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The Gangolli estimates for the coefficients of the Harish-Chandra expansions of the Eisenstein integrals on real reductive Lie groups

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1. Introduction

1.1 SUMMARY

Let G and $\mathfrak{g}$ be a Lie group and its Lie algebra. We denote by $\mathfrak{g}_c$ and G_c the complexification of $\mathfrak{g}$ and the complex adjoint group respectively. In this paper we assume that G is of class $\mathscr{H}$, that is, G satisfies the following three conditions: (1) $\mathfrak{g}$ is reductive and $Ad(G)\subset G_c$; (2) the center of the analytic subgroup corresponding to $[\mathfrak{g}, \mathfrak{g}]$ is finite; (3) the number of connected components of G is finite.

As is well known, the Eisenstein integrals on G, that is, the matrix elements of representations of principal series for G, play an essential role in harmonic analysis on G. Therefore it is very important to know the asymptotic behaviors of the Eisenstein integrals. In fact, the leading terms of the expansions of these integrals give the Harish-Chandra C-functions and are closely related to the Plancherel measure (cf. [5], [6]). The analysis of the Schwartz space on G needs only the leading terms as an approximation of the Eisenstein integrals and estimates of difference between them (cf. [1], [2], [6]). On the other hand, to carry out closer study of harmonic analysis on G, such as Paley-Wiener type theorems for various function spaces, one needs to know the asymptotic behavior of higher order, of the Eisenstein integrals and to estimate the approximations (cf. [2], [7], [8], [9], [10]). If we use our results to prove the Paley-Wiener type theorem for the L^p Schwartz spaces, as showed in [3], we can get it without the approximation theorems such as in [2], [8] or [10]. For an application, see Eguchi and Wakayama [4].

For each $\nu\in\mathscr{F}_C$, the zonal spherical function is defined by

$$\varphi_\nu(x) = \int_K e^{(\nu-\rho)(H(xk))}dk \quad (x\in G).$$

(The notation will be explained later.) When $x=h$ varies in the positive Weyl chamber A^+ of A, $\varphi_\nu(h)$ is expanded into an infinite series (cf. Harish-Chandra [5]) as

$$\varphi_\nu(h) = e^{-\rho(\log h)} \sum_{s \in W(A)} c(s\nu)\Phi(s\nu : h),$$

$$\Phi(\nu : h) = \sum_{\lambda \in L} \Gamma_\lambda(\nu-\rho) e^{(\nu-\lambda)(\log h)} \quad (h \in A^+).$$

Here $c(\cdot)$ is the Harish-Chandra c-function, Γ_λ $(\lambda \in L)$ are rational functions on $\mathscr{F}_C$ given by certain explicit recursion formulas and ν varies in a certain open dense subset ${}^*\mathscr{F}_C$ of $\mathscr{F}_C$. In his paper [8], Gangolli gave a remarkable estimate for the coefficients of the expansion as follows:

There exist absolute constants d, $D>0$ such that

$$|\Gamma_\lambda(\nu-\rho)| \leq Dm(\lambda)^d \tag{1.1}$$

for all $\nu \in \mathscr{R} = \{\nu = \xi + \eta : \xi \in \mathscr{F},\ -\eta \in cl(\mathscr{F}_R^+)\}$ and $\lambda \in L$.

This result is very fine as compared with the general estimate by Helgason [7].

For general Eisenstein integrals, Harish-Chandra [6] gave the same kind of series expansions. In the general case, when confined to $\mathscr{R}$, some singularities of the coefficients Γ arise from the double unitary representation of K. In this paper, using the fact that these singularities vanish away by multiplying a polynomial P, we give an estimate similar to (1.1) for $P\Gamma$ instead of Γ itself.

1.2 NOTATION AND PRELIMINARIES

We use the standard notation $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ for the ring of integers, the field of real numbers and the field of the complex numbers respectively. Let $\boldsymbol{R}^+$ denote the set of nonnegative real numbers and $\boldsymbol{Z}^+ = \boldsymbol{Z} \cap \boldsymbol{R}^+$. We write $(-1)^{1/2}$ for a square root of -1. We fix a Lie group G of class $\mathscr{H}$ and a maximal compact subgroup K in G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and the subalgebra of $\mathfrak{g}$ corresponding to K respectively. Let θ be the Cartan involution of G fixing all elements of K. Denote the Killing form on $\mathfrak{g}$ by $\langle \cdot, \cdot \rangle$. Then the quadratic form $-\langle X, \theta X \rangle$ $(X \in \mathfrak{g})$ defines a norm $\|\cdot\|$ on $\mathfrak{g}$. We also denote by the same symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the nondegenerate bilinear form and the norm on the real dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ defined by those on $\mathfrak{g}$ in natural way.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decomposition corresponding to θ, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{s}$ and A the corresponding analytic subgroup of G. We denote by $\mathscr{F}_R$, $\mathscr{F}_C$ and $\mathscr{F}$ the real dual space of $\mathfrak{a}$, its complexification and the subspace $(-1)^{1/2}\mathscr{F}_R$ of $\mathscr{F}_C$ respectively.

Let $G = KAN$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be the Iwasawa decompositions of G and $\mathfrak{g}$ respectively. If $x \in G$, x can be written uniquely as $x = \kappa(x) \exp H(x) n(x)$ $(\kappa(x) \in K,\ H(x) \in \mathfrak{a},\ n(x) \in N)$. Let M be the centralizer of A in K. Then $P = MAN$ is a minimal parabolic subgroup of G. We denote by $W(A)$ the Weyl group of (G, A). As usual, ρ is the element of $\mathscr{F}_R$ defined by $\rho(H) = (1/2) tr\, ad(H)|_{\mathfrak{n}}$ $(H \in \mathfrak{a})$.

Let Σ be the set of all roots of (P, A) and $\{\alpha_1,\dots, \alpha_l\}$ (here $l=\text{prk}\, P-\text{prk}\, G$) the set of all simple roots in Σ under the ordering on $\mathfrak{a}$ which is compatible with the above Iwasawa decomposition. Let $\mathfrak{a}^+$, $\mathscr{F}_R^+$ and A^+ be the positive Weyl chambers of $\mathfrak{a}$, $\mathscr{F}_R$ and A with respect to this ordering respectively. We denote by $cl(\mathscr{F}_R^+)$ the closure of $\mathscr{F}_R^+$ in $\mathscr{F}_R$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{a}\subset\mathfrak{h}$ and put $\mathfrak{h}_{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}$. Fix the ordering on the real dual psace $\mathfrak{h}^*=\mathfrak{h}_{\mathfrak{k}}+(-1)^{1/2}\mathfrak{a}$ of $\mathfrak{h}$ which is compatible with the one on $\mathfrak{a}$. We denote by P_+ the set of positive roots α of $(\mathfrak{g}_c, \mathfrak{h}_c)$ such that $\tilde{\alpha}=\alpha|_{\mathfrak{a}}\neq 0$. For each $\alpha\in P_+$, define the element $Q_{\tilde{\alpha}}\in\mathfrak{a}$ by $\alpha(H)=\langle Q_{\tilde{\alpha}}, H\rangle$ for all $H\in\mathfrak{a}$. For each $\alpha\in P_+$, we choose the root vectors $X_{\pm\alpha}\in\mathfrak{g}_c^{\pm\alpha}$ so that $\langle X, X_{-\alpha}\rangle=1$, and write them as $X_{\pm\alpha}=Y_{\pm\alpha}+Z_{\pm\alpha}$ ($Y_{\pm\alpha}\in\mathfrak{k}_c$, $Z_{\pm\alpha}\in\mathfrak{s}_c$).

Let $\mathfrak{G}$ be the universal enveloping algebra of $\mathfrak{g}_c$ and $\mathfrak{M}$, $\mathfrak{U}$ and $\mathfrak{R}$ the subalgebras of $\mathfrak{G}$ generated by $(1, \mathfrak{m}_c)$, $(1, \mathfrak{a}_c)$ and $(1, \mathfrak{k}_c)$ respectively. Let ω and $\omega_{\mathfrak{m}}$ denote the Casimir elements of $\mathfrak{G}$ and $\mathfrak{M}$ respectively. Choose an orthonormal basis $H_1,\dots, H_{l'}$ of $\mathfrak{a}$. Then ω can be written as

$$\omega = \omega_{\mathfrak{m}} + \sum_{i=1}^{l'} H_i^2 + \sum_{\alpha\in P_+} (X_\alpha X_{-\alpha}+X_{-\alpha}X_\alpha). \tag{1.2}$$

For each $D\in\mathfrak{G}$, we denote by $\mathfrak{l}_A(D)$ the radial component of D. In the sequel, we use the following expression of $\mathfrak{l}_A(\omega)$.

LEMMA 1.1. *The radial component $\mathfrak{l}_A(\omega)$ of the Casimir element ω of $\mathfrak{G}$ can be written as follows:*

$$\begin{aligned}\mathfrak{l}_A(\omega) &= \mathfrak{l}_A(\omega_{\mathfrak{m}}) + \delta'(\omega)\\ &\quad - 2 \sum_{\alpha\in P_+} (\sinh(\alpha))^{-2}(1\otimes 1\otimes Y_\alpha Y_{-\alpha}+ Y_\alpha Y_{-\alpha}\otimes 1\otimes 1)\\ &\quad + 4 \sum_{\alpha\in P_+} (\sinh(\alpha))^{-1}\coth(\alpha)(Y_\alpha\otimes 1\otimes Y_{-\alpha}).\end{aligned}$$

Here

$$\delta'(\omega) = \sum_{i=1}^{l'} H_i^2 + \sum_{\alpha\in P_+} \coth(\alpha) Q_{\tilde{\alpha}}$$

(cf. [11]).

2. Eisenstein Integrals and the Harish-Chandra Expansions

Let $\tau=(\tau_1, \tau_2)$ be a double unitary representation of K on a finite dimensional Hilbert space V. We denote by V_M the subspace of V comprised of all elements $v\in V$ such that $\tau_1(m)v=v\tau_2(m)$ for all $m\in M$. Then the Eisenstein integral on G is defined for $v\in V_M$ and $\nu\in\mathscr{F}_C$ by the integral:

$$E(v: \nu: x) = \int_K \tau_1(\kappa(xk))v\tau_2(k^{-1})e^{(\nu-\rho)(H(xk))}dk. \tag{2.1}$$

We define an endomorphism γ of $\mathscr{H} = \mathrm{Hom}_C(V_M, V_M)$ by

$$\gamma(T) = [\tau_2(\omega_m), T] \quad (T \in \mathscr{H}). \tag{2.2}$$

Then it is known that γ is a self-adjoint operator on $\mathscr{H}$ and all its eigenvalues are real. Let $\gamma_1,\dots, \gamma_t$ be the set of all distinct eigenvalues with multiplicities $m_1,\dots, m_t$ respectively. Let L denote the set of $\lambda = n_1\alpha_1 + \cdots + n_l\alpha_l$ ($n_i \in \boldsymbol{Z}^+$, $i = 1,\dots, l$) and put $L' = L \setminus \{0\}$. If $\lambda, \lambda' \in L$ and $\lambda - \lambda' \in L$ then we denote $\lambda \gg \lambda'$. For each $\lambda \in L$, define the functions Γ_λ on $\mathscr{F}_C$ with values in $\mathscr{H}$ recursively. If $\lambda = 0$, set $\Gamma_0 \equiv 1$; if $\lambda \neq 0$, then Γ_λ is given by the relation:

$$\begin{aligned}
&\{2\lambda - \langle \lambda, \lambda - 2\rho\rangle\}\Gamma_\lambda - \gamma(\Gamma_\lambda) \\
&= 2 \sum_{\alpha\in P_+} \sum_{n\geq 1} \{\tilde{\alpha} - \langle \tilde{\alpha}, \lambda - 2n\tilde{\alpha}\rangle\}\Gamma_{\lambda-2n\tilde{\alpha}} \\
&\quad + 8 \sum_{\alpha\in P_+} \sum_{n\geq 1} (2n-1)\tau_1(Y_\alpha)\tau_2(Y_{-\alpha})\Gamma_{\lambda-(2n-1)\tilde{\alpha}} \\
&\quad - 8 \sum_{\alpha\in P_+} \sum_{n\geq 1} n\{\tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha})\}\Gamma_{\lambda-2n\tilde{\alpha}}\,.
\end{aligned} \tag{2.3}$$

Here $\Gamma_\lambda \equiv 0$ for any λ which does not lie in L, so that the sum appearing on the right hand side are all finite.

For each $i \in \{1,\dots, t\}$ and $\lambda \in L'$, put

$$\sigma_{\lambda,i} = \{\nu \in \mathscr{F}_C \colon 2\langle \lambda, \nu\rangle = \langle \lambda, \lambda\rangle + \gamma_i\}$$

and let Υ and Υ_0 be the complement of the set $\cup_{\lambda\in L} \cup_i \sigma_{\lambda,i}$ in $\mathscr{F}_C$ and the subset of $\mathscr{F}_C$ comprised of all $\nu \in \mathscr{F}_C$ such that $w\nu \in \Upsilon$ for all $w \in W(A)$ respectively.

If $\mu \in \mathscr{F}_R$ and $h \in A$, for simplicity, we write h^μ for $e^{\mu(\log h)}$. The series in the following is called the Harish-Chandra expansion of the Eisenstein integral.

THEOREM 2.1 (Harish-Chandra). *Fix a* $\nu \in \Upsilon$ *and set*

$$\Phi(\nu: h) = \sum_{\lambda\in L} \Gamma_\lambda(\nu - \rho)h^{\nu-\lambda} \quad (h \in A^+).$$

Then the function $h \to \Phi(\nu: h)$ *is analytic on* A^+ *and satisfies the following differential equation*:

$$\Phi(\nu: h; e^\rho \circ \mathfrak{l}_A(\omega) \circ e^{-\rho}) = \Phi(\nu: h)\{\langle \nu, \nu\rangle - \langle \rho, \rho\rangle + \tau_2(\omega_m)\}. \tag{2.5}$$

Moreover, $h^\rho E(v: \nu: h)$ *is expanded as*

$$h^\rho E(v: \nu: h) = \sum_{w\in W(A)} \Phi(w\nu: h)C_\tau(w: \nu)v \quad (v \in V_M,\ h \in A^+,\ \nu \in \Upsilon_0), \tag{2.6}$$

where $C_\tau(w: \nu)$ are the Harish-Chandra C-functions, which are meromorphic on Υ_0 with values in $\mathscr{H}$ (cf. [11]).

We are interested in this series expansion and shall give an estimate for coefficients Γ_λ $(\lambda \in L)$.

3. The Estimate of the Coefficients Γ_λ

Recall first that $\gamma_1, \dots, \gamma_t$ are the set of all distinct eigenvalues of the endomorphism γ of $\mathscr{H}$ defined by (2.2) and $m_1, \dots, m_t$ denote their multiplicities. We assume that

$$\gamma_1 < \cdots < \gamma_s < 0 \le \gamma_{s+1} < \cdots < \gamma_t .$$

Let L_1' denote the finite set of all $\lambda \in L'$ such that $-\langle \lambda, \lambda \rangle \ge \gamma_1$. For each $\lambda \in L$, we define polynomials p_λ by

$$p_\lambda(\nu) = 1 \quad \text{if} \quad \lambda \in L \setminus L_1';$$

$$p_\lambda(\nu) = \prod_{1 \le i \le s, \langle \lambda, \lambda \rangle + \gamma_i \le 0} (2\langle \lambda, \nu \rangle - \langle \lambda, \lambda \rangle - \gamma_i)^{m_i} \quad \text{if} \quad \lambda \in L_1'$$

and set

$$d'(\lambda) = \sum_{1 \le i \le s, \langle \lambda, \lambda \rangle + \gamma_i \le 0} m_i .$$

We also put

$$P(\nu) = \prod_{\lambda \in L_1'} p_\lambda(\nu), \quad d = \sum_{\lambda \in L_1'} d'(\lambda);$$

$$P_\lambda(\nu) = \prod_{\lambda' \in L', \lambda' \ll \lambda} p_{\lambda'}(\nu), \quad d(\lambda) = \sum_{\lambda' \in L', \lambda' \ll \lambda} d'(\lambda')$$

for $\lambda \in L'$. Then remark that P is of finite degree and thus $d < \infty$. Recall that

$$\mathscr{R} = \{\xi + \eta \in \mathscr{F}_C : \xi \in \mathscr{F}, \ -\eta \in cl(\mathscr{F}_R^+)\}.$$

THEOREM 3.1. *There exist absolute constants D, $d_1 > 0$ such that*

$$\|P_\lambda(\nu)\Gamma_\lambda(\nu - \rho)\| \le D(1 + \|\nu\| + m(\lambda))^{2d} m(\lambda)^{d_1} \quad (\nu \in \mathscr{R}) \tag{3.1}$$

for all $\lambda \in L$. Here

$$m(\lambda) = n_1 + \cdots + n_l \quad \textit{if} \quad \lambda = n_1\alpha_1 + \cdots + n_l\alpha_l \in L.$$

In order to prove the result, we first introduce $\tilde{\Phi}$ and Ψ defined by

$$\tilde{\Phi}(\nu: h) = h^{-\rho}\Phi(\nu: h) \quad \text{and} \quad \Psi(\nu: h) = \Delta(h)^{1/2}\tilde{\Phi}(\nu: h) \quad (h \in A^+),$$

where Δ is defined by

$$\Delta(h) = h^{2\rho} \prod_{\alpha \in P_+} (1-h^{-2\alpha}) \quad (h \in A^+).$$

From the differential equation (2.5), it follows that $\tilde{\Phi}$ and Ψ satisfy:

$$\tilde{\Phi}(\nu: h; \mathfrak{l}_A(\omega)) = \tilde{\Phi}(\langle \nu, \nu \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_{\mathrm{m}})), \tag{3.2}$$

$$\Delta(h)^{1/2} \circ \mathfrak{l}_A(\omega) \circ \Delta(h)^{-1/2} \Psi = \Psi(\langle \nu, \nu \rangle - \langle \rho, \rho \rangle + \tau_2(\omega_{\mathrm{m}})). \tag{3.3}$$

On the other hand, by the definition of Ψ, we have

$$\Psi(\nu: h) = \prod_{\alpha \in P_+} (1-h^{-2\alpha})^{1/2} \Phi(\nu: h) \quad (h \in A^+),$$

which can be written in the following form by the binomial theorem

$$\begin{aligned} &= (\sum_{\sigma \in L} b_\sigma h^{-\sigma})(h^\nu \sum_{\mu \in L} \Gamma_\mu(\nu-\rho) h^{-\mu}) \\ &= h^\nu \sum_{\lambda \in L} (\sum_{\sigma, \mu \in L, \sigma+\mu=\lambda} b_\sigma \Gamma_\mu(\nu-\rho)) h^{-\lambda}. \end{aligned}$$

We remark here that the coefficients b_σ are absolute constants and thus independent of ν. Put $a_\lambda(\nu) = \sum_{\sigma+\mu=\lambda} b_\sigma \Gamma_\mu(\nu-\rho)$. Then we have

$$\Psi(\nu: h) = h^\nu \sum_{\lambda \in L} a_\lambda(\nu) h^{-\lambda} \quad (h \in A^+). \tag{3.4}$$

Conversely, if Ψ is written as (3.4), Φ can be written as follows:

$$\begin{aligned} \Phi(\nu: h) &= \prod_{\alpha \in P_+} (1-h^{-2\alpha})^{-1/2} \Psi(\nu: h) \\ &= (\sum_{\mu \in L} d_\mu h^{-\mu})(h^\nu \sum_{\sigma \in L} a_\sigma(\nu) h^{-\sigma}) = h^\nu \sum_{\lambda \in L} (\sum_{\sigma+\mu=\lambda} d_\mu a_\sigma(\nu)) h^{-\nu}. \end{aligned}$$

Therefore, we have

$$\Gamma_\lambda(\nu-\rho) = \sum_{\sigma+\mu=\lambda} d_\mu a_\sigma(\nu). \tag{3.5}$$

LEMMA 3.2. *If we write*

$$\prod_{\alpha \in P_+} (1-h^{-2\alpha})^{-1/2} = \sum_{\mu \in L} d_\mu h^{-\mu} \quad (h \in A^+).$$

then there exist constants R_1, $R_2 > 0$ such that, for any λ, σ, $\mu \in L$ satisfying $\lambda = \sigma + \mu$,

$$|d_\mu| \leq R_1 m(\lambda)^{R_2}. \tag{3.6}$$

Since the proof is elementary, it will be left to the reader.

From (3.5) and the last lemma, it follows that, for the proof of Theorem 3.1, it is sufficient to get the same kind of estimate for a_σ.

PROPOSITION 3.3. *Let $a_\sigma(v)$ be the coefficients in* (3.4). *Then there exist constants D', $d_1'>0$ such that*

$$\|P_\lambda(v)a_\lambda(v)\| \le D'(1+\|v\|+m(\lambda))^{2d}m(\lambda)^{d_1'}.$$

4. The Proof of Proposition 3.3

We first consider the series expansion of the operator $\Delta(h)^{1/2}\circ \mathfrak{l}_A(\omega)\circ \Delta(h)^{-1/2}$. By Lemma 1.1, it can be written as:

$$\begin{aligned}\Delta(h)^{1/2}\circ \mathfrak{l}_A(\omega)\circ\Delta(h)^{-1/2} &= \mathfrak{l}_A(\omega_{\mathrm{m}}) + \Delta(h)^{1/2}\circ\delta'(\omega)\circ\Delta(h)^{-1/2}\\ &\quad -2\sum_{\alpha\in P_+}(\sinh(\alpha))^{-2}(1\otimes 1\otimes Y_\alpha Y_{-\alpha}+Y_\alpha Y_{-\alpha}\otimes 1\otimes 1)\\ &\quad +4\sum_{\alpha\in P_+}(\sinh(\alpha))^{-1}\coth(\alpha)(Y_\alpha\otimes 1\otimes Y_{-\alpha}),\end{aligned}\tag{4.1}$$

where

$$\delta'(\omega) = \sum_{i=1}^{l'} H_i^2 + \sum_{\alpha\in P_+}\coth(\alpha)Q_{\tilde\alpha}.$$

LEMMA 4.1. *Write H for* $\log h$ $(h\in A^+)$. *Then*

$$\begin{aligned}\Delta(h)^{1/2}\circ \mathfrak{l}_A(\omega)\circ\Delta^{-1/2} &= \mathfrak{l}_A(\omega_{\mathrm{m}}) + \sum_{i=1}^{l'} H_i^2 - \langle\rho,\rho\rangle\\ &\quad +\sum_{\alpha\in P_+}\langle\tilde\alpha,\tilde\alpha\rangle\sum_{j\ge1} je^{-2j\alpha(H)} - \sum_{\alpha,\beta\in P_+,\alpha\ne\beta}\langle\tilde\alpha,\tilde\beta\rangle\sum_{j\ge1,k\ge0}e^{-2(j\alpha+k\beta)(H)}\\ &\quad -8\sum_{\alpha\in P_+}\sum_{j\ge1} je^{-2j\alpha(H)}(1\otimes1\otimes Y_\alpha Y_{-\alpha}+Y_\alpha Y_{-\alpha}\otimes1\otimes1)\\ &\quad +8\sum_{\alpha\in P_+}\sum_{j\ge1}(2j-1)e^{-(2j-1)\alpha(H)}(Y_\alpha\otimes1\otimes Y_{-\alpha}).\end{aligned}\tag{4.2}$$

To prove the lemma, we use the following relation.

LEMMA 4.2. *Let H be as in Lemma* 4.1. *Then we have*

$$\begin{aligned}\Delta(h)^{1/2}\circ\delta'(\omega)\circ\Delta(h)^{-1/2} &= \sum_{i=1}^{l'} H_i^2\\ &- \{\langle\rho,\rho\rangle - \sum_{\alpha\in P_+}\langle\tilde\alpha,\tilde\alpha\rangle\sum_{j\ge1} je^{-2j\alpha(H)} + \sum_{\alpha,\beta\in P_+,\alpha\ne\beta}\langle\tilde\alpha,\tilde\beta\rangle\sum_{j\ge1,k\ge0}e^{-2(j\alpha+k\beta)(H)}\}.\end{aligned}$$

PROOF. For simplicity, we put $\Delta=\Delta(h)$. From

$$H_i\circ\Delta = \sum_{\alpha\in P_+}\alpha(H_i)\coth\alpha(H)\Delta + \Delta\circ H_i,$$

it follows that

$$\delta'(\omega) = \sum_{i=1}^{l'} \Delta^{-1}\circ H_i\circ\Delta\circ H_i, \tag{4.3}$$

$$\Delta^{1/2}\circ\delta'(\omega)\circ\Delta^{-1/2} = \sum_{i=1}^{l'} \Delta^{-1/2}\circ H_i\circ\Delta\circ H_i\circ\Delta^{-1/2}.$$

Computing $H_i\circ\Delta^{-1/2}$ and $H_i\circ\Delta^{1/2}$, we see that the last expression equals

$$\sum_{i=1}^{l'} H_i^2 - \left\{\frac{1}{2}\sum_{i=1}^{l'} \Delta^{-1}\circ(H_i^2\Delta) - \frac{1}{4}\sum_{i=1}^{l'} [\Delta^{-1}\circ(H_i\Delta)]^2\right\}.$$

As is easily seen, this is equal to

$$\sum_i H_i^2 - \left\{\frac{1}{2}\sum_i H_i^2(\log\Delta) + \frac{1}{4}\sum_i (H_i\log\Delta)^2\right\}.$$

From the definition of Δ, we have

$$H_i\log\Delta = 2\{\rho(H)+\sum_{\alpha\in P_+}\alpha(H_i)\sum_{j\geq 1} e^{-2j\alpha(H)}\},$$

$$H_i^2(\log\Delta) = -4\sum_{\alpha\in P_+}\alpha(H_i)^2\sum_{j\geq 1} je^{-2j\alpha(H)}.$$

Hence we have

$$(H_i\log\Delta)^2 = 4\{\rho(H_i)^2+2\sum_{\alpha\in P_+}\rho(H_i)\alpha(H_i)\sum_{j\geq 1} e^{-2j\alpha(H)}$$

$$+\sum_{\alpha\in P_+}\alpha(H_i)^2\sum_{j,k\geq 1} e^{-2(j+k)\alpha(H)}+\sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\alpha(H_i)\beta(H_i)\sum_{j,k\geq 1} e^{-2(j\alpha+k\beta)(H)}\}.$$

Using the fact that $\sum_i\alpha(H_i)H_i=Q_{\tilde\alpha}$ and $\sum_i\rho(H_i)^2=\langle\rho,\rho\rangle$, we have also

$$\frac{1}{4}\sum_{i=1}^{l'}(H_i\log\Delta)^2 = \langle\rho,\rho\rangle + \sum_{\alpha\in P_+}\langle\tilde\alpha,\tilde\alpha\rangle\sum_{j\geq 1} je^{-2j\alpha(H)}$$

$$+\sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\langle\tilde\alpha,\tilde\beta\rangle\sum_{j\geq 1,k\geq 0} e^{-2(j\alpha+k\beta)(H)}.$$

Combining these, we get the desired expression.

PROOF OF LEMMA 4.1. Applying the series expansion:

$$(\sinh(\alpha))^{-2} = 4\sum_{l\geq 1} le^{-2l\alpha},$$

$$\sinh(\alpha)^{-1}\coth(\alpha) = 2\sum_{l\geq 1}(2l-1)e^{-(2l-1)\tilde\alpha}$$

and Lemma 4.2 to the right hand side of (4.1), we obtain

$$\Delta(h)^{1/2}\circ\mathfrak{l}_A(\omega)\circ\Delta(h)^{-1/2} = \mathfrak{l}_A(\omega_{\mathrm{m}}) + \sum_{i=1}^{l'} H_i^2 - \langle\rho,\rho\rangle$$

$$+\sum_{\alpha\in P_+}\langle\tilde\alpha,\tilde\alpha\rangle\sum_{j\geq 1} je^{-2j\alpha(H)} - \sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\langle\tilde\alpha,\tilde\beta\rangle\sum_{j\geq 1,k\geq 0} e^{-2(j\alpha+k\beta)(H)}$$

$$- 8 \sum_{\alpha \in P_+} \sum_{p \geq 1} p e^{-2p\alpha(H)}(1 \otimes 1 \otimes Y_\alpha Y_{-\alpha} + Y_\alpha Y_{-\alpha} \otimes 1 \otimes 1)$$
$$+ 8 \sum_{\alpha \in P_+} \sum_{q \geq 1} (2q-1) e^{-(2q-1)\alpha(H)}(Y_\alpha \otimes 1 \otimes Y_{-\alpha}).$$

This is the desired relation (4.2).

We differentiate $\Psi(\nu: h) = h^\nu \sum a_\lambda(\nu) h^{-\lambda}$ by $\Delta(h)^{1/2} \circ \underline{1}_A(\omega) \circ \Delta(h)^{-1/2}$ and use Lemma 4.1 and the differential equation (3.3). Then, comparing the coefficients of $h^{\nu-\lambda}$ in both side, we obtain the following recursive relation:

$$\begin{aligned}
&[2\langle \lambda, \nu \rangle - \langle \lambda, \lambda \rangle] a_\lambda(\nu) - \gamma(a_\lambda(\nu)) \\
&= \sum_{\alpha \in P_+} [\langle \tilde{\alpha}, \tilde{\alpha} \rangle - 8F_\alpha] \sum_{j \geq 1} j a_{\lambda - 2j\tilde{\alpha}}(\nu) - \sum_{\alpha, \beta \in P_+, \alpha \neq \beta} \langle \tilde{\alpha}, \tilde{\beta} \rangle \sum_{j \geq 1, k \geq 0} a_{\lambda - 2j\tilde{\alpha} - 2k\tilde{\beta}}(\nu) \\
&\quad + 8 \sum_{\alpha \in P_+} G_\alpha \sum_{j \geq 1} (2j-1) a_{\lambda - (2j-1)\tilde{\alpha}}(\nu),
\end{aligned} \tag{4.4}$$

where

$$F_\alpha = \tau_1(Y_\alpha Y_{-\alpha}) + \tau_2(Y_\alpha Y_{-\alpha}), \quad G_\alpha = \tau_1(Y_\alpha) \circ \tau_2(Y_{-\alpha}).$$

Conversely, if we define a series $\{a_\lambda\}_{\lambda \in L}$ by $a_0(\nu) = 1$ and (4.4), a_λ ($\lambda \in L$) are well defined for every generic ν. Using these a_λ ($\lambda \in L$), define Ψ by (3.4). Then obviously Ψ satisfies the differential equation (3.3). We shall obtain the estimate of $a_\lambda(\nu)$ by making use of the recursive relation (4.4).

Since $\gamma_1, \dots, \gamma_s$ are the set of all distinct negative eigenvalues of the endomorphism γ of $\mathscr{H}$, if we assume that all $a_{\lambda'}$ ($\lambda' \ll \lambda$) are defined and regard (4.4) as the defining formula of a_λ, we find that all singularities of a_λ in $\mathscr{R}$ are concentrated into P_λ.

We now put

$$Q_\lambda(\nu) = P_\lambda(\nu)(1 + \|\nu\| + \|\lambda\|)^{-2d(\lambda)},$$
$$q_\lambda(\nu) = p_\lambda(\nu)(1 + \|\nu\| + \|\lambda\|)^{-2d'(\lambda)}$$

and consider (4.4) multiplied by $Q_\lambda(\nu)$ instead of (4.4) itself:

$$\begin{aligned}
&[2\langle \lambda, \nu \rangle - \langle \lambda, \lambda \rangle] Q_\lambda(\nu) a_\lambda(\nu) - \gamma(Q_\lambda(\nu) a_\lambda(\nu)) \\
&= \sum_{\alpha \in P_+} [\langle \tilde{\alpha}, \tilde{\alpha} \rangle - 8F_\alpha] q_\lambda(\nu) \sum_{j \geq 1} j Q^1_{\lambda, j}(\nu) Q_{\lambda - 2j\tilde{\alpha}}(\nu) a_{\lambda - 2j\tilde{\alpha}}(\nu) \\
&\quad - \sum_{\alpha, \beta \in P_+, \alpha \neq \beta} \langle \tilde{\alpha}, \tilde{\beta} \rangle q_\lambda(\nu) \sum_{j \geq 1, k \geq 0} Q_{\lambda, j, k}(\nu) Q_{\lambda - 2j\tilde{\alpha} - 2k\tilde{\beta}}(\nu) a_{\lambda - 2j\tilde{\alpha} - 2k\tilde{\beta}}(\nu) \\
&\quad + 8 \sum_{\alpha \in P_+} G_\alpha q_\lambda(\nu) \sum_{j \geq 1} (2j-1) Q^2_{\lambda, j}(\nu) Q_{\lambda - (2j-1)\tilde{\alpha}}(\nu) a_{\lambda - (2j-1)\tilde{\alpha}}(\nu).
\end{aligned} \tag{4.5}$$

Here $Q^1_{\lambda, j}$, $Q_{\lambda, j, k}$ and $Q^2_{\lambda, j}$ are determined by

$$\begin{aligned}
Q_\lambda(\nu) q_\lambda(\nu)^{-1} &= Q_{\lambda, j, k}(\nu) Q_{\lambda - 2j\tilde{\alpha} - 2k\tilde{\beta}}(\nu) \\
&= Q^2_{\lambda, j}(\nu) Q_{\lambda - (2j-1)\tilde{\alpha}}(\nu) = Q^1_{\lambda, j}(\nu) Q_{\lambda - 2j\tilde{\alpha}}(\nu).
\end{aligned}$$

From the definition, it is clear that there exists a constant $C_1>0$ such that

$$|Q^1_{\lambda,j}(\nu)| < C_1, \quad |Q_{\lambda,j,k}(\nu)| < C_1, \quad |Q^2_{\lambda,j}(\nu)| < C_1 \tag{4.6}$$

for all $\lambda \in L'$ and $\nu \in \mathscr{F}_C$ and j, k. We define $b_\lambda(\nu)$ $(\lambda \in L)$ by

$$b_0(\nu) = 1 \quad \text{if} \quad \lambda = 0;$$
$$b_\lambda(\nu) = Q_\lambda(\nu) a_\lambda(\nu) \quad \text{if} \quad \lambda \in L'.$$

For simplicity, we also put

$$\gamma(\lambda\colon \nu) = (2\langle\lambda, \nu\rangle - \langle\lambda, \lambda\rangle)\boldsymbol{I} - \gamma,$$

where $\boldsymbol{I}$ denotes the identity operator on $\mathscr{H}$. Then (4.5) is written as

$$\begin{aligned}\gamma(\lambda\colon \nu) b_\lambda(\nu) = &\sum_{\alpha\in P_+} [\langle\tilde{\alpha}, \tilde{\alpha}\rangle - 8F_\alpha] q_\lambda(\nu) \sum_{j\geq 1} jQ^1_{\lambda,j}(\nu) b_{\lambda-2j\nu}(\nu) \\ &- \sum_{\alpha,\beta\in P_+, \alpha\neq\beta} \langle\tilde{\alpha}, \tilde{\beta}\rangle q_\lambda(\nu) \sum_{j\geq 1, k\geq 0} Q_{\lambda,j,k}(\nu) b_{\lambda-2j\tilde{\alpha}-2k\tilde{\beta}}(\nu) \\ &+ 8\sum_{\alpha\in P_+} G_\alpha q_\lambda(\nu) \sum_{j\geq 1} (2j-1) Q^2_{\lambda,j}(\nu) b_{\lambda-(2j-1)\tilde{\alpha}}(\nu).\end{aligned} \tag{4.7}$$

Note that $\mathscr{H}$ is a Hilbert space of finite dimension, say n, with respect to the inner product corresponding to the Hilbert-Schmidt norm $\|\cdot\|$ and fix an orthonormal basis $\mathscr{B} = \{\phi_1, \dots, \phi_n\}$ of $\mathscr{H}$. Let $A_{\gamma(\lambda:\nu)}$ be the matrix of the endomorphism $\gamma(\lambda\colon \nu)$ with respect to $\mathscr{B}$. Since the endomorphism γ is self-adjcint, there exists a unitary matrix B such that

$$BA_{\gamma(\lambda:\nu)}B^{-1} = \operatorname{diag}(a_1, \dots, a_1, \dots, a_t, \dots, a_t).$$

Here,

$$a_i = 2\langle\lambda, \nu\rangle - \langle\lambda, \lambda\rangle - \gamma_i \quad (i = 1, \dots, t).$$

We then obviously have

$$A^{-1}_{\gamma(\lambda:\nu)} = B^{-1} \operatorname{diag}(a_1^{-1}, \dots, a_1^{-1}, \dots, a_t^{-1}, \dots, a_t^{-1}) B.$$

Combining this with the fact that $\|B\| = n^{1/2}$, we obtain

$$\begin{aligned}\|p_\lambda(\nu) A^{-1}_{\gamma(\lambda:\nu)}\|_2^2 \leq n\{&|p_\lambda(\nu)|^2 \sum_{1\leq i\leq t, \|\lambda\|^2+\gamma_i>0} m_i |a_i|^{-2} \\ &+ \sum_{1\leq i\leq s, \|\lambda\|^2+\gamma_i\leq 0} \Big(\prod_{j=1, j\neq i}^{t} |a_j|^{2m_j}\Big) m_i |a_i|^{2(m_i-1)}\}.\end{aligned}$$

Since we can choose constants $C_2>0$ and $C_3>0$ so that

$$\|\lambda\| m(\lambda)^{-1} < C_2$$
$$|p_\lambda(\nu)|^2 < C_3(1 + \|\nu\| + \|\lambda\|)^{4d'(\lambda)},$$

we can find a constant $C_4>0$ such that

$$\|p_\lambda(\nu)A^{-1}_{\gamma(\lambda:\nu)}\|_2 < C_4(1+\|\nu\|+\|\lambda\|)^{2d'(\lambda)}m(\lambda)^{-2}.$$

Hence we have

$$\|q_\lambda(\nu)A^{-1}_{\gamma(\lambda:\nu)}\|_2 < C_5 m(\lambda)^{-2} \quad (\lambda\in L',\ \nu\in\mathscr{R}). \tag{4.8}$$

Putting

$$C_6 = C_1C_5 \max\{\|\langle\tilde\alpha,\ \tilde\alpha\rangle-8F_\alpha\|,\ 8\|G_\alpha\|,\ |\langle\tilde\alpha,\ \tilde\beta\rangle| : \alpha,\ \beta\in P_+\}$$

and combining (4.8) with (4.7), we obtain the following estimate for b_λ:

$$\begin{aligned}\|b_\lambda(\nu)\| &\le C_6 m(\lambda)^{-2}\{\sum_{\alpha\in P_+}\sum_{j\ge1}2j\|b_{\lambda-2j\tilde\alpha}(\nu)\|\\ &+\sum_{\alpha\in P_+}\sum_{j\ge1}(2j-1)\|b_{\lambda-(2j-1)\tilde\alpha}(\nu)\|+\sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\ \sum_{j\ge1,k\ge0}\|b_{\lambda-2j\tilde\alpha-2k\tilde\beta}(\nu)\|\}\\ &= C_6 m(\lambda)^{-2}\{\sum_{\alpha\in P_+}\sum_{j\ge1}j\|b_{\lambda-j\tilde\alpha}(\nu)\|+\sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\ \sum_{j\ge1,k\ge0}\|b_{\lambda-2j\tilde\alpha-2k\tilde\beta}(\nu)\|\}\\ &= m(\lambda)^{-2}\sum_{r=1}^{m(\lambda)-1}(S_1(r)+S_2(r)),\end{aligned}$$

where

$$S_1(r) = C_6\sum_{\alpha\in P_+}\ \sum_{m(\lambda-j\tilde\alpha)=r,\,j\ge1} j\|b_{\lambda-j\tilde\alpha}(\nu)\|,$$

$$S_2(r) = C_6\sum_{\alpha,\beta\in P_+,\alpha\neq\beta}\ \sum_{m(\lambda-2j\tilde\alpha-2k\tilde\beta)=r,\,j\ge1,\,k\ge0}\|b_{\lambda-2j\tilde\alpha-2k\tilde\beta}(\nu)\|.$$

Put now

$$H_0(\nu)=1,\quad H_r(\nu)=\sup_{\mu\in L',\,m(\mu)=r}\|b_\mu(\nu)\| \quad (r\ge1).$$

By an argument parallel to that in [8], we see that there exists a constant $C_7>0$ such that $S_1(r)$ and $S_2(r)$ are bounded by $C_7H_r(\nu)m(\lambda)$ and thus we can take a constant $C_8>0$ so that

$$\|b_\lambda(\nu)\| \le C_8(\sum_{r=1}^{m(\lambda)-1}H_r(\nu))m(\lambda)^{-1}.$$

Moreover, if we define a seires $\{D_r\}$ $(r\in \boldsymbol{Z}^+)$ by

$$D_0=1,\quad D_r=\frac{1}{r}C_8\sum_{s=0}^{r-1}D_s \quad (r\ge1),$$

then it is easy (cf. [8]) to see that

$$H_n(\nu)\le D_n \quad (\text{for all } n\in \boldsymbol{Z}^+ \text{ and } \nu\in\mathscr{R})$$

and that there exists a constant $C_9>0$ such that

$$D_n \leq C_9 n^{C_8-1} \quad (\text{for all } n \in Z^+).$$

This shows that

$$\|b_\lambda(\nu)\| \leq C_9 m(\lambda)^{C_8} \quad (\lambda \in L').$$

Since $d(\lambda) \leq d$ for all $\lambda \in L'$, we see from this that we can choose costants D, $d_1>0$ so that

$$\|P_\lambda(\nu) a_\lambda(\nu)\| \leq D(1+\|\nu\|+m(\lambda))^{2d} m(\lambda)^{d_1}.$$

This is the desired estimate for $P_\lambda a_\lambda$. This completes the proof of Proposition 3.3.

REMARK. When we study harmonic analysis on the Riemannian symmetric space G/K, only the Eisenstein integrals of special case $\tau=(\tau_1, 1)$ are related to the analysis. In these cases, since any singularity of Γ does not appear in $\mathscr{R}$, we can take $P(\nu)\equiv 1$.

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