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Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits

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§ 1. Introduction

Let G be a connected Lie group, σ an involution of G and H an open subgroup of $G^\sigma=\{x\in G\mid \sigma x=x\}$. Then the G-homogeneous manifold $H\backslash G$ is called an affine symmetric space. Suppose that G is a real semisimple Lie group. Let P be a minimal parabolic subgroup of G and P' a parabolic subgroup of G containing P. Then the double coset decomposition $H\backslash G/P$ is studied in [2], and [5], the relation between $H\backslash G/P'$ and $H\backslash G/P$ is studied in [3], and the closure relation for $H\backslash G/P$ is studied in [4].

Let θ be a Cartan involution of G such that $\sigma\theta=\theta\sigma$. Put $K=G^\theta$ and let H^{a} be the open subgroup of $G^{\sigma\theta}$ such that $K\cap H=K\cap H^{\mathrm{a}}$. Then $H^{\mathrm{a}}\backslash G$ is called the affine symmetric space associated to $H\backslash G$. Let A be a θ-stable split component of P and put $U=\{x\in K\mid xAx^{-1}$ is σ-stable$\}$.

There exists a natural one-to-one correspondence between the double coset decompositions $H\backslash G/P'$ and $H^{\mathrm{a}}\backslash G/P'$ given by $D\to D^{\mathrm{a}}=H^{\mathrm{a}}(D\cap U)P'$ for $H-P'$ double cosets D in G ([2], [3]). Moreover it follows easily from Corollary of Theorem in [4] that this correspondence reverses the closure relations for the double coset decompositions. In this paper we prove the following theorem.

THEOREM. *Let D_1 and D_2 be arbitrary $H-P'$ double cosets in G. Then we have the following.*

(i) $D_1^{\mathrm{cl}}\supset D_2\Leftrightarrow D_1\cap D_2^{\mathrm{a}}\neq\emptyset$.

(ii) *Let $I(D_1, D_2)$ be the set of all the $H-P'$ double cosets D in G such that $D_1^{\mathrm{cl}}\supset D^{\mathrm{cl}}\supset D_2$. Then*

$$(D_1\cap D_2^{\mathrm{a}})^{\mathrm{cl}}\cap D_2^{\mathrm{a}}=\bigcup_{D\in I(D_1,D_2)}D\cap D_2^{\mathrm{a}}.$$

(iii) *Let x be an element of U. Then $HxP'\cap H^{\mathrm{a}}xP'=(K\cap H)xP'$.*

(iv) *$D_1\cap D_2^{\mathrm{a}}$ is nonempty and closed in $G\Leftrightarrow D_1=D_2$.*

Example. Let G_1 be a connected semisimple Lie group, θ_1 a Cartan involution of G_1, $K_1=\{x\in G_1\mid \theta_1x=x\}$, and P_1 a minimal parabolic subgroup of G_1 with a θ_1-stable split component A_1. Let P_1' and P_1'' be parabolic subgroups of

G_1 containing P_1. Put $G=G_1\times G_1$, $H=\{(x, x)\in G \mid x\in G_1\}$, $H^a=\{(\theta_1 x, x)\in G \mid x\in G_1\}$ and $P'=P'_1\times P''_1$. Then we have natural bijections

$$H\backslash G/P' \xrightarrow{\sim} P'_1\backslash G_1/P''_1$$

and

$$H^a\backslash G/P' \xrightarrow{\sim} \theta_1(P'_1)\backslash G_1/P''_1$$

by the maps $(x, y)\to x^{-1}y$ and $(x, y)\to\theta_1(x^{-1})y$, respectively. Hence by the Bruhat decomposition of G_1, every $H-P'$ double coset and H^a-P' double coset have representatives in $W(A_1)\times 1$.

Consider the intersection $I=H(w, 1)P'\cap H^a(w', 1)P'$ for $w, w'\in W(A_1)$. Since $H\cap H^a=\{(x, x)\mid x\in K_1\}$ and since $G_1=K_1P_1$ by the Iwasawa decomposition of G_1, I contains elements of the form $(x, 1)$ with $x\in G_1$ if I is nonempty. We have easily

$$(x, 1)\in I \Longleftrightarrow x\in P'_1 w P''_1 \cap \theta_1(P'_1)w'P''_1.$$

Thus we have as a corollary of Theorem (i),

$$(1.1)\qquad (P'_1 w P''_1)^{cl} \supset P'_1 w' P''_1 \Longleftrightarrow P'_1 w P''_1 \cap \theta_1(P'_1)w'P''_1 \neq \emptyset.$$

Especially we have

$$(1.2)\qquad (P_1 w P_1)^{cl} \supset P_1 w' P_1 \Longleftrightarrow P_1 w P_1 \cap \theta_1(P_1)w'P_1 \neq \emptyset.$$

Remark. In [1], V. V. Deodhar studied explicitly the above type of intesection $P_1wP_1\cap \bar{P}_1w'P_1$ when G_1 is a semisimple algebraic group over an algebraically closed field. Here $\bar{P}_1=w_0P_1w_0^{-1}$ with the longest element w_0 of the Weyl group. He gave (1.2) as a corollary of his results in this case (replace $\theta_1(P_1)$ by $\bar{P}_1$).

The author is grateful to J. A. Wolf who suggested him the importance of the intersections of H-orbits and H^a-orbits on G/P. In fact, Theorem (iv) was conjectured by him.

§ 2. Notations and elementary lemmas

Let $\mathfrak{g}$ be the Lie algebra of G. Let σ and θ be the involutions of $\mathfrak{g}$ induced from the involutions σ and θ of G, respectively. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$, $\mathfrak{g}=\mathfrak{h}+\mathfrak{q}$ and $\mathfrak{g}=\mathfrak{h}^a+\mathfrak{q}^a$ be the decompositions of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ, θ and $\sigma\theta$, respectively.

Let $\mathfrak{a}$ be the Lie algebra of A. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$. Let Σ denote the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Then P can be written as

$$P = P(\mathfrak{a}, \Sigma^+) = Z_G(\mathfrak{a}) \exp \mathfrak{n}$$

with a positive system Σ^+ of Σ. Here $Z_G(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in G and $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}(\mathfrak{a}; \alpha)$ ($\mathfrak{g}(\mathfrak{a}; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X$ for all $Y \in \mathfrak{a}\}$).

LEMMA 1. *Let $\mathfrak{P}'$ be a parabolic subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}' = \mathfrak{g}$.*

PROOF. Let X be an element of $\mathfrak{P}'$. Then

$$\theta X = X - (X + \sigma X) + (\sigma X + \theta X) \in \mathfrak{P}' + \mathfrak{h} + \mathfrak{h}^a.$$

Hence $\theta\mathfrak{P}' \subset \mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}'$. Since $\mathfrak{P}' + \theta\mathfrak{P}' = \mathfrak{g}$, we have $\mathfrak{h} + \mathfrak{h}^a + \mathfrak{P}' \supset \mathfrak{g}$. Q.E.D.

LEMMA 2. *Let D_1 and D_2 be arbitrary $H-P'$ double cosets in G. Then we have the following.*

(i) $(D_1 \cap D_2^a)^{cl} \cap D_2^a = D_1^{cl} \cap D_2^a$.

(ii) $D_1^{cl} \supset D_2 \Rightarrow D_1 \cap D_2^a \neq \emptyset$.

PROOF. (i) It is clear that $(D_1 \cap D_2^a)^{cl} \cap D_2^a \subset D_1^{cl} \cap D_2^a$. Let x be an element of $D_1^{cl} \cap D_2^a$. Then we have only to show that $x \in (D_1 \cap D_2^a)^{cl}$. For any neighborhood V of the identity in H^a, the set $HVxP'$ contains a neighborhood of x in G by Lemma 1. Hence $D_1 \cap HVxP' \neq \emptyset$. Since $HD_1P' = D_1$, we have

$$D_1 \cap Vx \neq \emptyset.$$

On the other hand, $Vx \subset D_2^a$. Hence $(D_1 \cap D_2^a) \cap Vx \neq \emptyset$ and we have proved that $x \in (D_1 \cap D_2^a)^{cl}$.

(ii) is clear from (i) since $D_2 \cap D_2^a \neq \emptyset$. Q.E.D.

§ 3. Proof of Theorem (i) and (ii)

By Lemma 2 (i), Theorem (ii) follows from Theorem (i).

PROOF OF THEOREM (i). By Theorem 1 in [2], we can write $D_1 = HxP' \supset HxP$ with $x \in U$. Considering xPx^{-1} and $xP'x^{-1}$ as P and P', respectively, we may assume that $D_1 = HP'$ and that $\mathfrak{a}$ is σ-stable. By Lemma 2 (ii), we have only to prove the following.

$$D_1 \cap D_2^a \neq \emptyset \Longrightarrow D_1^{cl} \supset D_2.$$

Suppose that $D_1 \cap D_2^a \neq \emptyset$. Then $HP \cap D_2^a \neq \emptyset$ since $D_2^a P' = D_2^a$. Hence there exists an element y of $D_2^a \cap U = D_2 \cap U$ such that $HP \cap HyP^a \neq \emptyset$. On the other hand, if $(HP)^{cl} \supset HyP$ for some $y \in D_2$, then it is clear that $D_1^{cl} \supset D_2$. Thus we have only to prove the following.

(3.1) If $HP \cap H^a yP \neq \emptyset$ for $y \in U$, then $(HP)^{cl} \supset HyP$.

We will prove (3.1) by induction on the real rank of G ($=\dim \mathfrak{a}$). Suppose that $\mathfrak{a}\subset\mathfrak{q}$ and that $\mathrm{Ad}(y)\mathfrak{a}\subset\mathfrak{h}$. Then by [2] Proposition 1 and Proposition 2, HP is open in G and HyP is closed in G. By [4] Proposition, we have always $(HP)^{\mathrm{cl}}\supset HyP$. Hence we may assume that

$$\mathfrak{a}\cap\mathfrak{h}\neq\{0\} \tag{3.2}$$

or that

$$\mathrm{Ad}(y)\mathfrak{a}\cap\mathfrak{q}\neq\{0\}. \tag{3.3}$$

We first show that the case (3.3) is reduced to the case (3.2). Assume the condition (3.3). Then $\mathrm{Ad}(y)\mathfrak{a}\cap\mathfrak{h}^a\neq\{0\}$ since $\mathrm{Ad}(y)\mathfrak{a}\subset\mathfrak{p}$. Consider $\mathrm{Ad}(y)\mathfrak{a}$, $\mathfrak{h}^a$ and yPy^{-1} as $\mathfrak{a}$, $\mathfrak{h}$ and P in the case (3.2), respectively. Then we have in the proof of the case (3.2) that

$$H^a yPy^{-1}\cap HPy^{-1}\neq\emptyset\Longrightarrow(H^a yPy^{-1})^{\mathrm{cl}}\supset H^a Py^{-1}.$$

Hence

$$HP\cap H^a yP\neq\emptyset\Longrightarrow(H^a yP)^{\mathrm{cl}}\supset H^a P.$$

On the other hand, we have

$$(HP)^{\mathrm{cl}}\supset HyP\Longleftrightarrow(H^a yP)^{\mathrm{cl}}\supset H^a P$$

for $y\in U$ by Corollary of [4] Theorem. Thus the case (3.3) is reduced to the case (3.2) and so we may assume (3.2) in the following.

By [4] Theorem (iv), there exists a sequence $\alpha_1,\dots,\alpha_n$ of simple roots in Σ^+ such that

$$(HP)^{\mathrm{cl}}=H((L\cap H)P_L)^{\mathrm{cl}}wPL_{\alpha_n}\cdots L_{\alpha_1}. \tag{3.4}$$

Here $w=w_{\alpha_1}\cdots w_{\alpha_n}$, L is the analytic subgroup of G for $\mathfrak{l}=[\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}\cap\mathfrak{h}),\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}\cap\mathfrak{h})]$, $P_L=L\cap P$ ($=L\cap wPw^{-1}$) and $L_\alpha=Z_G(\mathfrak{a}^\alpha)$, $\mathfrak{a}^\alpha=\{Y\in\mathfrak{a}\mid\alpha(Y)=0\}$ for $\alpha\in\Sigma$.

LEMMA 3. *$HwP=(K\cap H)(L\cap H)_0wP$. ($(L\cap H)_0$ is the connected component of $L\cap H$ containing the identity.)*

PROOF. Put $L_1=Z_G(\mathfrak{a}\cap\mathfrak{h})$ and define a parabolic subgroup P_1 of G by $P_1=L_1wPw^{-1}$ as in [4] §1. Then $P_1\cap H_0$ is a parabolic subgroup of H_0 and we have $H_0=(K\cap H)_0(P_1\cap H)_0$ by the Iwasawa decomposition of H_0. On the other hand, $K\cap H$ intersects with every connected component of H since $H=(K\cap H)\cdot\exp(\mathfrak{p}\cap\mathfrak{h})$. Hence

$$H=(K\cap H)(P_1\cap H)_0. \tag{3.5}$$

Let $\mathfrak{n}_1$ be the nilpotent radical of the Lie algebra of P_1. Then $P_1=L_1 \exp \mathfrak{n}_1$ is a Langlands decomposition of P_1. Since L_1 and $\mathfrak{n}_1$ are σ-stable, we have

$$(3.6) \qquad (P_1 \cap H)_0 = (L_1 \cap H)_0 \exp(\mathfrak{n}_1 \cap \mathfrak{h}).$$

Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{l}_1$ of L_1. Then $\mathfrak{l}_1=\mathfrak{z}+\mathfrak{l}$. Since $\mathfrak{z}$ and $\mathfrak{l}$ are σ-stable, we have $\mathfrak{l}_1 \cap \mathfrak{h}=\mathfrak{z} \cap \mathfrak{h}+\mathfrak{l} \cap \mathfrak{h}$ and therefore

$$(3.7) \qquad (L_1 \cap H)_0 = (L \cap H)_0 \exp(\mathfrak{z} \cap \mathfrak{h}).$$

We get the desired formula from (3.5), (3.6) and (3.7) since $\exp \mathfrak{n}_1 \subset wPw^{-1}$ and $\exp \mathfrak{z} \subset wPw^{-1}$. Q.E.D.

Now we will continue the proof of Theorem (i). Suppose that $HP \cap H^a yP \neq \emptyset$. Since $HP \subset HwPL_{\alpha_n} \cdots L_{\alpha_1}$, we have

$$HwP \cap H^a yPL_{\alpha_1} \cdots L_{\alpha_n} \neq \emptyset.$$

By Lemma 3, we have

$$(3.8) \qquad (L \cap H)_0 \cap H^a yPL_{\alpha_1} \cdots L_{\alpha_n} w^{-1} \neq \emptyset.$$

Let y' be an element of the left hand side of (3.8) and y'' an element of $(L \cap H^a)_0 y' P_L \cap U$. Then

$$(3.9) \qquad H^a y'' wP \subset H^a yPL_{\alpha_1} \cdots L_{\alpha_n}$$

and

$$(3.10) \qquad (L \cap H)_0 P_L \cap (L \cap H^a)_0 y'' P_L \neq \emptyset.$$

Since $\sigma L=\theta L=L$ and $\dim(\mathfrak{l} \cap \mathfrak{a})<\dim \mathfrak{a}$, we have

$$((L \cap H)_0 P_L)^{cl} \supset (L \cap H)_0 y'' P_L$$

by the assumption of induction. By (3.4), we have

$$(3.11) \qquad (HP)^{cl} \supset H(L \cap H)_0 y'' P_L wPL_{\alpha_n} \cdots L_{\alpha_1}$$
$$\supset Hy'' wPL_{\alpha_n} \cdots L_{\alpha_1}.$$

Now consider the formula (3.9) which can be rewritten as

$$y \in H^a y'' wPL_{\alpha_n} \cdots L_{\alpha_1}.$$

As in the proof of [4] Theorem (vi), we can choose a $y_1 \in y'' wPL_{\alpha_n} \cdots L_{\alpha_1} \cap U$ so that $y \in H^a y_1 P$. Since $y \in U$, it follows from [2] Theorem 1 that $y \in (K \cap H) y_1 P$. Hence

(3.12) $$y \in (K \cap H) y'' w P L_{\alpha_n} \cdots L_{\alpha_1}.$$

From (3.11) and (3.12), we have $(HP)^{cl} \supset HyP$ as desired. Q. E. D.

§4 Proof of Theorem (iii) and (iv)

Theorem (iv) follows from (ii) and (iii). So we have only to prove (iii) in this section. Recall the definition of $P = P(\mathfrak{a}, \Sigma^+)$ in §2 and let Ψ denote the set of all the simple roots in Σ^+.

LEMMA 4. *Suppose that $H^a P$ is not open in G. Then there exists an $\alpha \in \Psi$ such that* $\dim H^a P_\alpha > \dim H^a P$ (*here P_α is the parabolic subgroup of G defined by $P_\alpha = PL_\alpha$*).

PROOF. By [2] Theorem 1, we may assume that $\sigma\mathfrak{a} = \mathfrak{a}$. By [2] Proposition 1, Σ^+ is not σ-compatible or $\mathfrak{a} \cap \mathfrak{h}$ is not maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$. First suppose that Σ^+ is not σ-compatible. Then by [4] Lemma 4 and Lemma 5, there exists an $\alpha \in \Psi$ such that $H^a P_\alpha = H^a P \cup H^a w_\alpha P$ and that $\dim H^a w_\alpha P > \dim H^a P$. Hence we may assume that Σ^+ is σ-compatible and that $\mathfrak{a} \cap \mathfrak{h}$ is not maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$.

Put $\mathfrak{l}_1 = \mathfrak{z}_\mathfrak{g}(\mathfrak{a} \cap \mathfrak{h})$. Suppose that there exists an $\alpha \in \Psi \cap \Sigma(\mathfrak{l}_1; \mathfrak{a})$ such that $\mathfrak{g}(\mathfrak{a}; \alpha) \cap \mathfrak{q}^a \neq \{0\}$. Here $\Sigma(\mathfrak{l}_1; \mathfrak{a})$ is the root system of the pair $(\mathfrak{l}_1, \mathfrak{a})$, and it is clear that $\alpha \in \Sigma(\mathfrak{l}_1; \mathfrak{a})$ if and only if $\alpha \in \Sigma$, $\sigma\alpha = -\alpha$. Then by [4] Lemma 3 (F), $\dim H^a P_\alpha > \dim H^a P$. Hence we may assume that

(4.1) $$\mathfrak{g}(\mathfrak{a}; \alpha) \cap \mathfrak{q}^a = \{0\} \quad \text{for all} \quad \alpha \in \Psi \cap \Sigma(\mathfrak{l}_1; \mathfrak{a}).$$

Let β be a root in $\Sigma(\mathfrak{l}_1; \mathfrak{a}) \cap \Sigma^+$ and write $\beta = \sum_{\alpha \in \Psi} n_\alpha \alpha$. Choose an element $Y \in \mathfrak{a} \cap \mathfrak{h}$ such that $\alpha(Y) > 0$ for all $\alpha \in \Sigma^+ - \Sigma(\mathfrak{l}_1; \mathfrak{a})$ by [4] Lemma 4. If $n_\alpha > 0$ for some $\alpha \in \Psi - \Sigma(\mathfrak{l}_1; \mathfrak{a})$, then $\beta(Y) > 0$. But since $\beta(Y) = 0$, we have proved that β is written as a linear combination of roots in $\Psi \cap \Sigma(\mathfrak{l}_1; \mathfrak{a})$. By (4.1) and Lemma 6 in §5, we have $\mathfrak{g}(\mathfrak{a}; \beta) \subset \mathfrak{h}^a$. Hence

$$\mathfrak{a} \cap \mathfrak{q} + \sum_{\beta \in \Sigma(\mathfrak{l}_1; \mathfrak{a})} \mathfrak{g}(\mathfrak{a}; \beta) \subset \mathfrak{h}^a.$$

Since $\mathfrak{z}_\mathfrak{g}(\mathfrak{a} \cap \mathfrak{h}) = \mathfrak{l}_1 = \mathfrak{z}_\mathfrak{k}(\mathfrak{a}) + \mathfrak{a} + \sum_{\beta \in \Sigma(\mathfrak{l}_1; \mathfrak{a})} \mathfrak{g}(\mathfrak{a}; \beta)$, $\mathfrak{a} \cap \mathfrak{h}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{h} = \mathfrak{p} \cap \mathfrak{q}^a$. But this is a contradiction to the assumption on $\mathfrak{a} \cap \mathfrak{h}$. Q. E. D.

LEMMA 5. *If HP is closed in G, then $HP = (K \cap H)P$.*

PROOF. If $HP = (K \cap H)xP$ for some $x \in HP$, then $HP = (K \cap H)P$. So taking a conjugate of P, we may assume that $\sigma\mathfrak{a} = \mathfrak{a}$. Since Σ^+ is σ-compatible, we can apply Lemma 3 for $w = 1$ to get

$$HP = (K \cap H)(L \cap H)_0 P.$$

Since $\mathfrak{a} \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$, we have $L \subset H^a$ by [4] Lemma 6 (i). Since $H \cap H^a = K \cap H$, we have $HP = (K \cap H)(L \cap H^a \cap H)_0 P = (K \cap H)P$ as desired.

Q. E. D.

PROOF OF THEOREM (iii). Choose $x' \in xP' \cap U$ so that $Hx'P$ has the minimum dimension among the $H-P$ double cosets contained in HxP'. Clearly $HxP' \cap H^a xP' = (Hx'P \cap H^a xP')P'$. Since $(Hx'P)^{cl} \cap HxP' = Hx'P$, it follows from Theorem (i) that $Hx'P \cap H^a xP' = Hx'P \cap H^a x'P$. So we have only to prove that

$$(4.2) \qquad Hx'P \cap H^a x'P = (K \cap H)x'P \qquad \text{for} \quad x' \in U.$$

We will prove (4.2) by induction on the codimension of $H^a x'P$. Rewriting $x'Px'^{-1}$ by P, we may assume that $x' = 1$ and that $\sigma\mathfrak{a} = \mathfrak{a}$.

Suppose that $H^a P$ is open in G. Then HP is closed in G by [2] §3 Corollary and $HP = (K \cap H)P \subset H^a P$ by Lemma 5. Hence we may assume that $H^a P$ is not open in G.

By Lemma 4, there exists an $\alpha \in \Psi$ such that $\dim H^a P_\alpha > \dim H^a P$. Then by [4] Lemma 3, there are two cases (B^a): $\sigma\theta\alpha \neq \pm\alpha$, $\sigma\theta\alpha \in \Sigma^+$ and (D^a): $\sigma\theta\alpha = \alpha$, $\mathfrak{g}(\mathfrak{a}; \alpha) \cap \mathfrak{q}^a \neq \{0\}$. Put $z = w_\alpha$ in the case (B^a) and put $z = c_\alpha$ in the case (D^a). Then we have $(HzP)^{cl} \cap HP_\alpha = HzP$ by [4] Lemma 3 (A) and (F) (since $\theta|_\mathfrak{a} = -1$, we have ($\mathrm{B}^a$)$=$(A): $\sigma\alpha \neq \pm\alpha$, $\sigma\alpha \notin \Sigma^+$ and (D^a)$=$(F): $\sigma\alpha = -\alpha$, $\mathfrak{g}(\mathfrak{a}; \alpha) \cap \mathfrak{q}^a \neq \{0\}$). Applying Theorem (i), we have

$$(4.3) \qquad HzP \cap H^a P_\alpha = HzP \cap H^a zP.$$

Let y be an element of $HP \cap H^a P$. Then we have only to show that $y \in (K \cap H)P$ since it is clear that $(K \cap H)P \subset HP \cap H^a P$. Let y' be an element of $HzP \cap yP_\alpha$. Then by (4.3) and the assumption of induction, we have

$$y' \in HzP \cap H^a P_\alpha \cap yP_\alpha = HzP \cap H^a zP \cap yP_\alpha = (K \cap H)zP \cap yP_\alpha$$

and therefore

$$y \in (K \cap H)zP_\alpha = (K \cap H)P_\alpha.$$

Since $y \in H^a P$, we have

$$y \in (K \cap H)P_\alpha \cap H^a P = (K \cap H)(P_\alpha \cap H^a)P = (K \cap H)JP.$$

Here J is the image of $P_\alpha \cap H^a$ under the projection $P_\alpha \to L_\alpha$ with respect to the Langlands decomposition $P_\alpha = L_\alpha \exp \mathfrak{n}_\alpha$. We consider the two cases (B^a) and (D^a) separately.

First consider the case (B^a). We have only to show that $J \subset L_\alpha \cap P$. Let L_α^s denote the analytic subgroup of G for the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}(\mathfrak{a}; \alpha) + \mathfrak{g}(\mathfrak{a}; -\alpha)$ as in [4] §3. Since $L_\alpha^s \cap J \supset \exp(\mathfrak{g}(\mathfrak{a}; \alpha) + \mathfrak{g}(\mathfrak{a}; 2\alpha))$, we have $L_\alpha^s -$

$(L_\alpha^s \cap J)w_\alpha(L_\alpha^s \cap P) \subset L_\alpha^s \cap P$ by the Bruhat decomposition of L_α^s. Since $L_\alpha/L_\alpha \cap P \simeq L_\alpha^s/L_\alpha^s \cap P$, we have $L_\alpha - Jw_\alpha(L_\alpha \cap P) \subset L_\alpha \cap P$. On the other hand, we have $J(L_\alpha \cap P) \cap Jw_\alpha(L_\alpha \cap P) = \emptyset$ since $H^a P \cap H^a w_\alpha P \neq \emptyset$. Hence $J \subset L_\alpha \cap P$.

Next consider the case (D^a). We have only to show that $J \subset (K \cap H)(L_\alpha \cap P)$. In this case, $J \supset L_\alpha^s \cap H^a$ and it follows easily from the proof of [4] Lemma 3 (D) that

$$L_\alpha = D(1) \cup D(w_\alpha) \cup D(c_\alpha) \cup D(c_\alpha^{-1}).$$

Here $D(x) = (L_\alpha^s \cap H^a)x(L_\alpha \cap P)$ for $x \in L_\alpha$. We also have

$$J(L_\alpha \cap P) = \begin{cases} D(1) & \text{if} \quad w_\alpha \notin N_{K\cap H}(\mathfrak{a})Z_K(\mathfrak{a}) \\ D(1) \cup D(w_\alpha) & \text{if} \quad w_\alpha \in N_{K\cap H}(\mathfrak{a})Z_K(\mathfrak{a}) \end{cases} \tag{4.4}$$

since $(H^a P \cup H^a w_\alpha P) \cap (H^a c_\alpha P \cup H^a c_\alpha^{-1} P) = \emptyset$. Since $D(1)$ and $D(w_\alpha)$ are closed in L_α, we have

$$D(x) = (L_\alpha^s \cap K \cap H)x(L_\alpha \cap P) \qquad \text{for} \quad x = 1 \quad \text{and} \quad w_\alpha \tag{4.5}$$

by Lemma 5 (Note that $L_\alpha/L_\alpha \cap P \simeq L_\alpha^s/L_\alpha^s \cap P$). From (4.4) and (4.5), we get

$$J(L_\alpha \cap P) \subset (K \cap H)(L_\alpha \cap P)$$

as desired. Q. E. D.

§5. Appendix

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan involution θ and the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and Σ the root system of the pair $(\mathfrak{g}, \mathfrak{a})$. Let Ψ be a fundemental system (the set of simple roots in a positive system of Σ) of Σ.

LEMMA 6. *Let $\mathfrak{s}$ be a θ-stable subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}(\mathfrak{a};\beta) \subset \mathfrak{s}$ for all $\beta \in \Psi$. Then $\mathfrak{g}(\mathfrak{a};\beta) \subset \mathfrak{s}$ for all $\beta \in \Sigma$.*

PROOF. Since $\mathfrak{g}(\mathfrak{a}; 2\beta) = [\mathfrak{g}(\mathfrak{a};\beta), \mathfrak{g}(\mathfrak{a};\beta)]$, we have only to prove $\mathfrak{g}(\mathfrak{a};\beta) \subset \mathfrak{s}$ for all $\beta \in \Sigma_0 = \{\beta \in \Sigma \mid 1/2\,\beta \notin \Sigma\}$ (the set of reduced roots in Σ). Let γ be a root in Ψ and X a nonzero element of $\mathfrak{g}(\mathfrak{a};\gamma)$. Then $w_\gamma = \exp c(X + \theta X) \in \exp \mathfrak{s}$ represents the reflection in $\mathfrak{a}$ with respect to γ for some $c \in \boldsymbol{R}$. Since $\mathfrak{g}(\mathfrak{a}; w_\gamma\beta) = \mathrm{Ad}(w_\gamma)\mathfrak{g}(\mathfrak{a};\beta)$ for $\beta \in \Sigma$, we have

$$\mathfrak{g}(\mathfrak{a}; w_\gamma\beta) \subset \mathfrak{s} \text{ if and only if } \mathfrak{g}(\mathfrak{a};\beta) \subset \mathfrak{s}. \tag{5.1}$$

Since the Weyl group W of Σ is generated by $\{w_\beta Z_K(\mathfrak{a}) \mid \beta \in \Psi\}$ and since $\Sigma_0 = W\Psi$, we get the desired assertion from (5.1). Q. E. D.

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