_1-1}(t)\leqq\sigma\zeta_{\alpha_1}(t) \qquad \text{for} \quad t\geqq 0 \tag{3.39}$$

and consider the set

$$Y_{\tau,\sigma}=\{y\in C[0,\infty):\tau\gamma_{\alpha_1}^{p_1-1}(t)\leqq y(t)\leqq\sigma\zeta_{\alpha_1}(t),\ t\geqq 0\}$$

and the mapping

$$(\Phi y)(t)=(-1)^\rho(HFy)(t),\quad t\geqq 0, \tag{3.40}$$

where $H=H_{\alpha_1}^{p_1}\cdots H_{\alpha_M}^{p_M}$. Letting $\tilde{H}=H_{\alpha_2}^{p_2}\cdots H_{\alpha_M}^{p_M}$ and $\tilde{H}^*=H_{\alpha_M}^{p_M}\cdots H_{\alpha_2}^{p_2}$, and using Lemmas 2.5, 2.4 and 2.7, we see that for any $y\in Y_{\tau,\sigma}$

$$\begin{aligned}(-1)^\rho(HFy)(t)&\leqq\zeta_{\alpha_1}(t)\int_0^\infty s^{n-1}\eta_{\alpha_1}^{p_1-1}(s)(\tilde{H}F^*y)(s)ds\\
&=\zeta_{\alpha_1}(t)\int_0^\infty s^{n-1}(\tilde{H}^*\eta_{\alpha_1}^{p_1-1})(s)(F^*y)(s)ds\\
&\leqq c_1\zeta_{\alpha_1}(t)\int_0^\infty s^{n-1}\eta_{\alpha_1}^{p_1-1}(s)(F^*y)(s)ds\\
&\leqq c_1\zeta_{\alpha_1}(t)\int_0^\infty s^{n-1}\eta_{\alpha_1}^{p_1-1}(s)f^*(s,\sigma\zeta_{\alpha_1}(s))ds,\quad t\geqq 0,\end{aligned} \tag{3.41}$$

where $c_1>0$ is a constant. On the other hand, from Lemmas 2.8 and 2.7 we have

$$\begin{aligned}(-1)^\rho(HFy)(t)&\geqq\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)(\tilde{H}F^*y)(s)ds\\
&=\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}(\tilde{H}^*\gamma_{\alpha_1})(s)(F^*y)(s)ds\end{aligned}$$

$$\geqq c_2\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)(F^*y)(s)ds \tag{3.42}$$
$$\geqq c_2\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)f^*(s, \tau\gamma_{\alpha_1}^{p_1-1}(s))ds, \quad t \geqq 0,$$

where $c_2>0$ is a constant. Here we have used the fact that $\gamma_\alpha(t)\equiv\eta_\alpha(t)$ for large t to botain the second inequality in (3.42). Combining (3.41) and (3.42) with (3.37) and (3.38), respectively, we infer that there exist a $\sigma>0$ (sufficiently large) and a $\tau>0$ (sufficiently small) such that (3.39) holds and

$$\tau\gamma_{\alpha_1}^{p_1-1}(t) \leqq (-1)^\rho(HFy)(t) \leqq \sigma\zeta_{\alpha_1}(t), \quad t \geqq 0,$$

for all $y \in Y_{\tau,\sigma}$. This shows that, for such σ and τ, Φ maps $Y_{\tau,\sigma}$ into itself. Since Φ is continuous and $\Phi(Y_{\tau,\sigma})$ is relatively compact, Φ has a fixed point $y \in Y_{\tau,\sigma}$: $y=\Phi y$. Lemma 2.5 implies that this y satisfies $\lim_{t\to\infty}(-1)^\rho(HFy)(t)/\zeta_{\alpha_1}(t)=0$. Therefore we conclude that $u(x)=y(|x|)$ gives an entire solution of (3.1) satisfying (3.34).

(ii) We now define the set $Y_{\tau,\sigma}$ by

$$Y_{\tau,\sigma} = \{y \in C[0, \infty): \tau\gamma_{\alpha_1}^{p_1-1}(t)\leqq y(t)\leqq\sigma\gamma_{\alpha_1}^{p_1-1}(t),\ t\geqq 0\},$$

where $\sigma\geqq\tau>0$. It is easy to verify that the following inequalities hold for all $y \in Y_{\tau,\sigma}$ and $t\geqq 0$:

$$(-1)^\rho(HFy)(t) \leqq \gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\Gamma_{\alpha_1}(s)(\tilde{H}F^*y)(s)ds$$
$$= \gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}(\tilde{H}^*\Gamma_{\alpha_1})(s)(F^*y)(s)ds$$
$$\leqq c_1\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\Gamma_{\alpha_1}(s)(F^*y)(s)ds \tag{3.43}$$
$$\leqq c_1\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\Gamma_{\alpha_1}(s)f^*(s, \sigma\gamma_{\alpha_1}^{p_1-1}(s))ds,$$

$$(-1)^\rho(HFy)(t) \geqq \gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)(\tilde{H}F^*y)(s)ds$$
$$= \gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}(\tilde{H}^*\gamma_{\alpha_1})(s)(F^*y)(s)ds$$
$$\geqq c_2\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)(F^*y)(s)ds \tag{3.44}$$
$$\geqq c_2\gamma_{\alpha_1}^{p_1-1}(t)\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)F^*(s, \tau\gamma_{\alpha_1}^{p_1-1}(s))ds,$$

where $c_1>0$ and $c_2>0$ are constants. In (3.43) and (3.44) the first inequalities follow from Lemma 2.8, the next equalities are imliped by Lemma 2.4 and the second inequalities are consequences of Lemma 2.7. Noting that

$$\lim_{\sigma\to\infty}\frac{1}{\sigma}\int_0^\infty s^{n-1}\Gamma_{\alpha_1}(s)f^*(s,\ \sigma\gamma_{\alpha_1}^{p_1-1}(s))ds=0$$

and

$$\lim_{\tau\to+0}\frac{1}{\tau}\int_0^\infty s^{n-1}\gamma_{\alpha_1}(s)f^*(s,\ \tau\gamma_{\alpha_1}^{p_1-1}(s))ds=0,$$

we see from (3.43) and (3.44) that $\sigma>0$ and $\tau>0$ can be chosen so that

$$\tau\gamma_{\alpha_1}^{p_1-1}(t)\leqq(-1)^p(HFy)(t)\leqq\sigma\gamma_{\alpha_1}^{p_1-1}(t),\quad t\geqq 0,$$

for all $y\in Y_{\tau,\sigma}$. The above observation shows that the mapping Φ defined by (3.40) possesses a fixed element y in $Y_{\tau,\sigma}$. Since it follows from Lemma 2.8 that $\lim_{t\to\infty}y(t)/\eta_{\alpha_1}^{p_1-1}(t)=\text{const.}>0$, the function $u(x)=y(|x|)$ is a desired decaying entire solution of equation (3.1). This completes the proof.

4. Example

To illustrate our main results presented in the preceding section, we consider the fourth order semilinear elliptic equation

(4.1) $$\Delta^2u-2a\Delta u+bu=q(|x|)u^p,\quad x\in R^n,\quad n\geqq 3,$$

where a and b are nonnegative constants, p is a constant with $p\neq 1$, and $q(t)$ is a continuous function in $[0,\infty)$. In this case $f(t,u)=q(t)u^p$, for which (f_1) is satisfied. Assumption (f_3) or (f_4) holds according as $p>1$ or $p<1$, and (f_5) or (f_6) holds according as $p\geqq 0$ or $p\leqq 0$.

We assume throughout this section that $a^2\geqq b$, in which case we have

$$\Delta^2-2a\Delta+b=(\Delta-\alpha^2)(\Delta-\beta^2),$$

where

$$\alpha^2=a-(a^2-b)^{1/2},\quad \beta^2=a+(a^2-b)^{1/2}.$$

The unperturbed linear equation

$$\Delta^2u-2a\Delta u+bu=0$$

has the positive entire solutions

$$\{\zeta_\alpha(|x|),\ \zeta_\beta(|x|)\}\quad\text{if}\quad a^2>b;$$

$$\{\zeta_\alpha(|x|),\ \zeta_\alpha^1(|x|)\}\quad\text{if}\quad a^2=b,$$

and the positive solutions

$$\{\eta_\alpha(|x|),\ \eta_\beta(|x|)\} \quad \text{if} \quad a^2 > b;$$

$$\{\eta_\alpha(|x|),\ \eta_\alpha^1(|x|)\} \quad \text{if} \quad a^2 = b$$

in $R^n \setminus \{0\}$. The asymptotic behavior of these functions is described in (2.3)–(2.6) and Lemmas 2.2 and 2.3.

(i) Suppose that $a^2 > b$. It is easy to see that the condition $F^*(\lambda\zeta_\alpha) \in L_\alpha^1[0, \infty)$ is equivalent to

$$\int^\infty s^{-(p-1)(n-1)/2} e^{(p-1)\alpha s} |q(s)| ds < \infty \qquad \text{for} \quad b > 0, \tag{4.2}$$

$$\int^\infty s|q(s)|ds < \infty \qquad \text{for} \quad b = 0, \tag{4.3}$$

and the condition $F^*(\lambda\zeta_\beta) \in L_\beta^0[0, \infty)$ is equivalent to

$$\int^\infty s^{-(p-1)(n-1)/2} e^{(p-1)\beta s} |q(s)| ds < \infty. \tag{4.4}$$

Therefore, Theorems 3.1 and 3.2 show that, for equation (4.1),

$$T(\zeta_\alpha) \cap T(\zeta_\beta) \supset (0, \tau_0) \qquad \text{for} \quad p > 1,$$

$$T(\zeta_\alpha) \cap T(\zeta_\beta) \supset (\tau_0, \infty) \qquad \text{for} \quad p < 1,$$

provided conditions (4.2)–(4.4) are satisfied. From Theorems 3.3 and 3.4 it follows that, under (4.2) or (4.3),

$$T(\zeta_\alpha) = (0, \infty) \quad \text{if} \quad p < 1 \quad \text{and} \quad q(t) \geqq 0 \qquad \text{for} \quad t \geqq 0,$$

and under (4.4),

$$T(\zeta_\beta) = (0, \infty) \quad \text{if} \quad p < 1 \quad \text{and} \quad q(t) \leqq 0 \qquad \text{for} \quad t \geqq 0.$$

According to Theorem 3.8 we see that if $0 < p < 1$ and $q(t) > 0$ for $t \geqq 0$, then condition (4.2) or (4.3) guarantees the existence of a positive radial entire solution $u(x)$ of (4.1) such that $\lim_{|x| \to \infty} u(x)/\zeta_\alpha(|x|) = 0$. Note that the condition $F^*(\lambda\gamma_\alpha) \in \tilde{L}_\alpha[0, \infty)$ reduces to

$$\int^\infty s^{-(p-1)(n-1)/2} e^{-(p-1)\alpha s} |q(s)| ds < \infty \qquad \text{for} \quad b > 0, \tag{4.5}$$

$$\int^\infty s^{n-1-p(n-2)} |q(s)| ds < \infty \qquad \text{for} \quad b = 0. \tag{4.6}$$

Theorem 3.8–(ii) implies that if $0 < p < 1$ and $q(t) > 0$ for $t \geqq 0$ and if (4.5) or (4.6) holds, then equation (4.1) has a radial decaying entire solution $u(x)$ such that $\lim_{|x|\to\infty} u(x)/\eta_\alpha(|x|) = \text{const.} > 0$. On the other hand, the conditions $F^*(\lambda\zeta_\alpha) \notin L^0_\alpha[0, \infty)$ and $F^*(\lambda\zeta_\beta) \notin L^0_\beta[0, \infty)$ become

$$(4.7) \qquad \int^\infty s^{-(p-1)(n-1)/2} e^{(p-1)\alpha s} |q(s)| ds = \infty \qquad \text{for} \quad b > 0,$$

$$(4.8) \qquad \int^\infty s|q(s)| ds = \infty \qquad \text{for} \quad b = 0,$$

and

$$(4.9) \qquad \int^\infty s^{-(p-1)(n-1)/2} e^{(p-1)\beta s} |q(s)| ds = \infty,$$

respectively. In case $0<p<1$ and $q(t) \leqq 0$ for $t \geqq 0$, Theorem 3.6 implies that, under condition (4.4) and either (4.7) or (4.8), equation (4.1) has a radial entire solution $u(x)$ such that $\lim_{|x|\to\infty} u(x)/\zeta_\alpha(|x|) = \infty$ and $\lim_{|x|\to\infty} u(x)/\zeta_\beta(|x|) = 0$. Finally, in case $p \leqq 0$ and $q(t) \geqq 0$ for $t \geqq 0$, Theorem 3.7 shows that (4.9) is a sufficient condition for (4.1) to have a radial entire solution $u(x)$ with the property $\lim_{|x|\to\infty} u(x)/\zeta_\beta(|x|) = \infty$.

(ii) Suppose that $a^2 = b$. Then, the conditions $F^*(\lambda\zeta_\alpha) \in L^1_\alpha[0, \infty)$ and $F^*(\lambda\zeta^1_\alpha) \in L^0_\alpha[0, \infty)$ are equivalent to

$$\int^\infty s^{1-(p-1)(n-1)/2} e^{(p-1)\alpha s} |q(s)| ds < \infty \qquad \text{for} \quad b > 0,$$

$$\int^\infty s^3 |q(s)| ds < \infty \qquad \text{for} \quad b = 0 \quad \text{and} \quad n \geqq 5,$$

and

$$\int^\infty s^{p-(p-1)(n-1)/2} e^{(p-1)\alpha s} |q(s)| ds < \infty \qquad \text{for} \quad b > 0,$$

$$\int^\infty s^{1+2p} |q(s)| ds < \infty \qquad \text{for} \quad b = 0 \quad \text{and} \quad n \geqq 5,$$

respectively, while $F^*(\lambda\gamma^1_\alpha) \in \tilde{L}_\alpha[0, \infty)$ is equivalent to

$$\int^\infty s^{p-(p-1)(n-1)/2} e^{-(p-1)\alpha s} |q(s)| ds < \infty \qquad \text{for} \quad b > 0,$$

$$\int^\infty s^{n-1-p(n-4)} |q(s)| ds < \infty \qquad \text{for} \quad b = 0 \quad \text{and} \quad n \geqq 5.$$

With the help of these integral conditions one can establish criteria for the existence of positive entire solutions of equation (4.1) with various asymptotic properties, as described in Theorems 3.1 through 3.8. The details are left to the reader.

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References

[1] N. Fukagai, Existence and uniqueness of entire solutions of second order sublinear elliptic equations, Funkcial. Ekvac. **29** (1986), 151–165.

[2] N. Fukagai, Positive entire solutions of semilinear elliptic equations, Math. Ann. **274** (1986), 75–93.

[3] N. Kawano, On bounded entire solutions of semilinear elliptic equations, Hiroshima Math. J. **14** (1984), 125–158.

[4] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions of even order semilinear elliptic equations in the plane, Proc. Royal Soc. Edinburgh, to appear.

[5] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions of even order semilinear elliptic equations, preprint.

[6] T. Kusano, M. Naito and C. A. Swanson, Entire solutions of a class of even order quasilinear elliptic equations, Math. Z. **195** (1987), 151–163.

[7] T. Kusano and C. A. Swanson, Positive entire solutions of semilinear biharmonic equations, Hiroshima Math. J. **17** (1987), 13–28.

[8] H. Usami, On strongly increasing entire solutions of even order semilinear elliptic equations, Hiroshima Math. J. **17** (1987), 175–217.

[9] W. Walter, Ganze Lösungen der Differentialgleichung $\Delta^p u = f(u)$, Math. Z. **67** (1957), 32–37.

[10] W. Walter, Zur Existenz ganzer Lösungen der Differentialgleichung $\Delta^p u = e^u$, Arch. Math. **9** (1958), 308–312.

[11] W. Walter and H. Rhee, Entire solutions of $\Delta^p u = f(r, u)$, Proc. Royal Soc. Edinburgh **82A** (1979), 189–192.

[12] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge UP, London, 1944.

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