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Almost complex and almost contact structures in fibred Riemannian spaces

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Introduction

The idea of foliations of manifolds emerged from works of C. Ehresmann and G. Reeb in the 1940's. One of directions of its study in differential geometry and topology has been to research possibility of fibering of manifolds, and many works in this field were synthetically reviewed by H. B. Lawson Jr. [5] and B. L. Reinhart [13].

Another direction is to research structures and properties of fibred space, and it goes back to a unified field theory in a 5-dimensional Riemannian space due to Th. Kaluza and O. Klein. Fibred Riemannian spaces were first considered by Y. Muto [7] and treated by B. L. Reinhart [12] in the name of foliated Riemannian manifolds. B. O'Neill [11] called such a foliation a Riemannian submersion and gave its structure equations and in the almost same time K. Yano and S. Ishihara [20, 21, 22] developed an extensive theory of fibred Riemannian spaces. M. Ako [1] and T. Okubo [10] studied fibred spaces with almost complex or almost Hermitian structure. These works were synthetically reported in S. Ishihara and M. Konishi's monograph [4].

In connection with almost contact structure, S. Tanno [17] and Y. Ogawa [9] investigated principal bundles over almost complex spaces having a 1-dimensonal structure group. Generalizing Calabi-Eckmann's example, S. Morimoto [6] defined an almost complex structure in the product of two almost contact spaces and obtained a condition on the normality, and S. Goldberg and K. Yano [3] researched similar properties for the product of two framed manifolds.

On the other hand, the tangent bundle of a Riemannian space can be endowed with a Riemannian metric, see S. Sasaki [14, 15], and with an almost complex structure associated with the Riemannian connection, see S. Tachibana and M. Okumura [16]. P. Dombroski [2], T. Nagano [8], S. Tanno [18], Y. Tashiro [19], I. Yokote [23] and other geometers investigated properties of tangent bundles related with metric and almost complex structure.

In this point of view, the purpose of the present paper is to study relations among structures of fibred spaces with almost Hermitian or almost contact metric structures and those of base spaces and fibres, and to apply results to tangent bundles of Riemannian spaces. In §§1–3, we shall explain preliminaries and fundamental equations on fibred Riemannian spaces with projectable metric. In §4, we shall treated fibred almost Hermitian spaces with anti-invariant fibres and obtain a condition for such a fibred space to be Kaehlerian. It will be shown in §5 that the tangent bundle of a Riemannian space is one treated in §4. In §§6 and 7, we shall consider a fibred almost contact metric space with invariant fibres tangent to the structure vector. §8 is devoted to the study of relations among the almost complex structure induced in the total space from the lift of an almost contact structure of the base space and that of each fibre. Finally, in the last section, we shall deal with the tangent bundles of almost contact manifolds with nearly Kaehlerian or almost Kaehlerian structure.

§1. Preliminaries on fibred Riemannian spaces

Let $\{\tilde{M}, M, \tilde{g}, \pi\}$ be a fibred space, where $\{\tilde{M}, \tilde{g}\}$ is the total space with Riemannian metric $\tilde{g}$, M the base space and π the projection $\tilde{M}\rightarrow M$ of maximum rank everywhere. We suppose that the dimensions of $\tilde{M}$ and M are r and n respectively. The fibre at any point P of M, the inverse image $\pi^{-1}(P)$, is of dimension $s=r-n$ and denoted by $\bar{M}(P)$ or generally by $\bar{M}$. We suppose that every fibre is connected. In the total space $\{\tilde{M}, \tilde{g}\}$, the horizontal distribution D is by definition of dimension n and perpendicular to the tangent space of the fibre $\bar{M}$ at each point.

The *horizontal* and *vertical* parts of a tensor field $\tilde{T}$ in the total space $\tilde{M}$ are tensor fields of the same type as $\tilde{T}$ and denoted by $\tilde{T}^H$ and $\tilde{T}^V$ respectively. For a function $\tilde{f}$, a vector field $\tilde{X}$ and a 1-form $\tilde{\omega}$ in $\tilde{M}$, we have

$$\tilde{f}=\tilde{f}^H=\tilde{f}^V, \quad \tilde{X}=\tilde{X}^H+\tilde{X}^V, \quad \tilde{\omega}=\tilde{\omega}^H+\tilde{\omega}^V.$$

If $\tilde{T}=\tilde{T}^H$ or $\tilde{T}=\tilde{T}^V$, then $\tilde{T}$ is said to be *horizontal* or *vertical* respectively.

We shall denote by $\mathscr{L}_{\tilde{X}}$ the Lie derivation with respect to a vector field $\tilde{X}$ in $\tilde{M}$. If a tensor field $\tilde{T}$ in $\tilde{M}$ possesses the property

$$(\mathscr{L}_V\tilde{T}^H)^H=0$$

for any vertical vector field V, then $\tilde{T}$ is said to be *projectable*. A function $\tilde{f}$ in $\tilde{M}$ is projectable if and only if $\mathscr{L}_V\tilde{f}=V\tilde{f}=0$ for any vertical vector field V, that is, $\tilde{f}$ is constant along each fibre. Then there is in the base space M a unique function f such that $\tilde{f}=\pi^*f$, π^* being the induced map of the projection π. The function f is called the *projection* of $\tilde{f}$ and denoted by $f=p\tilde{f}$. Conversely, given a function f in the base space M, the π^*f is a projectable function in $\tilde{M}$, which is called the *lift* of f and denoted by f^L.

If the metric tensor $\tilde{g}$ is projectable in a fibred space $\{\tilde{M}, M, \tilde{g}, \pi\}$, the space is called a *fibred Riemannian space*.

Several kinds of indices will run on the following ranges respectively:

$$h,\ j,\ i,\ k,\ l = 1, 2, \ldots\ldots\ldots\ldots, \quad r,$$
$$a,\ b,\ c,\ d,\ e = 1, 2, \ldots, n,$$
$$\alpha,\ \beta,\ \gamma,\ \delta,\ \varepsilon = \qquad n+1, \ldots, r,$$
$$A, B, C, D, E = 1, 2, \ldots, n, n+1, \ldots, r.$$

The summation convetion will be used on their own ranges.

We consider coordinate neighborhoods $(\tilde{U}, z^h)$ in $\tilde{M}$ and (U, x^a) in M such that $\pi(\tilde{U})=U$, where z^h and x^a are coordinates in $\tilde{U}$ and U respectively. The projection π: $\tilde{M}\to M$ can be expressed by certain equations

$$x^a(\pi(\tilde{P})) = x^a(z^h(\tilde{P}))$$

for any point $\tilde{P}\in\tilde{U}$, or generally

(1.1) $$x^a = x^a(z^h)$$

which are differentiable functions of coordinates z^h in $\tilde{U}$ with Jacibian $(\partial x^a/\partial z^i)$ of maximum rank n. Take a fibre $\bar{M}$ such that $\bar{M}\cap\tilde{U}\neq\emptyset$. Then there are in $\bar{M}\cap\tilde{U}$ local coordinates y^α, and (x^a, y^α) form a coordinate system in $\tilde{U}$. Differentiating (1.1) in z^i, we put

(1.2) $$E_i{}^a = \partial_i x^a, \quad \partial_i = \partial/\partial z^i.$$

For each fixed index a, $E_i{}^a$ are components of a local covector field E^a defined in $\tilde{U}$. On the other hand, each fibre $\bar{M}(\tilde{P})$ at $\tilde{P}\in\bar{M}\cap\tilde{U}$ is parameterized as $z^h = z^h(x^a, y^\alpha)$, and we put

(1.3) $$C^h{}_\alpha = \partial_\alpha z^h, \quad \partial_\alpha = \partial/\partial y^\alpha,$$

which are components of a local vector field C_α in $\tilde{U}$ for each fixed index α. The vector fields C_α form a natural frame tangent to $\bar{M}(\tilde{P})$. We have

(1.4) $$E_i{}^a C^i{}_\beta = 0.$$

We denote by $\tilde{g}_{ji}$ components of $\tilde{g}$ in $(\tilde{U}, z^h)$ and put $(\tilde{g}^{ih})=(\tilde{g}_{ji})^{-1}$. Components of the induced metric tensor $\bar{g}$ of the fibre $\bar{M}$ are given by

(1.5) $$\bar{g}_{\gamma\beta} = \tilde{g}_{ji} C^j{}_\gamma C^i{}_\beta.$$

If we put $(\bar{g}^{\beta\alpha})=(\bar{g}_{\gamma\beta})^{-1}$ and

(1.6) $$C_i{}^\alpha = \tilde{g}_{ih}\bar{g}^{\alpha\beta} C^h{}_\beta,$$

then, for each fixed index α, $C_i{}^\alpha$ are components of a local covector field C^α in $\tilde{U}$.

The covector fields E^a and C^α form a base of the cotangent space of $\tilde{M}$ at each point $\tilde{P}$ in $\tilde{U}$. We choose vector fields E_b such that E_b together with C_β form the base, denoted by Σ, of the tangent space of $\tilde{M}(\tilde{P})$ dual to the base (E^a, C^α). We write (E^A) or (E_B) for (E^a, C^α) or (E_b, C_β) in all, if necessary. Then they satisfy the relations

$$(1.7)\qquad E_i{}^A E^i{}_B = \delta_B^A: \quad \begin{array}{ll} E_i{}^a E^i{}_b = \delta_b^a, & E_i{}^a C^i{}_\beta = 0, \\ C_i{}^\alpha E^i{}_b = 0, & C_i{}^\alpha C^i{}_\beta = \delta_\beta^\alpha, \end{array}$$

and

$$(1.8)\qquad E_i{}^A E^h{}_A = E_i{}^a E^h{}_a + C_i{}^\alpha C^h{}_\alpha = \delta_i^h.$$

If we put

$$(1.9)\qquad g_{cb} = \tilde{g}_{ji} E^j{}_c E^i{}_b \quad \text{and} \quad (g^{ba}) = (g_{cb})^{-1},$$

then we obtain

$$(1.10)\qquad E^h{}_a = \tilde{g}^{hi} g_{ab} E_i{}^b.$$

The local vector fields E_b and the covector fields E^a are horizontal. The local vector fields C_β and the covector fields C^α are vertical.

Any tensor field in $\tilde{M}$, say $\tilde{T}$ of type (1, 2), is represented in $\tilde{U}$ in the form

$$(1.11)\qquad \begin{aligned} \tilde{T} &= T_{CB}{}^A E^C \otimes E^B \otimes E_A \\ &= T_{cb}{}^a E^c \otimes E^b \otimes E_a + T_{cb}{}^\alpha E^c \otimes E^b \otimes C_\alpha + T_{c\beta}{}^a E^c \otimes C^\beta \otimes E_a + \cdots \\ &\quad + T_{\gamma\beta}{}^a C^\gamma \otimes C^\beta \otimes E_a + T_{\gamma\beta}{}^\alpha C^\gamma \otimes C^\beta \otimes C_\alpha \end{aligned}$$

with respect to the base Σ, where $T_{cb}{}^a$, $T_{cb}{}^\alpha$, $T_{c\beta}{}^a, \dots, T_{\gamma\beta}{}^a$ and $T_{\gamma\beta}{}^\alpha$ are local functions in $\tilde{U}$. The first term $T_{cb}{}^a E^c \otimes E^b \otimes E_a$ in the right hand side determines a global tensor field in $\tilde{M}$, which is the horizontal part $\tilde{T}^H$ of $\tilde{T}$, and the last term $T_{\gamma\beta}{}^\alpha C^\gamma \otimes C^\beta \otimes C_\alpha$ determines a global tensor field, which is the vertical part $\tilde{T}^V$ of $\tilde{T}$.

§2. Connections

Denoting by $\mathscr{L}_\beta$ the Lie derivation with respect to the vector field C_β, we have

$$(2.1)\qquad \mathscr{L}_\beta E^a = 0 \quad \text{and} \quad \mathscr{L}_\beta C_\alpha = [C_\beta, C_\alpha] = 0.$$

Applying $\mathscr{L}_\beta$ to (1.7), we see that $\mathscr{L}_\beta E_b$ is vertical and $\mathscr{L}_\beta C^\alpha$ is horizontal and we may put

(2.2)
$$\mathscr{L}_\beta E_b = [C_\beta, E_b] = - P_{b\beta}{}^\alpha C_\alpha,$$
$$\mathscr{L}_\beta C^\alpha = P_{c\beta}{}^\alpha E^c,$$

where $P_{c\beta}{}^\alpha$ are local functions in $\tilde{U}$.

Assume that a tensor field $\tilde{T}$ in $\tilde{M}$ is of type (1.2) and represented as (1.11). Then, by use of (2.1) and (2.2), we have

$$(\mathscr{L}_\beta \tilde{T}^H)^H = (\mathscr{L}_\beta(T_{cb}{}^a E^c \otimes E^b \otimes E_a))^H$$
$$= (\partial_\beta T_{cb}{}^a) E^c \otimes E^b \otimes E_a.$$

Therefore *a tensor field* $\tilde{T}$ *in* $\tilde{M}$ *is projectable if and only if* $\partial_\beta T_{cb}{}^a = 0$, *i.e., the local functions* $T_{cb}{}^a$ *are all constant along each fibre* $\overline{M} \cap \tilde{U}$. Then the projections $pT_{cb}{}^a$ are functions in the neighborhood U of the base space, and determine a global tensor field T of type (1,2), which has $T_{cb}{}^a$ as components with respect to the coordinates x^a. The tensor field T is called the *projection* of $\tilde{T}$ and denoted by $T = p\tilde{T}$.

Given in M a tensor field T having components $T_{cb}{}^a$ with respect to (U, x^a), we can have the tensor field $\tilde{T}$ in $\tilde{M}$ defined by

$$\tilde{T} = (\pi^* T_{cb}{}^a) E^c \otimes E^b \otimes E_a$$

in $\tilde{U} = \pi^{-1}(U)$, which is called the *lift* of T and denoted by $\tilde{T} = T^L$.

We put $\partial_c = E^j{}_c \partial_j$. If a function $\tilde{f}$ in $\tilde{M}$ is projectable and $f = p\tilde{f}$, then so is $\partial_c \tilde{f}$ and we have

$$p(\partial_c \tilde{f}) = \partial f / \partial x^c.$$

An affine connection $\tilde{\nabla}$ in $\tilde{M}$ is said to be *projectable* if $\tilde{\nabla}_{Y^L} X^L$ is a projectable vector field for arbitrary vector fields X and Y in M. Given an arbitrary affine connection $\tilde{\nabla}$ in $\tilde{M}$, we put

(2.3)
$$\tilde{\nabla}_j E^h{}_B = \Gamma^A_{CB} E_j{}^C E^h{}_A$$

or

(2.4)
$$\Gamma^A_{CB} = E^j{}_C (\tilde{\nabla}_j E^h{}_B) E_h{}^A,$$

which are coefficients of the connection $\tilde{\nabla}$ with respect to the base Σ and local functions in $\tilde{U}$. An affine connection $\tilde{\nabla}$ is projectable in $\tilde{M}$ if and only if the functions Γ^a_{cb} given by (2.4) with $A = a$, $B = b$, $C = c$ are all projectable in each $\tilde{U}$. Then, putting

$$\nabla_Y X = p(\tilde{\nabla}_{Y^L} X^L)$$

for any vector fields X and Y in M, we can define an affine connection ∇ in the base space M, which is called the *projection* of $\tilde{\nabla}$ and denoted by $\nabla = p\tilde{\nabla}$. The projection ∇ has $\Gamma^a_{cb} = p\tilde{\Gamma}^a_{cb}$ as coefficients in the neighborhood $U = \pi(\tilde{U})$. The projection of the torsion tensor of $\tilde{\nabla}$ coincides with the torsion tensor of the projection $\nabla = p\tilde{\nabla}$.

We now suppose that $\{\tilde{M}, M, \tilde{g}, \pi\}$ is a fibred Riemannian space, i.e., the metric tensor $\tilde{g}$ is projectable. Since E_a are perpendicular to C_α, $\tilde{g}$ has a local form

$$\tilde{g} = g_{cb}E^c \otimes E^b + \bar{g}_{\gamma\beta}C^\gamma \otimes C^\beta \tag{2.5}$$

in $\tilde{U}$. The projection $g = p\tilde{g}$ has components g_{cb} in U and defines a Riemannian metric in the base space M. If we put $\bar{g} = \tilde{g}^V = \bar{g}_{\gamma\beta}C^\gamma \otimes C^\beta$, then $\tilde{g}$ is written as

$$\tilde{g} = \tilde{g}^H + \tilde{g}^V = g^L + \bar{g}. \tag{2.6}$$

Let $\tilde{\nabla}$ be the Riemannian connection of $\tilde{g}$ in $\tilde{M}$. Then the projection $\nabla = p\tilde{\nabla}$ is the Riemannian connection of the metric $g = p\tilde{g}$ in M and the coefficients Γ^a_{cb} are equal to those of g in (U, x^a), i.e.,

$$\tilde{\Gamma}^a_{cb} = \Gamma^a_{cb} = \frac{1}{2} g^{ae}(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{cb}).$$

Taking account of the local form (2.5) of $\tilde{g}$ and (1.7), (2.1), (2.2), the expression (2.3) of the Riemannian connection $\tilde{\nabla}$ in $\tilde{M}$ can be separately rewritten in the forms

$$\begin{aligned} \tilde{\nabla}_j E^h{}_b &= \Gamma^a_{cb}E_j{}^c E^h{}_a - L_{cb}{}^\alpha E_j{}^c C^h{}_\alpha + L_b{}^a{}_\gamma C_j{}^\gamma E^h{}_a - h_\gamma{}^\alpha{}_b C_j{}^\gamma C^h{}_\alpha, \\ \tilde{\nabla}_j C^h{}_\beta &= L_c{}^a{}_\beta E_j{}^c E^h{}_a - (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha)E_j{}^c C^h{}_\alpha + h_{\gamma\beta}{}^a C_j{}^\gamma E^h{}_a + \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C^h{}_\alpha \end{aligned} \tag{2.7}$$

and the covariant derivatives of the covectors $E_i{}^a$ and $C_i{}^\alpha$

$$\begin{aligned} \tilde{\nabla}_j E_i{}^a &= -\Gamma^a_{cb}E_j{}^c E_i{}^b - L_c{}^a{}_\beta(E_j{}^c C_i{}^\beta + C_j{}^\beta E_i{}^c) - h_{\gamma\beta}{}^a C_j{}^\gamma C_i{}^\beta, \\ \tilde{\nabla}_j C_i{}^\alpha &= L_{cb}{}^\alpha E_j{}^c E_i{}^b + (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha)E_j{}^c C_i{}^\beta + h_\gamma{}^\alpha{}_b C_j{}^\gamma E_i{}^b - \bar{\Gamma}^\alpha_{\gamma\beta} C_j{}^\gamma C_i{}^\beta, \end{aligned} \tag{2.8}$$

where $L_{cb}{}^\alpha$, $h_{\gamma\beta}{}^a$ and $\bar{\Gamma}^\alpha_{\gamma\beta}$ are local functions in $\tilde{U}$ and

$$L_c{}^a{}_\beta = L_{cb}{}^\alpha g^{ba}\bar{g}_{\alpha\beta}, \quad h_\gamma{}^\alpha{}_b = h_{\gamma\beta}{}^a \bar{g}^{\beta\alpha} g_{ba}.$$

For later convenience, we use $L_{cb}{}^\alpha$ and $h_{\gamma\beta}{}^a$ in place of $-h_{cb}{}^\alpha$ and $-L_{\gamma\beta}{}^a$ in [4].

We put $\tilde{\nabla}_c = E^j{}_c \tilde{\nabla}_j$ and $\tilde{\nabla}_\gamma = C^j{}_\gamma \tilde{\nabla}_j$. From (2.7), we have

$$\tilde{\nabla}_\gamma E_b = L_b{}^a{}_\gamma E_a - h_\gamma{}^\alpha{}_b C_\alpha, \quad \tilde{\nabla}_\gamma C_\beta = h_{\gamma\beta}{}^a E_a + \bar{\Gamma}^\alpha_{\gamma\beta} C_\alpha. \tag{2.9}$$

The second equation is Gauss' equation of each fibre $\bar{M}$ as a submanifold in $\tilde{M}$. Hence, along each fibre $\bar{M}$ in $\tilde{M}$, $h_{\gamma\beta}{}^a$ are components of the second fundamental tensor with respect to the normal vector E_a, and $\bar{\Gamma}^{\gamma}_{\beta\alpha}$ are coefficients of the Riemannian connection $\bar{\nabla}$ of the induced metric $\bar{g}$ in $\bar{M}$. Therefore, we see that

$$h_{\gamma\beta}{}^{a} = h_{\beta\gamma}{}^{a} \tag{2.10}$$

and

$$\bar{\Gamma}^{\alpha}_{\gamma\beta} = \frac{1}{2}\bar{g}^{\alpha\varepsilon}(\partial_{\gamma}\bar{g}_{\beta\varepsilon}+\partial_{\beta}\bar{g}_{\gamma\varepsilon}-\partial_{\varepsilon}\bar{g}_{\gamma\beta}). \tag{2.11}$$

The first equation of (2.9) is Weingarten's equation of each fibre $\bar{M}$ in $\tilde{M}$, and $L_{ba}{}^{\gamma}$ are coefficients of the normal connection of $\bar{M}$ and satisfy

$$L_{cb}{}^{\alpha} + L_{bc}{}^{\alpha} = 0. \tag{2.12}$$

It follows from (2.7) together with (2.1) and (2.2) that

$$[E_c, E_b] = -2L_{cb}{}^{\alpha}C_{\alpha}, \quad [C_{\gamma}, C_{\beta}] = 0, \quad [E_c, C_{\beta}] = P_{c\beta}{}^{\alpha}C_{\alpha}, \tag{2.13}$$

which are equivalent to

$$\partial_c\partial_b - \partial_b\partial_c = -2L_{cb}{}^{\alpha}\partial_{\alpha}, \quad \partial_{\gamma}\partial_{\beta} = \partial_{\beta}\partial_{\gamma}, \quad \partial_c\partial_{\beta} - \partial_{\beta}\partial_c = P_{c\beta}{}^{\alpha}\partial_{\alpha}, \tag{2.14}$$

respectively. Denoting by $\mathscr{L}_c$ the Lie derivation with respect to E_c, we have

$$\begin{aligned} &\mathscr{L}_c E_b = 2L_{bc}{}^{\alpha}C_{\alpha}, \quad \mathscr{L}_c C_{\beta} = P_{c\beta}{}^{\alpha}C_{\alpha}, \\ &\mathscr{L}_c E^a = 0, \quad \mathscr{L}_c C^{\alpha} = 2L_{cb}{}^{\alpha}E^b - P_{c\beta}{}^{\alpha}C^{\beta}. \end{aligned} \tag{2.15}$$

The tensor field $L=(L_{cb}{}^{\alpha})$ is called the *structure tensor* of the fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. From (2.13, 1) we see that, *in a fibred Riemannian space, the horizontal distribution D is integrable if and only if the structure tensor L vanishes identically.*

Applying Jacobi's identity to triple combinations of the vector fields E_b and C_{β}, we have the identities

$$\begin{aligned} &(\partial_d L_{cb}{}^{\alpha} + P_{d\beta}{}^{\alpha}L_{cb}{}^{\beta}) + (\partial_c L_{bd}{}^{\alpha} + P_{c\beta}{}^{\alpha}L_{bd}{}^{\beta}) \\ &\qquad + (\partial_b L_{dc}{}^{\alpha} + P_{b\beta}{}^{\alpha}L_{dc}{}^{\beta}) = 0, \end{aligned} \tag{2.16}$$

$$2\partial_{\gamma}L_{cb}{}^{\alpha} - (\partial_c P_{b\gamma}{}^{\alpha} - \partial_b P_{c\gamma}{}^{\alpha} + P_{c\beta}{}^{\alpha}P_{b\gamma}{}^{\beta} - P_{b\beta}{}^{\alpha}P_{c\gamma}{}^{\beta}) = 0, \tag{2.17}$$

$$\partial_{\gamma}P_{d\beta}{}^{\alpha} - \partial_{\beta}P_{d\gamma}{}^{\alpha} = 0. \tag{2.18}$$

Using (2.5), (2.8) and (2.15), we obtain the equations

$$\begin{aligned} (\mathscr{L}_{X^L}g^V)^V = (\mathscr{L}_{X^L}g)^V &= X^b(\partial_b\bar{g}_{\gamma\beta} - P_{b\gamma}{}^{\varepsilon}\bar{g}_{\varepsilon\beta} - P_{b\beta}{}^{\varepsilon}\bar{g}_{\gamma\varepsilon})C^{\gamma}C^{\beta} \\ &= -2(h_{\gamma\beta}{}^{a}X_a)C^{\gamma}C^{\beta}, \end{aligned} \tag{2.19}$$

for any vector field $X=X^b\partial_b$ in M, $X_a=g_{ab}X^b$ being covariant components of X, and hence the identity

$$\partial_b\bar{g}_{\gamma\beta} - P_{b\gamma}{}^{\varepsilon}\bar{g}_{\varepsilon\beta} - P_{b\beta}{}^{\varepsilon}\bar{g}_{\gamma\varepsilon} = -2h_{\gamma\beta}{}^a g_{ab}. \tag{2.20}$$

Let γ be a curve through a point P in the base space M and X be the tangent vector field of γ. There is a unique curve $\tilde{\gamma}$ through a point $\tilde{P}\in\pi^{-1}(P)$ such that the tangent vector field is the lift X^L. The curve $\tilde{\gamma}$ is called the *horizontal lift* of γ passing through $\tilde{P}$. If a curve γ joins points P and Q in M, then the horizontal lifts of γ through all points of the fibre $\overline{M}(P)$ define a fibre mapping Φ_γ: $\overline{M}(P)\rightarrow\overline{M}(Q)$, which is called the *horizontal mapping covering* γ.

If the horizontal mapping convering any curve in M is an isometry of fibres, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a fibred Riemannian space *with isometric fibres.* A necessary and sufficient condition for $\tilde{M}$ to have isometric fibres is

$$(\mathscr{L}_{X^L}\tilde{g}^V)^V = 0$$

for any vector field X in M, or, by means of (2.19),

$$h_{\gamma\beta}{}^a = 0. \tag{2.21}$$

If $L_{cb}{}^{\alpha}=0$ and $h_{\gamma\beta}{}^a=0$, then the fibred Riemannian space $\tilde{M}$ is locally a Riemannian product of the base space and a fibre, and is said to be *locally trivial.*

If the horizontal mapping covering any curve in M is a conformal mapping of fibres, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a fibred Riemannian space *with conformal fibres.* A condition for $\tilde{M}$ to have conformal fibres is

$$h_{\gamma\beta}{}^a = \bar{g}_{\gamma\beta}A^a, \tag{2.22}$$

where $A=A^aE_a$ is the mean curvature vector along each fibre in $\tilde{M}$.

§ 3. Curvature tensors and structure equations

The curvature tensor $\tilde{K}$ of a fibred Riemannian space $\tilde{M}$ is defined by

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \tag{3.1}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{M}$. If we put

$$\tilde{K}(E_D, E_C)E_B = \tilde{K}_{DCB}{}^A E_A = \tilde{K}_{DCB}{}^a E_a + \tilde{K}_{DCB}{}^{\alpha}C_\alpha, \tag{3.2}$$

then $\tilde{K}_{DCB}{}^A$ are components of the curvature tensor $\tilde{K}$ with respect to the base Σ. Denoting by $\tilde{K}_{kji}{}^h$ components of $\tilde{K}$ in $(\tilde{U}, z^h)$, we have the relations

$$\tilde{K}_{DCB}{}^A = \tilde{K}_{kji}{}^h E^k{}_D E^j{}_C E^i{}_B E_h{}^A. \tag{3.3}$$

Substituting (2.7) into the definition (3.1) of the curvature tensor $\tilde{K}$, we have the structure equations of a fibred Riemannian space as follows [4]:

(3.4) $\tilde{K}_{dcb}{}^{a} = K_{dcb}{}^{a} - L_{d}{}^{a}{}_{\varepsilon}L_{cb}{}^{\varepsilon} + L_{c}{}^{a}{}_{\varepsilon}L_{db}{}^{\varepsilon} + 2L_{dc}{}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon}$,

(3.5) $\tilde{K}_{dcb}{}^{\alpha} = -{}^{*}\nabla_{d}L_{cb}{}^{\alpha} + {}^{*}\nabla_{c}L_{db}{}^{\alpha} - 2L_{dc}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b}$,

(3.6) $\tilde{K}_{dc\beta}{}^{\alpha} = {}^{*}\nabla_{c}h_{\beta}{}^{\alpha}{}_{d} - {}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{c} + 2{}^{**}\nabla_{\beta}L_{dc}{}^{\alpha} + L_{de}{}^{\alpha}L_{c}{}^{e}{}_{\beta} - L_{ce}{}^{\alpha}L_{d}{}^{e}{}_{\beta}$
$- h_{\varepsilon}{}^{\alpha}{}_{d}h_{\beta}{}^{\varepsilon}{}_{c} + h_{\varepsilon}{}^{\alpha}{}_{c}h_{\beta}{}^{\varepsilon}{}_{d}$,

(3.7) $\tilde{K}_{d\gamma b}{}^{a} = {}^{*}\nabla_{d}L_{b}{}^{a}{}_{\gamma} - L_{d}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{b} + L_{db}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} - L_{b}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{d}$,

(3.8) $\tilde{K}_{d\gamma b}{}^{\alpha} = -{}^{*}\nabla_{d}h_{\gamma}{}^{\alpha}{}_{b} + {}^{**}\nabla_{\gamma}L_{db}{}^{\alpha} + L_{d}{}^{e}{}_{\gamma}L_{eb}{}^{\alpha} + h_{\gamma}{}^{\varepsilon}{}_{d}h_{\varepsilon}{}^{\alpha}{}_{b}$,

(3.9) $\tilde{K}_{\delta\gamma b}{}^{a} = L_{\delta\gamma b}{}^{a} + h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon}{}^{a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon}{}^{a}$,

(3.10) $\tilde{K}_{\delta\gamma\beta}{}^{a} = {}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} - {}^{**}\nabla_{\gamma}h_{\delta\beta}{}^{a}$,

(3.11) $\tilde{K}_{\delta\gamma\beta}{}^{\alpha} = \bar{K}_{\delta\gamma\beta}{}^{\alpha} + h_{\delta\beta}{}^{e}h_{\gamma}{}^{\alpha}{}_{e} - h_{\gamma\beta}{}^{e}h_{\delta}{}^{\alpha}{}_{e}$,

where we have put

(3.12) $K_{dcb}{}^{a} = \partial_{d}\Gamma_{cb}^{a} - \partial_{c}\Gamma_{db}^{a} + \Gamma_{de}^{a}\Gamma_{cb}^{e} - \Gamma_{ce}^{a}\Gamma_{db}^{e}$,

(3.13) ${}^{*}\nabla_{d}L_{cb}{}^{\alpha} = \partial_{d}L_{cb}{}^{\alpha} - \Gamma_{dc}^{e}L_{eb}{}^{\alpha} - \Gamma_{db}^{e}L_{ce}{}^{\alpha} + Q_{d\varepsilon}{}^{\alpha}L_{cb}{}^{\varepsilon}$,

(3.14) ${}^{*}\nabla_{d}L_{c}{}^{a}{}_{\beta} = \partial_{d}L_{c}{}^{a}{}_{\beta} + \Gamma_{de}^{a}L_{c}{}^{e}{}_{\beta} - \Gamma_{dc}^{e}L_{e}{}^{a}{}_{\beta} - Q_{d\beta}{}^{\varepsilon}L_{c}{}^{a}{}_{\varepsilon}$,

(3.15) ${}^{*}\nabla_{d}h_{\gamma\beta}{}^{a} = \partial_{d}h_{\gamma\beta}{}^{a} + \Gamma_{de}^{a}h_{\gamma\beta}{}^{e} - Q_{d\gamma}{}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - Q_{d\beta}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a}$,

(3.16) ${}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{b} = \partial_{d}h_{\beta}{}^{\alpha}{}_{b} - \Gamma_{db}^{e}h_{\beta}{}^{\alpha}{}_{e} + Q_{d\varepsilon}{}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - Q_{d\beta}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b}$,

$Q_{c\beta}{}^{\alpha}$ being defined by

$$Q_{c\beta}{}^{\alpha} = P_{c\beta}{}^{\alpha} - h_{\beta}{}^{\alpha}{}_{c},$$

and

(3.17) ${}^{**}\nabla_{\delta}L_{cb}{}^{\alpha} = \partial_{\delta}L_{cb}{}^{\alpha} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}L_{cb}{}^{\varepsilon} + L_{c}{}^{e}{}_{\delta}L_{eb}{}^{\alpha} + L_{b}{}^{e}{}_{\delta}L_{ce}{}^{\alpha}$,

(3.18) ${}^{**}\nabla_{\delta}L_{b}{}^{a}{}_{\beta} = \partial_{\delta}L_{b}{}^{a}{}_{\beta} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon} - L_{e}{}^{a}{}_{\delta}L_{b}{}^{e}{}_{\beta} + L_{b}{}^{e}{}_{\delta}L_{e}{}^{a}{}_{\beta}$,

(3.19) ${}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} = \partial_{\delta}h_{\gamma\beta}{}^{a} - \bar{\Gamma}_{\delta\gamma}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} - L_{e}{}^{a}{}_{\delta}h_{\gamma\beta}{}^{e}$,

(3.20) ${}^{**}\nabla_{\delta}h_{\beta}{}^{\alpha}{}_{b} = \partial_{\delta}h_{\beta}{}^{\alpha}{}_{b} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} + L_{b}{}^{e}{}_{\delta}h_{\beta}{}^{\alpha}{}_{e}$,

(3.21) $L_{\delta\gamma b}{}^{a} = \partial_{\delta}L_{b}{}^{a}{}_{\gamma} - \partial_{\gamma}L_{b}{}^{a}{}_{\delta} + L_{e}{}^{a}{}_{\delta}L_{b}{}^{e}{}_{\gamma} - L_{e}{}^{a}{}_{\gamma}L_{b}{}^{e}{}_{\delta}$,

(3.22) $\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \partial_{\delta}\bar{\Gamma}_{\gamma\beta}^{\alpha} - \partial_{\gamma}\bar{\Gamma}_{\delta\beta}^{\alpha} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}\bar{\Gamma}_{\gamma\beta}^{\varepsilon} - \bar{\Gamma}_{\gamma\varepsilon}^{\alpha}\bar{\Gamma}_{\delta\beta}^{\varepsilon}$.

Among these, the functions $K_{dcb}{}^{a}$ are projectable in $\tilde{U}$ and its projections, denoted by $K_{dcb}{}^{a}$ too, are components of the curvature tensor K of the base space $\{M, g\}$. On each fibre $\overline{M}$, the functions $\overline{K}_{\delta\gamma\beta}{}^{\alpha}$ are components of the curvature tensor $\overline{K}$ of the induced Riemannian metric $\bar{g}$ and $L_{\delta\gamma b}{}^{a}$ those of the curvature tensor of the normal bundle of $\overline{M}$ in $\tilde{M}$. The components $\tilde{K}_{DCB}{}^{A}$ satisfy the same algebraic equations as those $\tilde{K}_{kji}{}^{h}$ satisfy. By means of the first Bianchi identity, we have the identity

$$(3.23) \qquad {}^*\nabla_d L_{cb}{}^{\alpha} + {}^*\nabla_c L_{bd}{}^{\alpha} + {}^*\nabla_b L_{dc}{}^{\alpha} + L_{dc}{}^{\varepsilon} h_{\varepsilon}{}^{\alpha}{}_{b} + L_{cb}{}^{\varepsilon} h_{\varepsilon}{}^{\alpha}{}_{d} + L_{bd}{}^{\varepsilon} h_{\varepsilon}{}^{\alpha}{}_{c} = 0,$$

which can be also obtained from (2.16).

Denote by $\tilde{S}_{CB}$ components of the Ricci tensor $\tilde{S}$ of $\{\tilde{M}, \tilde{g}\}$ with respect to the base Σ in $\tilde{U}$, and by S_{cb} and $\overline{S}_{\gamma\beta}$ components of the Ricci tensors S and $\overline{S}$ of the base space $\{M, g\}$ in (U, x^{a}) and each fibre $\{\overline{M}, \bar{g}\}$ in $(\overline{U}, y^{\alpha})$ respectively. Then we have

$$(3.24) \qquad \tilde{S}_{cb} = S_{cb} - 2L_{ce}{}^{\varepsilon} L_{b}{}^{e}{}_{\varepsilon} + h_{\delta}{}^{\varepsilon}{}_{c} h_{\varepsilon}{}^{\delta}{}_{b} + \frac{1}{2}({}^*\nabla_c h_{\varepsilon}{}^{\varepsilon}{}_{b} + {}^*\nabla_b h_{\varepsilon}{}^{\varepsilon}{}_{c}),$$

$$(3.25) \qquad \tilde{S}_{\gamma b} = {}^{**}\nabla_{\gamma} h_{\varepsilon}{}^{\varepsilon}{}_{b} - {}^{**}\nabla_{\varepsilon} h_{\gamma}{}^{\varepsilon}{}_{b} + {}^*\nabla_e L_{b}{}^{e}{}_{\gamma} - 2h_{\gamma}{}^{\varepsilon}{}_{e} L_{b}{}^{e}{}_{\varepsilon},$$

$$(3.26) \qquad \tilde{S}_{\gamma\beta} = \overline{S}_{\gamma\beta} - h_{\gamma\beta}{}^{e} h_{\varepsilon}{}^{\varepsilon}{}_{e} + {}^*\nabla_e h_{\gamma\beta}{}^{e} - L_{a}{}^{e}{}_{\gamma} L_{e}{}^{a}{}_{\beta}.$$

Denoting by $\tilde{\kappa}$, κ and $\bar{\kappa}$ the scalar curvatures of $\tilde{M}, M$ and each fibre $\overline{M}$ respectively, we have the relation

$$(3.27) \qquad \tilde{\kappa} = \kappa^{L} + \bar{\kappa} - L_{cb\varepsilon} L^{cb\varepsilon} + h_{\gamma\beta e} h^{\gamma\beta e} - h_{\gamma}{}^{\gamma}{}_{e} h_{\beta}{}^{\beta e}.$$

§ 4. Fibred almost Hermitian spaces with anti-invariant fibres

Now we suppose that the total space $\tilde{M}$ is an almost Hermitian space with almost complex structure $\tilde{J}=(\tilde{J}_{i}{}^{h})$ and each fibre $\overline{M}$ is anti-invariant. Then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a *fibred almost Hermitian space with anti-invariant fibres.* The dimension r of the total space $\tilde{M}$ is equal to $r=2n$, and indices $\alpha, \beta, \ldots, \varepsilon$ run on the range $n+1, \ldots, 2n$. Since the transform of the tangent space of each fibre $\overline{M}$ by $\tilde{J}$ is normal to $\overline{M}$, we may put

$$(4.1) \qquad \tilde{J}_{i}{}^{h} C^{i}{}_{\beta} = J_{\beta}{}^{a} E^{h}{}_{a}, \quad \tilde{J}_{i}{}^{h} E^{i}{}_{b} = J_{b}{}^{\alpha} C^{h}{}_{\alpha}.$$

The almost complex structure $\tilde{J}$ with respect to the base Σ in $\tilde{U}$ has the local form

$$(4.2) \qquad \tilde{J} = J_{\beta}{}^{a} C^{\beta} \otimes E_{a} + J_{b}{}^{\alpha} E^{b} \otimes C_{\alpha}$$

and the components satisfy the equations

$$(4.3) \qquad J_{\beta}{}^{a} J_{b}{}^{\beta} = -\delta_{b}^{a}, \quad J_{b}{}^{\alpha} J_{\beta}{}^{b} = -\delta_{\beta}^{\alpha}.$$

Covariant components of $\tilde{J}$ are given by

$$J_{ba}=0,\quad J_{\beta a}=J_\beta{}^e g_{ea},\quad J_{b\alpha}=J_b{}^\varepsilon \bar{g}_{\varepsilon\alpha},\quad J_{\beta\alpha}=0 \tag{4.4}$$

and satisfy the relations

$$J_{\beta a}+J_{a\beta}=0. \tag{4.5}$$

Components of the covariant derivative $\tilde{\nabla}\tilde{J}$ with respect to Σ are given by $(\tilde{\nabla}_j\tilde{J}_{ih})E^j{}_C E^i{}_B E^h{}_A$, and we can obtain the following expressions by means of (2.7) and (4.4):

$$\begin{aligned}
(\tilde{\nabla}_j\tilde{J}_{ih})E^j{}_c E^i{}_b E^h{}_a &= L_{cb}{}^\varepsilon J_{\varepsilon a}+L_{ca}{}^\varepsilon J_{b\varepsilon},\\
(\tilde{\nabla}_j\tilde{J}_{ih})C^j{}_\gamma E^i{}_b E^h{}_a &= h_\gamma{}^\varepsilon{}_b J_{\varepsilon a}+h_\gamma{}^\varepsilon{}_a J_{b\varepsilon},\\
(\tilde{\nabla}_j\tilde{J}_{ih})E^j{}_c C^i{}_\beta E^h{}_a &= \partial_c J_{\beta a}-\Gamma^e_{ca}J_{\beta e}-Q_{c\beta}{}^\varepsilon J_{\varepsilon a},\\
(\tilde{\nabla}_j\tilde{J}_{ih})C^j{}_\gamma C^i{}_\beta E^h{}_a &= \partial_\gamma J_{\beta a}-\bar{\Gamma}^\varepsilon_{\gamma\beta}J_{\varepsilon a}-L_a{}^e{}_\gamma J_{\beta e},\\
(\tilde{\nabla}_j\tilde{J}_{ih})E^j{}_c C^i{}_\beta C^h{}_\alpha &= -L_c{}^e{}_\beta J_{e\alpha}-L_c{}^e{}_\alpha J_{\beta e},\\
(\tilde{\nabla}_j\tilde{J}_{ih})C^j{}_\gamma C^i{}_\beta C^h{}_\alpha &= -h_{\gamma\beta}{}^e J_{e\alpha}-h_{\gamma\alpha}{}^e J_{\beta e}.
\end{aligned} \tag{4.6}$$

If the total space $\tilde{M}$ is almost Kaehlerian, i.e., $\tilde{J}$ satisfies the equation

$$\tilde{\nabla}_j\tilde{J}_{ih}+\tilde{\nabla}_i\tilde{J}_{hj}+\tilde{\nabla}_h\tilde{J}_{ji}=0,$$

then we have

$$\begin{aligned}
&L_{cb}{}^\varepsilon J_{\varepsilon a}+L_{ba}{}^\varepsilon J_{\varepsilon c}+L_{ac}{}^\varepsilon J_{\varepsilon b}=0,\\
&\partial_b J_{\gamma a}-P_{b\gamma}{}^\varepsilon J_{\varepsilon a}=\partial_a J_{\gamma b}-P_{a\gamma}{}^\varepsilon J_{\varepsilon b},\\
&\partial_\gamma J_{\beta a}=\partial_\beta J_{\gamma a}.
\end{aligned} \tag{4.7}$$

We see from (4.7, 3) that there are locally functions ψ_a in $\tilde{U}$ such that $J_{\beta a}=\partial_\beta\psi_a$. By means of (2.14, 3), we have

$$\partial_b J_{\gamma a}=\partial_b\partial_\gamma\psi_a=\partial_\gamma\partial_b\psi_a+P_{b\gamma}{}^\varepsilon\partial_\varepsilon\psi_a, \tag{4.8}$$

and hence the equation (4.7, 2) is equivalent to

$$\partial_\gamma\partial_b\psi_a=\partial_\gamma\partial_a\psi_b, \tag{4.9}$$

which means that $\partial_b\psi_a-\partial_a\psi_b$ is projectable. Therefore we can state the following

PROPOSITION 4.1. *A fibred almost Hermitian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ with structure $\tilde{J}$ and anti-invariant fibres is almost Kaehlerian if and only if the equation* (4.7, 1) *holds and there are local functions ψ_a in each neighborhood $\tilde{U}$ such that $J_{\beta a}=\partial_\beta\psi_a$ and $\partial_b\psi_a-\partial_a\psi_b$ are projectable.*

Now we suppose that the total space $\tilde{M}$ is Kaehlerian, i.e., $\tilde{\nabla}_j\tilde{J}_{ih}=0$. Then, from (4.6, 1), we see $L_{cb}{}^{\varepsilon}J_{\varepsilon a}$ is symmetric in a and b. On the other hand, $L_{cb}{}^{\alpha}$ is skew-symmetric in b and c, and hence we have $L_{cb}{}^{\varepsilon}J_{\varepsilon a}=0$ or

$$L_{cb}{}^{\alpha} = 0. \tag{4.10}$$

By means of the remainders of (4.6), we have

$$\begin{aligned}
&h_{\gamma}{}^{\varepsilon}{}_{b}J_{\varepsilon a} = h_{\gamma}{}^{\varepsilon}{}_{a}J_{\varepsilon b},\\
&h_{\gamma\beta}{}^{e}J_{e\alpha} = h_{\gamma\alpha}{}^{e}J_{e\beta},\\
&\partial_c J_{\beta a} - \Gamma^{e}_{ca}J_{\beta e} + (h_{\beta}{}^{\varepsilon}{}_{c} - P_{c\beta}{}^{\varepsilon})J_{\varepsilon a} = 0,\\
&\partial_{\gamma}J_{\beta a} - \bar{\Gamma}^{\alpha}_{\gamma\beta}J_{\alpha a} = 0.
\end{aligned} \tag{4.11}$$

Computing $[C_{\delta}, C_{\gamma}]J_{\beta a}=0$, we have $\bar{K}_{\delta\gamma\beta}{}^{\alpha}J_{\alpha a}=0$ or

$$\bar{K}_{\delta\gamma\beta}{}^{\alpha} = 0$$

and, by means of (3.11), $\tilde{K}_{\delta\gamma\beta\alpha}$ are expressed as

$$\tilde{K}_{\delta\gamma\beta\alpha} = h_{\delta\beta}{}^{e}h_{\gamma\alpha e} - h_{\gamma\beta}{}^{e}h_{\delta\alpha e}. \tag{4.12}$$

Since $J_{\beta a}=\partial_{\beta}\psi_a$, the equation (4.11, 3) turns to

$$\partial_{\gamma}\partial_{b}\psi_{a} = \Gamma^{e}_{ba}J_{\gamma e} - h_{\gamma}{}^{\varepsilon}{}_{b}J_{\varepsilon a}.$$

Computing $[C_{\delta}, C_{\gamma}]\partial_b\psi_a=0$ and taking account of (3.10), we have $\tilde{K}_{\delta\gamma b}{}^{\alpha}J_{\alpha a}=0$ or $\tilde{K}_{\delta\gamma b}{}^{\alpha}=0$. In this case, the curvature tensor of the normal connection of each fibre $\bar{M}$ vanishes, and we have

$$\tilde{K}_{\delta\gamma ba} = h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon a}, \tag{4.13}$$

by means of (3.8). The covariant components $\tilde{K}_{kjih}$ of the curvature tensor $\tilde{K}$ of $\tilde{M}$ are hybrid in the pairs (h, i) and (j, k) of indices, so are the components $\tilde{K}_{DCBA}$ with respect to $\tilde{J}=(J_B{}^A)$. Therefore we have

$$K_{dcba} = K_{\delta\gamma ba}J_d{}^{\delta}J_c{}^{\gamma} = (h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon a})J_d{}^{\delta}J_c{}^{\gamma}. \tag{4.14}$$

On the other hand, we have $\tilde{K}_{dcba}=K_{dcba}$. Thus we have the following

PROPOSITION 4.2. *If a fibred almost Hermitian space $\tilde{M}$ with anti-invariant fibres is Kaehlerian, then the structure tensor L vanishes identically, each fibre $\bar{M}$ is locally Euclidean and the curvature tensor K of the base spacee M is given by* (4.14).

In particular, if $\tilde{M}$ has isometric or conformal fibres, then the total space $\tilde{M}$ itself and the base space M are locally Euclidean.

§5. Tangent bundles of Riemannian spaces

Let $\{M, g\}$ be a Riemannian space and $\tilde{M}=T(M)$ the space of the tangent bundle of M. If (U, x^a) is a local coordinate neighborhood in M and we denote by y^a cartesian coordinates in the tangent space $T_P(M)$, the fibre $\overline{M}(P)$, with respcet to the natural base $\{\partial_a\}$, then (x^a, y^a) are local coordinates in $\tilde{U}=\pi^{-1}(U)$. We shall write $\bar{a}, \bar{b},\dots, \bar{e}$ for indices $\alpha, \beta,\dots, \varepsilon$ belonging to the fibres $\overline{M}$ and running on the range $n+1,\dots, 2n$. Then the projection $\pi\colon \tilde{M}\to M$ is expressed by

$$x^a(x, y) = x^a$$

and the immersion of each fibre $\overline{M}$ into $\tilde{M}$ by

$$x^a = \text{const.} \quad \text{and} \quad x^{\bar{a}} = y^a.$$

Therefore the covectors E^a and the vectors $C_{\bar{b}}$ in $\tilde{U}$ have components

$$E_i{}^a = \delta_i^a, \quad C^h{}_{\bar{b}} = \delta_{\bar{b}}^h. \tag{5.1}$$

Let Γ_{cb}^a be coefficients of the Riemannian connection ∇ in U of the base space M. Putting

$$\delta y^a = dy^a + \Gamma_c^a dx^c, \quad \Gamma_c^a = \Gamma_{cb}^a y^b, \tag{5.2}$$

S. Sasaki [14] introduced a Riemannian metric $\tilde{g}=(\tilde{g}_{ih})$ in $\tilde{M}$ by

$$\tilde{g}_{ih}dz^i dz^h = g_{ba}dx^b dx^a + g_{ba}\delta y^b \delta y^a \tag{5.3}$$

with respect to the coordinates $(z^h)=(x^a, y^a)$ in $\tilde{U}$. Covariant components of the metric tensor $\tilde{g}$ in $\tilde{U}$ are given by

$$(\tilde{g}_{ih}) = \begin{pmatrix} g_{ba}+g_{dc}\Gamma_b^d\Gamma_a^c & \Gamma_{ab} \\ \Gamma_{ba} & g_{ba} \end{pmatrix} \tag{5.4}$$

where we have put $\Gamma_{ba}=\Gamma_b^e g_{ea}$. The vector fields E_b and the covector fields $C^{\bar{a}}$ have components

$$(E_b{}^h) = \begin{pmatrix} \delta_b^a \\ -\Gamma_b^a \end{pmatrix}, \quad (C_i{}^{\bar{a}}) = (\Gamma_b^a, \delta_b^a) \tag{5.5}$$

in $(\tilde{U}, z^h)$.

Components of the metric tensor $\tilde{g}$ defined by (5.3) are given by

$$
(5.6)\qquad \begin{aligned}
\tilde{g}_{ba} &= \tilde{g}_{ih}E^i{}_bE^h{}_a = g_{ba},\\
\tilde{g}_{\bar{b}a} &= \tilde{g}_{ih}C^i{}_{\bar{b}}E^h{}_a = 0,\\
\tilde{g}_{\bar{b}\bar{a}} &= \tilde{g}_{ih}C^i{}_{\bar{b}}C^h{}_{\bar{a}} = g_{ba},
\end{aligned}
$$

that is

$$
(5.7)\qquad \tilde{g} = g_{ba}E^b \otimes E^a + g_{ba}C^{\bar{b}} \otimes C^{\bar{a}},
$$

and the metric tensor $\tilde{g}$ induced in $T(M)$ is projectable.

By simple computations, we have

$$
(5.8)\qquad \begin{aligned}
&[E_c, E_b] = -K_{cb.}{}^aC_{\bar{a}}, \quad [C_{\bar{c}}, C_{\bar{b}}] = 0\\
&[E_c, C_{\bar{b}}] = \Gamma^a_{cb}C_{\bar{a}},
\end{aligned}
$$

where $K_{dcb}{}^a$ are components of the curvature tensor K of the base space $\{M, g\}$ and we have put

$$
K_{dc.}{}^a = K_{dcb}{}^a y^b.
$$

Thus the structure tensor $L_{cb}{}^{\bar{a}}$ and $P_{c\bar{b}}{}^{\bar{a}}$ are equal to

$$
(5.9)\qquad 2L_{cb}{}^{\bar{a}} = K_{cb.}{}^a, \quad P_{c\bar{b}}{}^{\bar{a}} = \Gamma^a_{cb}.
$$

Substituting (5.6, 3) and (5.9, 2) into (2.19), we see

$$
(\mathscr{L}_{X^L}g^V)^V = X^c(\partial_c g_{\bar{b}\bar{a}} - \Gamma^e_{cb}g_{ea} - \Gamma^e_{ca}g_{be})C^{\bar{b}}C^{\bar{a}} = 0.
$$

Therefore the tangent bundle $\tilde{M}$ has isometric fibres and we have $h_{\bar{c}\bar{b}}{}^a = 0$. On the other hand, noting that the functions g_{ba} in $\tilde{U}$ are projectable, applying $\partial_{\bar{c}}$ to (5.6, 3), and using (2.7), we have

$$
\partial_{\bar{c}}\bar{g}_{\bar{b}\bar{a}} = \bar{\Gamma}^{\bar{e}}_{\bar{c}\bar{b}}g_{ea} + \bar{\Gamma}^{\bar{e}}_{\bar{c}\bar{a}}g_{eb} = 0.
$$

Since $\bar{\Gamma}^{\bar{a}}_{\bar{c}\bar{b}}$ is symmetric in $\bar{b}$ and $\bar{c}$, we have $\Gamma^{\bar{a}}_{\bar{c}\bar{b}} = 0$. Therefore we have

PROPOSITION 5.1. *The space of the tangent bundle $T(M)$ of a Riemannian space $\{M, g\}$ has a structure of fibred Riemannian space with the metric tensor g defined by* (5.3) *and isometric fibres.*

Substituting (5.9) into the structure equations (3.4) to (3.11), we can express the curvature tensor $\tilde{K}$ of $\tilde{M}$ by use of the components $K_{dcb}{}^a$ of the curvature tensor K of the base space M and $K_{dc.}{}^a = K_{dcb}{}^a y^b$. In particular, the expression (3.13) is now equal to

$$
{}^*\nabla_d L_{cb}{}^{\bar{a}} = \frac{1}{2}\nabla_d K_{cb.}{}^a,
$$

and we have

$$K_{dcb}{}^{\bar a} = \frac{1}{2} \nabla_b K_{dc.}{}^{a}, \quad K_{\bar d \bar c \bar b}{}^{a} = K_{\bar d \bar c \bar b}{}^{\bar a} = 0.$$

S. Tachibana and M. Okumura [16] introduced an almost complex structure $\tilde J$ in the tangent bundle $\tilde M = TM$, covariant components $\tilde J_{ji}$ of which are

$$(5.10) \qquad (\tilde J_{ji}) = \begin{pmatrix} \Gamma_{cb} - \Gamma_{bc} & g_{cb} \\ -g_{cb} & 0 \end{pmatrix}$$

with respect to a coordinate neighborhood $(\tilde U, z^h)$. Its components $\tilde J_{CB}$ with respcet to the $\Sigma = \{E_b, C_{\bar b}\}$ are given by $\tilde J_{CB} = \tilde J_{ji} E^j{}_C E^i{}_B$ and have expressions

$$(5.11) \qquad \begin{aligned} J_{cb} &= 0, \quad J_{\bar c b} = g_{cb}, \\ J_{c\bar b} &= -\, g_{cb}, \quad J_{\bar c \bar b} = 0, \end{aligned}$$

and the components $\tilde J_C{}^A = \tilde J_{CB} \tilde g^{BA}$ of type (1, 1) are

$$(5.12) \qquad \begin{aligned} J_c{}^a &= 0, \quad J_{\bar c}{}^a = \delta^a_c, \\ J_c{}^{\bar a} &= -\, \delta^a_c, \quad J_{\bar c}{}^{\bar a} = 0. \end{aligned}$$

Hence each fibre $\bar M = T_p(M)$ is anti-invariant with respect to the almost complex structure $\tilde J$ defined above in $\tilde M$.

Let $\tilde Y$ be the vector field in $\tilde M$ defined by

$$(5.13) \qquad \tilde Y^a = 0, \quad \tilde Y^{\bar a} = y^a.$$

Then $\tilde Y$ is vertical and has the same components $(0, y^a)$ with respect to the base Σ. If we put the transform $\tilde\theta = \tilde J(\tilde Y)$, then the vector $\tilde\theta$ has components

$$(5.14) \qquad \theta^a = y^a, \quad \theta^{\bar a} = 0,$$

and covariant components

$$\theta_b = y_b = g_{ba} y^a, \quad \theta_{\bar b} = 0.$$

We denote the 1-form by

$$\tilde\theta = \theta_B dz^B = y_b dx^b.$$

The almost complex structure $\tilde J$ can be interpreted as one associated with the derived form $d\theta$, and hence it is almost Kaehlerian. Then we have

PROPOSITION 5.2. *The space of the tangent bundle $\tilde M$ of a Riemannian space $\{M, g\}$ has an almost Kaehlerian structure $\tilde J$, and isometric and anti-invariant fibres with respect to the structure $\tilde J$.*

If the structure $\tilde J$ defined above in $\tilde M$ is Kaehlerian, then combining Proposition 4.2 and 5.2, we can show the following proposition due to S. Tachibana and M. Okumura [16]:

PROPOSITION 5.3. *The almost Kaehlerian structure $\tilde{J}$ induced above in the tangent bundle $\tilde{M}$ of a Riemannian space $\{M, g\}$ is Kaehlerian if and only if the base space M is locally Euclidean. Then the tangent bundle $\tilde{M}$ is locally trivial.*

§ 6. Fibred almost contact metric spaces

We suppose that a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ has an almost contact structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta})$ associated with Riemannian metric $\tilde{g}$, i.e., $\tilde{\phi}$, $\tilde{\xi}$ and $\tilde{\eta}$ satisfy the relations

$$(6.1)\qquad \begin{aligned} &\tilde{\phi}^2\tilde{X} = -\tilde{X} + \tilde{\eta}(\tilde{X})\tilde{\xi}, \quad \tilde{\phi}(\tilde{\xi}) = 0, \quad \tilde{\eta}(\tilde{\phi}\tilde{X}) = 0, \quad \tilde{\eta}(\tilde{\xi}) = 1, \\ &\tilde{g}(\tilde{X}, \tilde{Y}) = g(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}) + \tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) \end{aligned}$$

for any vectors $\tilde{X}$, $\tilde{Y}$ in $\tilde{M}$. In addition, we suppose that $\tilde{\phi}$ is projectable and each fibre $\bar{M}$ is $\tilde{\phi}$-invariant and tangent to the structure vector $\tilde{\xi}$. We call such a space a *fibred almost contact metric space with $\tilde{\phi}$-invariant fibres tangent to $\tilde{\xi}$*, and shall confine ourselves to such a space in §§6 and 7.

We can easily verify that the distribution D spanned by E_a is also $\tilde{\phi}$-invariant, $\tilde{\phi}$ has the local form

$$(6.2)\qquad \tilde{\phi} = \phi_b{}^a E^b \otimes E_a + \bar{\phi}_\beta{}^\alpha C^\beta \otimes C_\alpha,$$

$J = (\phi_b{}^a)$ defines an almost complex structure in the base space M, and $(\bar{\phi}, \bar{g}, \bar{\xi}, \bar{\eta})$ an almost contact metric structure in each fibre $\bar{M}$, where $\bar{\phi} = \bar{\phi}_\beta{}^\alpha C^\beta \otimes C_\alpha$ and $\bar{g}$ is the induced metric tensor in $\bar{M}$. Hence the base space M and the fibre $\bar{M}$ are even- and odd-dimensional respectively.

By means of (2.7) and noting $\eta_b = 0$, components of the covariant derivative $\tilde{\nabla}\tilde{\eta}$ of the covector $\tilde{\eta}$ with respect to the base Σ are given by

$$(6.3)\qquad \begin{aligned} &(\tilde{\nabla}_j\tilde{\eta}_i)E^j{}_c E^i{}_b = L_{cb}{}^\alpha \bar{\eta}_\alpha, \\ &(\tilde{\nabla}_j\tilde{\eta}_i)C^j{}_\gamma E^i{}_b = h_\gamma{}^\alpha{}_b \bar{\eta}_\alpha, \\ &(\tilde{\nabla}_j\tilde{\eta}_i)E^j{}_c C^i{}_\beta = \partial_c \bar{\eta}_\beta + (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha)\bar{\eta}_\alpha, \\ &(\tilde{\nabla}_j\tilde{\eta}_i)C^j{}_\gamma C^i{}_\beta = \bar{\nabla}_\gamma \bar{\eta}_\beta. \end{aligned}$$

Now we consider the case where $(\tilde{\phi}, \tilde{g}, \tilde{\xi}, \tilde{\eta})$ is a contact metric structure, i.e., they satisfy the equation

$$(6.4)\qquad \tilde{\phi}_{ji} = \frac{1}{2}(\partial_j\tilde{\eta}_i - \partial_i\tilde{\eta}_j) = \frac{1}{2}(\tilde{\nabla}_j\tilde{\eta}_i - \tilde{\nabla}_i\tilde{\eta}_j).$$

Then, using (6.2) and (6.3), we have the equations

$$\phi_{cb} = L_{cb}{}^{\alpha}\bar{\eta}_{\alpha},$$
(6.5)
$$\tilde{\phi}_{c\beta} = \partial_c\bar{\eta}_{\beta} - P_{c\beta}{}^{\alpha}\bar{\eta}_{\alpha} = 0,$$
$$\bar{\phi}_{\gamma\beta} = \frac{1}{2}(\bar{\nabla}_{\gamma}\bar{\eta}_{\beta} - \bar{\nabla}_{\beta}\bar{\eta}_{\gamma}).$$

By means of (2.14), (2.17) and (6.5), we have

$$\partial_{\gamma}(L_{cb}{}^{\alpha}\bar{\eta}_{\alpha}) = 2L_{cb}{}^{\alpha}\bar{\phi}_{\gamma\alpha}.$$

In order for $\tilde{\phi}$ to be projectable, it is necessary and sufficient that

$$L_{cb}{}^{\alpha}\bar{\phi}_{\gamma\alpha} = 0$$

or

$$L_{cb}{}^{\alpha} = \phi_{cb}\xi^{\alpha}, \tag{6.6}$$

by using of the fact that $(\bar{\phi}, \bar{g}, \xi, \bar{\eta})$ is almost contact. From (6.5, 1) and (6.5, 2), we have

$$\partial_d\phi_{cb} = (\partial_d L_{cb}{}^{\alpha} + P_{d\varepsilon}{}^{\alpha} L_{cb}{}^{\varepsilon})\bar{\eta}_{\alpha}, \tag{6.7}$$

and by means of (2.14),

$$\partial_d\phi_{cb} + \partial_c\phi_{bd} + \partial_b\phi_{dc} = 0. \tag{6.8}$$

Thus the structure ϕ in the base space M is almost Kaehlerian. The equation (6.5, 3) means that the structure $(\bar{\phi}, \bar{g}, \xi, \bar{\eta})$ on each fibre $\bar{M}$ is contact metric.

Conversely, suppose that the equations (6.6) and (6.8) are valid. Then, substituting (6.6) into (2.14) and using (6.8), we obtain

$$\phi_{cb}(\partial_d\xi^{\alpha} + P_{d\beta}{}^{\alpha}\xi^{\beta}) + \phi_{bd}(\partial_c\xi^{\alpha} + P_{c\beta}{}^{\alpha}\xi^{\beta}) + \phi_{dc}(\partial_b\xi^{\alpha} + P_{b\beta}{}^{\alpha}\xi^{\beta}) = 0.$$

Contracting this equation with ϕ^{cb}, we see that

$$\partial_d\xi^{\alpha} + P_{d\beta}{}^{\alpha}\xi^{\beta} = 0 \tag{6.9}$$

provided $n > 2$.

Thus we have the following

PROPOSITION 6.1. *Suppose that a fibred almost contact metric space $\tilde{M}$ with structure $(\tilde{\phi}, \tilde{g}, \xi, \tilde{\eta})$ has $\tilde{\phi}$-invariant fibres tangent to ξ. In order that the structure in $\tilde{M}$ is contact metric, it is necessary and sufficient that the almost Hermitian structure (J, g) of the base space M is almost Kaehlerian, the structure $(\bar{\phi}, \bar{g}, \xi, \bar{\eta})$ of each fibre $\bar{M}$ is contact metric, the equation* (6.5, 2) *is valid and the structure tensor is equal to*

$$L_{cb}{}^{\alpha} = \phi_{cb}\xi^{\alpha}.$$

Next we suppose that, in addition to the assumptions of Proposition 6.1, the total space $\tilde{M}$ has isometric fibres, i.e., $h_{\gamma\beta}{}^{a}=0$, and is an η-Einstein,

$$\tilde{S}_{ji} = a\tilde{g}_{ji} + b\tilde{\eta}_j\tilde{\eta}_i,$$

a and b being constants. Then, taking account of (3.24), (3.25) and (3.26), we have

$$S_{cb} = (a+2)g_{cb},$$

$$\bar{S}_{\gamma\beta} = a\bar{g}_{\gamma\beta} + (b-n)\bar{\eta}_\gamma\bar{\eta}_\beta,$$

and

$$*\nabla_a L_c{}^a{}_\beta = (\nabla_a\phi_c{}^a)\bar{\eta}_\beta + \phi_c{}^a(\partial_a\bar{\eta}_\beta - P_{a\beta}{}^\varepsilon\bar{\eta}_\varepsilon) = (\nabla_a\phi_c{}^a)\bar{\eta}_\beta = 0.$$

Thus we have the following

PROPOSITION 6.2. *If a fibred contact metric space $\tilde{M}$ with isometric and $\tilde{\phi}$-invariant fibres tangent to $\tilde{\xi}$ is an η-Einstein space, then the base space M is an Einstein space, each fibre $\bar{M}$ an η-Einstein space, and the fundamental form $\theta=\frac{1}{2}\phi_{cb}dx^c\wedge dx^b$ is harmonic.*

If, in addition to the assumptions of Proposition 6.1, $\tilde{M}$ is a K-contact space, i.e.,

$$\tilde{\phi}_{ji} = \tilde{\nabla}_j\tilde{\eta}_i,$$

then it follows from (6.3) that, in addition to the properties of Proposition 6.1, we have

$$h_\gamma{}^\alpha{}_b\bar{\eta}_\alpha = h_{\gamma\alpha b}\xi^\alpha = 0$$

and

$$\bar{\phi}_{\gamma\beta} = \bar{\nabla}_\gamma\bar{\eta}_\beta.$$

Therefore we have

PROPOSITION 6.3. *A fibred contact metric space $\tilde{M}$ with $\tilde{\phi}$-invariant fibres tangent to $\tilde{\xi}$ is a K-contact space if and only if the base space M is almost Kaehlerian, each fibre $\bar{M}$ is a K-contact space, the structure tensor is given by (6.6) and $h_{\gamma\beta}{}^a$ satisfies the equation*

(6.10) $$h_{\gamma\beta}{}^a\xi^\beta = 0.$$

§7. Fibred Sasakian spaces

By means of (2.7) and the local form (6.2), components of the covariant derivative $\tilde{\nabla}\tilde{\phi}$ with respect to the base Σ are given by

$$
(7.1)\qquad
\begin{aligned}
(\tilde{\nabla}_j\tilde{\phi}_{ih})E^j{}_cE^i{}_bE^h{}_a &= \nabla_c\phi_{ba},\\
(\tilde{\nabla}_j\tilde{\phi}_{ih})C^j{}_\gamma E^i{}_bE^h{}_a &= -L_b{}^e{}_\gamma\phi_{ea} - L_a{}^e{}_\gamma\phi_{be},\\
(\tilde{\nabla}_j\tilde{\phi}_{ih})E^j{}_cC^i{}_\beta E^h{}_a &= -L_c{}^e{}_\beta\phi_{ea} + L_{ca}{}^\varepsilon\bar{\phi}_{\beta\varepsilon},\\
(\tilde{\nabla}_j\tilde{\phi}_{ih})C^j{}_\gamma C^i{}_\beta E^h{}_a &= -h_{\gamma\beta}{}^e\phi_{ea} + h_\gamma{}^\varepsilon{}_a\bar{\phi}_{\beta\varepsilon},\\
(\tilde{\nabla}_j\tilde{\phi}_{ih})E^j{}_cC^i{}_\beta C^h{}_\alpha &= \partial_c\bar{\phi}_{\beta\alpha} + (h_\beta{}^\varepsilon{}_c - P_{c\beta}{}^\varepsilon)\bar{\phi}_{\varepsilon\alpha}\\
&\qquad + (h_\alpha{}^\varepsilon{}_c - P_{c\alpha}{}^\varepsilon)\bar{\phi}_{\beta\varepsilon},\\
(\tilde{\nabla}_j\tilde{\phi}_{ih})C^j{}_\gamma C^i{}_\beta C^h{}_\alpha &= \bar{\nabla}_\gamma\bar{\phi}_{\beta\alpha}.
\end{aligned}
$$

A fibred contact metric space $\tilde{M}$ is called a *fibred Sasakian space* if the structure $(\tilde{\phi}, \tilde{g}, \tilde{\xi}, \tilde{\eta})$ is Sasakian, that is,

$$\tilde{\nabla}_j\tilde{\phi}_{ih} = \tilde{\eta}_i\tilde{g}_{jh} - \tilde{\eta}_h\tilde{g}_{ji}.$$

Referring this equation to the base Σ, we have the equations

$$
(7.2)\qquad
\begin{aligned}
&\nabla_c\phi_{ba} = 0,\\
&L_b{}^e{}_\gamma\phi_{ea} + L_a{}^e{}_\gamma\phi_{be} = 0,\\
&L_{ca}{}^\varepsilon\bar{\phi}_{\beta\varepsilon} - L_c{}^e{}_\beta\phi_{ea} = \bar{\eta}_\beta g_{ca},\\
&h_\gamma{}^\varepsilon{}_a\bar{\phi}_{\beta\varepsilon} - h_{\gamma\beta}{}^e\phi_{ea} = 0,\\
&\partial_c\bar{\phi}_{\beta\alpha} + (h_\beta{}^\varepsilon{}_c - P_{c\beta}{}^\varepsilon)\bar{\phi}_{\varepsilon\alpha} + (h_\alpha{}^\varepsilon{}_c - P_{c\alpha}{}^\varepsilon)\bar{\phi}_{\beta\varepsilon} = 0,\\
&\bar{\nabla}_\gamma\bar{\phi}_{\beta\alpha} = \bar{\eta}_\beta\bar{g}_{\gamma\alpha} - \bar{\eta}_\alpha\bar{g}_{\gamma\beta}.
\end{aligned}
$$

The second and third relations are fulfilled by use of $L_{cb}{}^\alpha = \phi_{cb}\xi^\alpha$. Since $h_{\gamma\beta}{}^a$ is symmetric in β and γ, it follows from the fourth that

$$(7.3)\qquad h_\gamma{}^\varepsilon{}_a\bar{\phi}_{\varepsilon\beta} = h_\beta{}^\varepsilon{}_a\bar{\phi}_{\varepsilon\gamma}.$$

Two times of the left hand side of the fifth is equal to

$$
\begin{aligned}
&\partial_c(\partial_\beta\bar{\eta}_\alpha - \partial_\alpha\bar{\eta}_\beta) - P_{c\beta}{}^\varepsilon(\partial_\varepsilon\bar{\eta}_\alpha - \partial_\alpha\bar{\eta}_\varepsilon) + P_{c\alpha}{}^\varepsilon(\partial_\varepsilon\bar{\eta}_\beta - \partial_\beta\bar{\eta}_\varepsilon)\\
&= \partial_\beta\partial_c\bar{\eta}_\alpha - \partial_\alpha\partial_c\bar{\eta}_\beta + P_{c\beta}{}^\varepsilon\partial_\alpha\bar{\eta}_\varepsilon - P_{c\alpha}{}^\varepsilon\partial_\beta\bar{\eta}_\varepsilon\\
&= \partial_\beta(P_{c\alpha}{}^\varepsilon\bar{\eta}_\varepsilon) - \partial_\alpha(P_{c\beta}{}^\varepsilon\bar{\eta}_\varepsilon) + P_{c\beta}{}^\varepsilon\partial_\alpha\bar{\eta}_\varepsilon - P_{c\alpha}{}^\varepsilon\partial_\beta\bar{\eta}_\varepsilon\\
&= (\partial_\beta P_{c\alpha}{}^\varepsilon - \partial_\alpha P_{c\beta}{}^\varepsilon)\bar{\eta}_\varepsilon = 0
\end{aligned}
$$

by means of (2.14), (2.18) and (6.5). Therefore the first, fourth and the last equation of (7.2) are essential, and we can state the following

PROPOSITION 7.1. *A fibred contact metric space $\tilde{M}$ with $\tilde{\phi}$-invariant fibres tangent to $\tilde{\xi}$ is Sasakian if and only if the structure (J, g) of the base space M is*

Kaehlerian, the structure $(\bar{\phi}, \bar{g}, \bar{\xi}, \bar{\eta})$ *of each fibre is Sasakian, and the equations* (6.6), (7.2, 4), (7.2, 5) *hold.*

Since we have

$$ {}^*\nabla_d L_{cb}{}^\alpha = 0, \quad {}^{**}\nabla_\delta L_{cb}{}^\alpha = \phi_{cb}\bar{\phi}_\delta{}^\alpha $$

in a fibred Sasakian space, the structure equations (3.4) to (3.9) are reduced to

$$ (7.4) \quad \begin{aligned} \tilde{K}_{dcba} &= K_{dcba} - \phi_{da}\phi_{cb} + \phi_{ca}\phi_{db} + 2\phi_{dc}\phi_{ba}, \\ \tilde{K}_{dcb\alpha} &= 0, \\ \tilde{K}_{dc\beta\alpha} &= -{}^*\nabla_d h_{\beta\alpha c} + {}^*\nabla_c h_{\beta\alpha d} - h_{\varepsilon\alpha d}h_\beta{}^\varepsilon{}_c + h_{\varepsilon\alpha c}h_\beta{}^\varepsilon{}_d + 2\phi_{dc}\bar{\phi}_{\beta\alpha}, \\ \tilde{K}_{d\gamma ba} &= 0, \\ \tilde{K}_{d\gamma b\alpha} &= -{}^*\nabla_d h_{\gamma\alpha b} + h_\gamma{}^\varepsilon{}_d h_{\varepsilon\alpha b} + \phi_{db}\bar{\phi}_{\gamma\alpha} - g_{db}\bar{\eta}_\gamma\bar{\eta}_\alpha, \\ \tilde{K}_{\delta\gamma ba} &= 2\phi_{ba}\bar{\phi}_{\delta\gamma} + h_\delta{}^\varepsilon{}_b h_{\gamma\varepsilon a} - h_\gamma{}^\varepsilon{}_b h_{\delta\varepsilon a}, \\ \tilde{K}_{\delta\gamma\beta a} &= {}^{**}\nabla_\delta h_{\gamma\beta a} - {}^{**}\nabla_\gamma h_{\delta\beta a}, \\ \tilde{K}_{\delta\gamma\beta\alpha} &= \bar{K}_{\delta\gamma\beta\alpha} + h_{\delta\beta}{}^e h_{\gamma\alpha e} - h_{\gamma\beta}{}^e h_{\delta\alpha e}. \end{aligned} $$

Since $\tilde{K}_{d\gamma b\alpha} = \tilde{K}_{b\alpha d\gamma}$, we have

$$ (7.5) \quad {}^*\nabla_d h_{\beta\alpha c} = {}^*\nabla_c h_{\beta\alpha d}. $$

§8. Almost complex structure in a fibred Riemannian space

In this section, we consider a fibred Riemannian space such that the base space M and each fibre $\bar{M}$ are almost contact spaces. Let $n=2k+1$ and $s=2l+1$ and we denote the almost contact structure of M and its lift in the total space $\tilde{M}$ by (ϕ, ξ, η) and that of each fibre $\bar{M}$ by $(\bar{\phi}, \bar{\xi}, \bar{\eta})$. Then we can define an almost complex structure $\tilde{J}$ on the total space $\tilde{M}$ by

$$ (8.1) \quad \tilde{J}E_b = \phi_b{}^a E_a + \eta_b\bar{\xi}, \quad \tilde{J}C_\beta = -\bar{\eta}_\beta\xi + \bar{\phi}_\beta{}^\alpha C_\alpha $$

where (ϕ, ξ, η) is independent on the fibre. We call $\tilde{J}$ the *induced almost complex structure* on $\tilde{M}$.

By means of (2.7), (2.8) and (8.1), we can derive

$$ (8.2) \quad \begin{aligned} (\tilde{\nabla}_c\tilde{J})E_b = {} & (\nabla_c\phi_b{}^a - L_{cb}{}^\varepsilon\bar{\eta}_\varepsilon\xi^a + L_c{}^a{}_\varepsilon\eta_b\bar{\xi}^\varepsilon)E_a \\ & + \{(\nabla_c\eta_b)\bar{\xi}^\alpha + ({}^*\nabla_c\bar{\xi}^\alpha)\eta_b + L_{cb}{}^\varepsilon\bar{\phi}_\varepsilon{}^\alpha - L_{ce}{}^\alpha\phi_b{}^e\}C_\alpha, \end{aligned} $$

$$ (8.3) \quad \begin{aligned} (\tilde{\nabla}_c\tilde{J})C_\beta = {} & (-L_c{}^e{}_\beta\phi_e{}^a - \xi^a{}^*\nabla_c\bar{\eta}_\beta - \bar{\eta}_\beta\nabla_c\xi^a + L_c{}^a{}_\varepsilon\bar{\phi}_\beta{}^\varepsilon)E_a \\ & + ({}^*\nabla_c\bar{\phi}_\beta{}^\alpha - L_c{}^b{}_\beta\eta_b\bar{\xi}^\alpha + L_{ce}{}^\alpha\xi^e\bar{\eta}_\beta)C_\alpha, \end{aligned} $$

$$(8.4)\qquad (\tilde{\nabla}_\gamma \tilde{J})E_b = ({}^*\nabla_\gamma \phi_b{}^a - h_\gamma{}^\varepsilon{}_b \bar{\eta}_\varepsilon \xi^a + h_{\gamma\varepsilon}{}^a \bar{\xi}^\varepsilon \eta_b)E_a + (\bar{\xi}^{\alpha **}\nabla_\gamma \eta_b + \eta_b{}^{**}\nabla_\gamma \bar{\xi}^\alpha + h_\gamma{}^\varepsilon{}_b \bar{\phi}_\varepsilon{}^\alpha - h_\gamma{}^\alpha{}_e \phi_b{}^e)C_\alpha,$$

$$(8.5)\qquad (\tilde{\nabla}_\gamma \tilde{J})C_\beta = \{(-{}^{**}\nabla_\gamma \bar{\eta}_\beta)\xi^a - \bar{\eta}_\beta({}^{**}\nabla_\gamma \xi^a) - h_{\gamma\beta}{}^e \phi_e{}^a + h_{\gamma\varepsilon}{}^a \bar{\phi}_\beta{}^\varepsilon\}E_a + ({}^{**}\nabla_\gamma \bar{\phi}_\beta{}^\alpha - h_{\gamma\beta}{}^e \eta_e \bar{\xi}^\alpha + h_\gamma{}^\alpha{}_e \bar{\eta}_\beta \xi^e)C_\alpha,$$

where we have put

$$(8.6)\qquad \nabla_c \phi_b{}^a = \partial_c \phi_b{}^a + \Gamma^a_{ce}\phi_b{}^e - \Gamma^e_{cb}\phi_e{}^a,$$

$$(8.7)\qquad \nabla_c \eta_b = \partial_c \eta_b - \Gamma^a_{cb}\eta_a,$$

$$(8.8)\qquad {}^*\nabla_c \bar{\xi}^\gamma = \partial_c \bar{\xi}^\gamma + \bar{\xi}^\alpha Q_{c\alpha}{}^\gamma,$$

$$(8.9)\qquad {}^*\nabla_c \bar{\eta}_\gamma = \partial_c \bar{\eta}_\gamma - Q_{c\gamma}{}^\varepsilon \bar{\eta}_\varepsilon,$$

$$(8.10)\qquad {}^*\nabla_c \bar{\phi}_\beta{}^\alpha = \partial_c \bar{\phi}_\beta{}^\alpha + \bar{\phi}_\beta{}^\varepsilon Q_{c\varepsilon}{}^\alpha - \bar{\phi}_\varepsilon{}^\alpha Q_{c\beta}{}^\varepsilon,$$

$$(8.11)\qquad {}^{**}\nabla_\gamma \phi_b{}^a = \partial_\gamma \phi_b{}^a + L_{eb\gamma}\phi^{ea} - L^a{}_{e\gamma}\phi_b{}^e,$$

$$(8.12)\qquad {}^{**}\nabla_\gamma \eta_b = \partial_\gamma \eta_b - L_b{}^e{}_\gamma \eta_e,$$

$$(8.13)\qquad {}^{**}\nabla_\gamma \bar{\xi}^\alpha = \partial_\gamma \bar{\xi}^\alpha + \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{\xi}^\varepsilon,$$

$$(8.14)\qquad {}^{**}\nabla_\gamma \bar{\eta}_\beta = \partial_\gamma \bar{\eta}_\beta - \bar{\Gamma}^\varepsilon_{\gamma\beta}\bar{\eta}_\varepsilon,$$

$$(8.15)\qquad {}^{**}\nabla_\gamma \xi^d = \partial_\gamma \xi^d + L^d{}_{e\gamma}\xi^e,$$

$$(8.16)\qquad {}^{**}\nabla_\gamma \bar{\phi}_\beta{}^\alpha = \partial_\gamma \bar{\phi}_\beta{}^\alpha + \bar{\Gamma}^\alpha_{\gamma\varepsilon}\bar{\phi}_\beta{}^\varepsilon - \bar{\Gamma}^\varepsilon_{\gamma\beta}\bar{\phi}_\varepsilon{}^\alpha.$$

Now we suppose that the induced structure $\tilde{J}$ in $\tilde{M}$ is Kaehlerian. Then we get

$$(8.17)\qquad \nabla_c \phi_b{}^a - L_{cb}{}^\varepsilon \bar{\eta}_\varepsilon \xi^a + L_c{}^a{}_\varepsilon \eta_b \bar{\xi}^\varepsilon = 0,$$

$$(8.18)\qquad (\nabla_c \eta_b)\bar{\xi}^\alpha + ({}^*\nabla_c \bar{\xi}^\alpha)\eta_b + L_{cb}{}^\varepsilon \bar{\phi}_\varepsilon{}^\alpha - L_{ce}{}^\alpha \phi_b{}^e = 0,$$

$$(8.19)\qquad {}^*\nabla_c \bar{\phi}_\beta{}^\alpha - L_c{}^e{}_\beta \eta_e \bar{\xi}^\alpha + L_{ce}{}^\alpha \xi^e \bar{\eta}_\beta = 0,$$

$$(8.20)\qquad {}^{**}\nabla_\gamma \phi_b{}^a - h_\gamma{}^\varepsilon{}_b \bar{\eta}_\varepsilon \xi^a + h_{\gamma\varepsilon}{}^a \bar{\xi}^\varepsilon \eta_b = 0,$$

$$(8.21)\qquad ({}^{**}\nabla_\gamma \bar{\eta}_\beta)\xi^a + \bar{\eta}_\beta({}^{**}\nabla_\gamma \xi^a) + h_{\gamma\beta}{}^e \phi_e{}^a - h_{\gamma\varepsilon}{}^a \bar{\phi}_\beta{}^\varepsilon = 0,$$

$$(8.22)\qquad {}^{**}\nabla_\gamma \bar{\phi}_\beta{}^\alpha - h_{\gamma\beta}{}^a \eta_a \bar{\xi}^\alpha + h_\gamma{}^\alpha{}_e \bar{\eta}_\beta \xi^e = 0.$$

If $\bar{M}$ is a Sasakian manifold, ${}^{**}\nabla_\gamma \bar{\phi}_{\beta\alpha} = \eta_\beta \bar{g}_{\gamma\alpha} - \eta_\alpha \bar{g}_{\gamma\beta}$, then it follows from (8.22) that

$$\bar{\eta}_\gamma(\bar{g}_{\beta\alpha} + h_{\beta\alpha}{}^a \eta_a) - \bar{\eta}_\beta(\bar{g}_{\gamma\alpha} + h_{\gamma\alpha e}\xi^e) = 0,$$

and we have

$$\bar{g}_{\beta\alpha} + h_{\beta\alpha}{}^{e}\eta_e = \lambda\bar{\eta}_\beta\bar{\eta}_\alpha \tag{8.23}$$

λ being a proportional factor. If, in addition, $\tilde{M}$ has isometric fibres, $h_{\beta\alpha}{}^{a}=0$, then

PROPOSITION 8.1. *If $\tilde{M}$ is a fibred Kaehlerian manifold with isometric fibres and the fibre $\bar{M}$ is a Sasakian manifold, then the dimension s of the fibre $\bar{M}$ should be one.*

If we assume that $\tilde{M}$ is a Kaehlerian space with conformal fibres, i.e., $h_{\alpha\beta}{}^{a}=A^a\bar{g}_{\beta\alpha}$, then the transvection of (8.23) with $\bar{\eta}^\beta\bar{\eta}^\alpha$ yields $\lambda=1+A^a\eta_a$ and we have

$$(A^a\eta_a+1)(\bar{g}_{\alpha\beta}-\bar{\eta}_\alpha\bar{\eta}_\beta) = 0.$$

Provided $s>1$, $\lambda=1+A^a\eta_a=0$ and we obtain $h_{\gamma\beta}{}^{a}\eta_a=-\bar{g}_{\gamma\beta}$. Conversely, if $h_{\gamma\beta}{}^{a}\eta_a=-\bar{g}_{\gamma\beta}$, we have $\bar{\nabla}_\alpha\bar{\eta}_\beta=\bar{\phi}_{\alpha\beta}$ and $\bar{\nabla}_\alpha\bar{\phi}_\beta{}^{\gamma}=\delta_\alpha{}^{\gamma}\bar{\eta}_\beta-\bar{g}_{\alpha\beta}\bar{\xi}^\gamma$ by means of (8.21) and (8.22). Hence we can state that

PROPOSITION 8.2. *If $\tilde{M}$ is a fibred Kaehlerian manifold with conformal fibres and $s>1$, then a necessary and sufficient condition for $\bar{M}$ to be Sasakian is $h_{\gamma\beta}{}^{a}\eta_a=-\bar{g}_{\gamma\beta}$.*

On the other hand, if the base space M is a contact manifold, i.e., $2\phi_{cb}=\nabla_c\eta_b-\nabla_b\eta_c$, then transvecting (8.18) with $\bar{\eta}_\alpha$, we see

$$2\phi_{cb} + (L_{ba}{}^{\alpha}\phi_c{}^{a}-L_{ca}{}^{\alpha}\phi_b{}^{a})\bar{\eta}_\alpha = 0,$$

and transvecting this equation with $\phi_d{}^{c}$,

$$2(-g_{db}+\eta_d\eta_b) = (L_{ca}{}^{\alpha}\phi_d{}^{c}\phi_b{}^{a}-L_{db}{}^{\alpha}-L_{ba}{}^{\alpha}\xi^a\eta_d)\bar{\eta}_\alpha,$$

moreover, transvecting with ξ^b, and noting the skew-symmetry of $L_{ba}{}^{\alpha}$ in b and a, we have $L_{db}{}^{\alpha}\xi^b\bar{\eta}_\alpha=0$. Hence the above equation reduces to

$$2(-g_{db}+\eta_d\eta_b) = (L_{ca}{}^{\alpha}\phi_d{}^{c}\phi_b{}^{a}-L_{db}{}^{\alpha})\bar{\eta}_\alpha.$$

However the left hand side is symmetric and the right is skew-symmetric in b and d respectively, and the both sides should be equal to 0. Therefore we have

PROPOSITION 8.3. *Let $\tilde{M}$ is a fibred Kaehlerian manifold and the base space M is a contact manifold, then the dimension of M should be one.*

When $\tilde{M}$ is a Kaehlerian manifold with conformal fibres and $\bar{M}$ is a contact manifold, the equation (8.21) implies

$$\bar{\nabla}_\alpha\bar{\eta}_\beta - h_{\alpha\varepsilon}{}^{d}\bar{\phi}_\beta{}^{\varepsilon}\eta_d = 0,$$

from which we get

$$\bar{V}_\alpha \bar{\eta}_\beta - \bar{V}_\beta \bar{\eta}_\alpha - h_{\alpha\varepsilon}{}^d \bar{\phi}_\beta{}^\varepsilon \xi_d + h_{\beta\varepsilon}{}^d \bar{\phi}_\alpha{}^\varepsilon \xi_d = 0,$$

that is,

$$2\bar{\phi}_{\alpha\beta} - h_{\alpha\varepsilon}{}^d \bar{\phi}_\beta{}^\varepsilon \xi_d + h_{\beta\varepsilon}{}^d \bar{\phi}_\alpha{}^\varepsilon \xi_d = 0.$$

By use of $h_{\alpha\beta}{}^a = \bar{g}_{\alpha\beta} A^a$, we get $2\bar{\phi}_{\alpha\beta}(1 + A^d \eta_d) = 0$. Hence by the same argument as that of Proposition 8.2, we have the following.

PROPOSITION 8.4. *Let $\tilde{M}$ be a fibred Kaehlerian manifold with conformal fibres. Then the contact manifold $\bar{M}$ induces the Sasakian structure provided $s > 1$.*

Combining Propositions 8.3 and 8.4, we obtain

PROPOSITION 8.5. *Let $\tilde{M}$ be a fibred Kaehlerian manifold with conformal fibres. Then the manifolds M and $\bar{M}$ are contact manifolds if and only if* (a) *$n = 1$ and $s = 1$ or* (b) *$n = 1$ and $A^a \eta_a = -1$.*

Next, if we assume that the total space $\{\tilde{M}, \tilde{g}\}$ is a Kaehlerian manifold of constant holomorphic sectional curvature $\tilde{k}$, then the equations (3.4)–(3.11) turn to

$$K_{dcb}{}^a = \frac{\tilde{k}}{4}(\delta_d^a g_{cb} - \delta_c^a g_{db} + \phi_d{}^a \phi_{cb} - \phi_c{}^a \phi_{db} - 2\phi_{dc}\phi_b{}^a) + L_d{}^a{}_\varepsilon L_{cb}{}^\varepsilon - L_c{}^a{}_\varepsilon L_{db}{}^\varepsilon - 2L_{dc}{}^\varepsilon L_b{}^a{}_\varepsilon, \tag{8.24}$$

$$\kappa = \frac{\tilde{k}}{4}(n^2 + 2n - 3) + 3\|L_{cd}{}^\varepsilon\|^2, \tag{8.25}$$

$$*\nabla_c L_{db}{}^\alpha - *\nabla_d L_{cb}{}^\alpha = \frac{\tilde{k}}{4}(\phi_{cb}\eta_d - \phi_{db}\eta_c - 2\phi_{dc}\eta_b)\bar{\xi}^\alpha + 2L_{dc}{}^\varepsilon h_\varepsilon{}^\alpha{}_b, \tag{8.26}$$

$$*\nabla_d L_b{}^a{}_\gamma - L_d{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_b + L_{db}{}^\varepsilon h_{\gamma\varepsilon}{}^a - L_b{}^a{}_\varepsilon h_\gamma{}^\varepsilon{}_d = \frac{\tilde{k}}{4}(-\phi_d{}^a \bar{\eta}_\gamma \xi_b + \phi_{db}\bar{\eta}_\gamma \xi^a - 2\phi_b{}^a \eta_d \bar{\xi}_\gamma), \tag{8.27}$$

$$-*\nabla_d h_\beta{}^\alpha{}_c + *\nabla_c h_\beta{}^\alpha{}_d + 2{**}\nabla_\beta L_{dc}{}^\alpha - L_{de}{}^\alpha L_c{}^e{}_\beta + L_{ce}{}^\alpha L_d{}^e{}_\beta - h_\varepsilon{}^\alpha{}_d h_\beta{}^\varepsilon{}_c + h_\varepsilon{}^\alpha{}_c h_\beta{}^\varepsilon{}_d = -\frac{\tilde{k}}{2}\phi_{dc}\bar{\phi}_\beta{}^\alpha, \tag{8.28}$$

$$-*\nabla_d h_\gamma{}^\alpha{}_b + {**}\nabla_\gamma L_{db}{}^\alpha + L_d{}^e{}_\gamma L_{eb}{}^\alpha + h_\gamma{}^\varepsilon{}_d h_\varepsilon{}^\alpha{}_b = \frac{\tilde{k}}{4}(-3\eta_b\eta_d \bar{\xi}^\alpha \bar{\eta}_\gamma - \bar{\phi}_\gamma{}^\alpha \phi_{db}), \tag{8.29}$$

$$\bar{K}_{\delta\gamma\beta}{}^\alpha = \frac{\tilde{k}}{4}(\delta_\delta^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma^\alpha \bar{g}_{\delta\beta} + \bar{\phi}_\delta{}^\alpha \bar{\phi}_{\gamma\beta} - \bar{\phi}_\gamma{}^\alpha \bar{\phi}_{\delta\beta} - 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha) - h_{\delta\beta}{}^e h_\gamma{}^\alpha{}_e + h_{\gamma\beta}{}^e h_\delta{}^\alpha{}_e, \tag{8.30}$$

$$**\nabla_\delta h_{\gamma\beta}{}^a - **\nabla_\gamma h_{\delta\beta}{}^a = \frac{\tilde{k}}{4}(-\bar{\eta}_\delta\bar{\Phi}_{\gamma\beta}+\bar{\eta}_\gamma\bar{\Phi}_{\delta\beta}+2\bar{\Phi}_{\delta\gamma}\bar{\eta}_\beta)\xi^a, \tag{8.31}$$

$$L_{\delta\gamma b}{}^a = -\frac{\tilde{k}}{2}\bar{\Phi}_{\delta\gamma}\phi_b{}^a - h_\delta{}^\varepsilon{}_b h_{\gamma\varepsilon}{}^a + h_\gamma{}^\varepsilon{}_b h_{\delta\varepsilon}{}^a. \tag{8.32}$$

If, in addition, $\tilde{M}$ have conformal fibres, then the equation (8.31) is reduced to

$$\bar{g}_{\gamma\beta}(**\nabla_\delta A^a) - \bar{g}_{\delta\beta}(**\nabla_\gamma A^a) = \frac{\tilde{k}}{4}(-\bar{\eta}_\delta\bar{\Phi}_{\gamma\beta}+\bar{\eta}_\gamma\bar{\Phi}_{\delta\beta}+2\bar{\Phi}_{\delta\gamma}\bar{\eta}_\beta)\xi^a \tag{8.33}$$

the transvection with $\bar{g}^{\gamma\beta}$ gives $(s-1)\,(**\nabla_\delta A^a)=0$, provided $s>1$, and moreover we see $\tilde{k}=0$. Hence the equations (8.24), (8.25), (8.27), (8.30) and (8.32) reduce to

$$K_{dcb}{}^a = L_d{}^a{}_\varepsilon L_{cb}{}^\varepsilon - L_c{}^a{}_\varepsilon L_{db}{}^\varepsilon - 2L_{dc}{}^\varepsilon L_b{}^a{}_\varepsilon, \tag{8.34}$$

$$\kappa = 3\|L_{cd}{}^\varepsilon\|^2, \tag{8.35}$$

$$*\nabla_d L_{cb}{}^\alpha = L_{db}{}^\alpha A_c - L_{dc}{}^\alpha A_b + L_{cb}{}^\alpha A_d, \tag{8.36}$$

$$\bar{K}_{\delta\gamma\beta}{}^\alpha = \|A_e\|^2(\bar{g}_{\gamma\beta}\delta_\delta{}^\alpha - \bar{g}_{\delta\beta}\delta_\gamma{}^\alpha) \tag{8.37}$$

and

$$L_{\delta\gamma b}{}^a = 0 \tag{8.38}$$

respectively. Thus we have the following

THEOREM 8.6. *If, for a fibred space $\tilde{M}$ with conformal fibres of dimension $s>1$, the induced almost complex structure in $\tilde{M}$ is Kaehlerian one of constant holomorphic sectional curvature $\tilde{k}$, then $\tilde{M}$ is locally Euclidean,*

(a) *the base space M has the curvature tensor of the form* (8.34),

(b) $\kappa\geq 0$,

(c) *each fibre is a space of constant curvature,*

(d) *the curvature tensor of the normal connection of each fibre vanishes, and*

(e) *the mean curvature vector A is parallel with respect to the normal connection along each fibre.*

§9. Tangent bundle over an almost contact manifold

Let $\{M, g\}$ be an almost contact manifold and $\tilde{M}$ the space of the tangent bundle TM of M, and use the same notations and indices as those in §5. Then the induced almost complex structure $\tilde{J}$ on $\tilde{M}$ is defined by

$$\tilde{J} = \begin{pmatrix} \phi_b{}^a & \xi^a\bar{\eta}_{\bar{b}} \\ -\eta_b\bar{\xi}^{\bar{a}} & \bar{\Phi}_{\bar{b}}{}^{\bar{a}} \end{pmatrix}$$

where $\bar{\eta}_{\bar{b}}=\eta_b$, $\bar{\xi}^{\bar{a}}=\xi^a$, $\bar{\phi}_{\bar{b}}{}^{\bar{a}}=\phi_b{}^a$ and the quantities ϕ, ξ, η are independent of the fibre.

As seen in §5, the tangent bundle $\tilde{M}$ has isometric fibres and the local scalar fields $h_{\bar{c}\bar{b}}{}^a$ and $\bar{\Gamma}^{\bar{a}}_{\bar{b}\bar{c}}$ vanish. Therefore, by virtue of (2.7), (2.8) and (8.2)–(8.5), we obtain the equations

$$(9.1)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)E^j{}_cE^i{}_bE_h{}^a = \nabla_b\phi_c{}^a + \nabla_c\phi_b{}^a + L_c{}^a{}_{\bar{\alpha}}\eta_b\bar{\xi}^{\bar{\alpha}} + L_b{}^a{}_{\bar{\alpha}}\eta_c\bar{\xi}^{\bar{\alpha}},$$

$$(9.2)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)E^j{}_cE^i{}_bC_h{}^{\bar{a}} = (\nabla_c\eta_b+\nabla_b\eta_c)\bar{\xi}^{\bar{a}} + ({}^*\nabla_b\bar{\xi}^{\bar{a}})\eta_c + ({}^*\nabla_c\bar{\xi}^{\bar{a}})\eta_b - L_{ce}{}^{\bar{a}}\phi_b{}^e - L_{be}{}^{\bar{a}}\phi_c{}^e,$$

$$(9.3)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)E^j{}_cC^i{}_{\bar{b}}E_h{}^a = -L_c{}^e{}_{\bar{b}}\phi_e{}^a - ({}^*\nabla_c\bar{\eta}_{\bar{b}})\xi^a - \bar{\eta}_{\bar{b}}(\nabla_c\xi^a) + L_{c\bar{e}}{}^a\bar{\phi}_{\bar{b}}{}^{\bar{e}} + {}^{**}\nabla_{\bar{b}}\phi_c{}^a,$$

$$(9.4)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)E^j{}_cC^i{}_{\bar{b}}C_h{}^{\bar{a}} = {}^*\nabla_c\bar{\phi}_{\bar{b}}{}^{\bar{a}} - L_c{}^e{}_{\bar{b}}\eta_e\bar{\xi}^{\bar{a}} + L_{ce}{}^{\bar{a}}\bar{\eta}_{\bar{b}}\xi^e + ({}^{**}\nabla_{\bar{b}}\eta_c)\bar{\xi}^{\bar{a}},$$

$$(9.5)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)C^j{}_{\bar{c}}C^i{}_{\bar{b}}E_h{}^a = -\bar{\eta}_{\bar{b}}({}^{**}\nabla_{\bar{c}}\xi^a) - \bar{\eta}_{\bar{c}}({}^{**}\nabla_{\bar{b}}\xi^a),$$

$$(9.6)\qquad (\tilde{\nabla}_j\tilde{J}_i{}^h+\tilde{\nabla}_i\tilde{J}_j{}^h)C_{\bar{c}}{}^jC_{\bar{b}}{}^iC_h{}^{\bar{a}} = 0.$$

If $\tilde{M}$ is nearly Kaehlerian, that is,

$$\tilde{\nabla}_j\tilde{J}_i{}^h + \tilde{\nabla}_i\tilde{J}_j{}^h = 0,$$

then the equation (9.2) is reduced to

$$(9.7)\qquad (\nabla_c\eta_b+\nabla_b\eta_c)\bar{\xi}^{\bar{a}} + ({}^*\nabla_b\bar{\xi}^{\bar{a}})\eta_c + ({}^*\nabla_c\bar{\xi}^{\bar{a}})\eta_b - L_{ce}{}^{\bar{a}}\phi^e{}_b - L_{be}{}^{\bar{a}}\phi_c{}^e = 0.$$

Transvecting this equation with ξ^b, then we get

$$(9.8)\qquad \xi^b(\nabla_b\eta_c)\bar{\xi}^{\bar{a}} + \xi^b({}^*\nabla_b\bar{\xi}^{\bar{a}})\eta_c + ({}^*\nabla_c\bar{\xi}^{\bar{a}}) - L_{be}{}^{\bar{a}}\phi_c{}^e\xi^b = 0,$$

and transvecting again with ξ^c, $\xi^c({}^*\nabla_c\bar{\xi}^{\bar{a}})=0$. Therefore (9.8) leads to

$$(9.9)\qquad {}^*\nabla_c\bar{\xi}^{\bar{a}} = L_{be}{}^{\bar{a}}\phi_c{}^e\xi^b,$$

or equivalently

$$(9.10)\qquad \phi_d{}^c({}^*\nabla_c\bar{\xi}^{\bar{a}}) = -L_{bd}{}^{\bar{a}}\xi^b$$

by the skew-symmetry of L and the equation (6.1, 1). Consequently we obtain $L_{bd}{}^{\bar{a}}\xi^b\bar{\eta}_{\bar{a}}=0$.

On the other hand, the equation (9.5) turns to

$$(9.11)\qquad L_e{}^c{}_{\bar{a}}\xi^e + L_e{}^c{}_{\bar{b}}\bar{\xi}^{\bar{b}}\xi^e\bar{\eta}_{\bar{a}} = 0,$$

by virtue of (8.15). If we combine (9.11) with $L_{bd}{}^{\bar{a}}\xi^b\bar{\eta}_{\bar{a}}=0$, then we have

$$L_{cb}{}^{\bar{a}}\xi^b = 0 \tag{9.12}$$

and, by means of (9.9),

$$*\nabla_c\xi^{\bar{a}} = 0 \tag{9.13}$$

Substituting (9.12) and $**\nabla_{\bar{b}}\,\eta_c=0$ into the expression (9.4) equated to zero, we see

$$*\nabla_c\phi_{\bar{b}}{}^{\bar{a}} = 0. \tag{9.14}$$

Since $Q_{c\bar{b}}{}^{\bar{a}}=P_{c\bar{b}}{}^{\bar{a}}=\Gamma^a_{cb}$ in the present case, the equations (9.13) and (9.14) are equivalent to

$$\nabla_c\xi^a = 0, \quad \nabla_c\phi_b{}^a = 0 \tag{9.15}$$

on the base space M, respectively. Hence, from the equation (9.2), we have

$$L_{ce}{}^{\bar{a}}\phi_b{}^e + L_{be}{}^{\bar{a}}\phi_c{}^e = 0. \tag{9.16}$$

Applying Ricci's formula to the equations (9.15), we have

$$K_{dcb}{}^a\xi^b = 0, \tag{9.17}$$

$$K_{dce}{}^a\phi_b{}^e - K_{dcb}{}^e\phi_e{}^a = 0 \tag{9.18}$$

and, substituting $2L_{ce}{}^{\bar{a}}=K_{ced}{}^a y^d$ into (9.16),

$$K_{dcbe}\phi_a{}^e + K_{dcae}\phi_b{}^e = 0. \tag{9.19}$$

By means of (9.18) and (9.19), we can see $K_{dcb}{}^e\phi_e{}^a=0$, and this together with (9.17) gives $K_{dcb}{}^a=0$. Thus we have

THEOREM 9.1. *The almost complex structure $\tilde{J}$ induced above in the tangent bundle of an almost contact space M is nearly Kaehlerian if and only if the base space M is locally Euclidean.*

Next, we investigate the case where the induced almost complex structure of the tangent bundle of an almost contact manifold is almost Kaehlerian. The condition $\tilde{\nabla}_j\tilde{J}_{ih}+\tilde{\nabla}_i\tilde{J}_{hj}+\tilde{\nabla}_h\tilde{J}_{ji}=0$ splits essentially into the three following equations:

$$\nabla_c\phi_{ba} + \nabla_b\phi_{ac} + \nabla_a\phi_{cb} - 2(L_{cb}{}^{\bar{e}}\bar{\eta}_{\bar{e}}\eta_a + L_{ba}{}^{\bar{e}}\bar{\eta}_{\bar{e}}\eta_c + L_{ac}{}^{\bar{e}}\bar{\eta}_{\bar{e}}\eta_b) = 0, \tag{9.20}$$

$$\begin{aligned}(\nabla_c\eta_b - \nabla_b\eta_c)\bar{\eta}_{\bar{a}} + (*\nabla_c\bar{\eta}_{\bar{a}})\eta_b - (*\nabla_b\bar{\eta}_{\bar{a}})\eta_c + L_{cb}{}^{\bar{e}}\bar{\phi}_{\bar{e}\bar{a}} \\ - L_{ce\bar{a}}\phi_b{}^e - L_b{}^e{}_{\bar{a}}\phi_{ec} + L_{bc\bar{e}}\bar{\phi}_{\bar{a}}{}^{\bar{e}} + **\nabla_{\bar{a}}\phi_{cb} = 0,\end{aligned} \tag{9.21}$$

$$(9.22)\quad {}^*\nabla_c\bar{\phi}_{\bar{b}\bar{a}} + ({}^{**}\nabla_{\bar{a}}\eta_c)\bar{\eta}_{\bar{b}} - ({}^{**}\nabla_{\bar{b}}\eta_c)\bar{\eta}_{\bar{a}} - L_c{}^e{}_{\bar{b}}\eta_e\bar{\eta}_{\bar{a}} + L_{ce\bar{a}}\xi^e\bar{\eta}_{\bar{b}} = 0,$$

by use of (8.11) and (8.12), the equations (9.21) and (9.22) reduce to

$$(9.23)\qquad (\nabla_c\eta_b - \nabla_b\eta_c)\bar{\eta}_{\bar{a}} + ({}^*\nabla_c\bar{\eta}_{\bar{a}})\eta_b - ({}^*\nabla_b\bar{\eta}_{\bar{a}})\eta_c + 2L_{cb\bar{d}}\bar{\phi}_{\bar{a}}{}^{\bar{d}} = 0$$

and

$$(9.24)\qquad {}^*\nabla_c\phi_{\bar{b}\bar{a}} = 0.$$

Transvection of (9.23) with $\bar{\xi}^{\bar{a}}$ implies $\nabla_a\eta_c = \nabla_c\eta_a$. The remainder of (9.23) is equivalent to

$$(\nabla_a\eta_b)\eta_c - (\nabla_c\eta_b)\eta_a + K_{aced}\phi_b{}^d y^e = 0,$$

by use of (5.9) and (8.9). Separating the terms containing y^e or not, we have $\nabla_b\eta_a = 0$ and $K_{dcb}{}^e\phi_e{}^a = 0$. By the same argument of the Theorem 9.1, we have the following

THEOREM 9.2. *The almost complex structure $\tilde{J}$ induced above in the tangent bundle of an almost contact space M is almost Kaehlerian if and only if the base space M is locally Euclidean.*

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