HIROSHIMA MATH. J.
18 (1988), 15–29

Deterministic version lemmas in ergodic theory of random dynamical systems

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(Received March 30, 1987)

1. Introduction

In this paper all measurable spaces are assumed to be standard. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let σ be a P-preserving transformation on Ω. For measurable spaces $(S, \mathscr{B}_S)$ and $(M, \mathscr{B}_M)$ consider a $\mathscr{B}_S \times \mathscr{B}_M \mid \mathscr{B}_M$-measurable map f: $(s, x) \to f_s x$ and a stationary sequence of S-valued random variables $\{\xi_n\}_{n=1}^{\infty}$ which are defined as $\xi_n = \xi \circ \sigma^{n-1}$ $(n \geqq 1)$ for some S-valued random variable ξ. A random dynamical system $X = (\{X_n\}_{n=0}^{\infty}, f, \xi)$ with state space $(M, \mathscr{B}_M)$ and parameter space $(S, \mathscr{B}_S)$ is defined as a sequence of randomly iterated transformations:

$$(1.1) \qquad \begin{cases} X_n(\omega)x = f_{\xi_n(\omega)} X_{n-1}(\omega)x & (n \geqq 1) \\ X_0(\omega)x = x \in M. \end{cases}$$

We often write $X = \{X_n\}_{n=0}^{\infty}$ instead of $X = (\{X_n\}_{n=0}^{\infty}, f, \xi)$. The ergodic properties of such a random dynamical systems are studied in [1], [3], [4], [5], [6], [7], and [9]. In the present paper we assume that the random variables ξ_n's are mutually independent and they generate $\mathscr{F}$, i.e., $\mathscr{F} = \mathscr{F}(\xi_1, \xi_2, \ldots)$. We have to notice that there is no work, except [7], which is concerned with the case when ξ_n's are not independent as far as we know. Under the above assumption, we shall show the deterministic version lemmas in Section 3, and we shall give their applications to the study of asymptotic properties of random dynamical systems. Precisely, as it is known in [4], [5] and [9], the random orbit $X_x = \{X_n(\omega)x\}_{n=0}^{\infty}$ becomes a Markov process starting from $x \in M$ and it is closely related to the skew product transformation T_X on $M \times \Omega$ which is defined by

$$(1.2) \qquad T_X(x, \omega) = (X_1(\omega)x, \sigma\omega) \qquad \text{for} \quad (x, \omega) \in M \times \Omega.$$

For example, a probability measure μ on M is X-invariant if and only if the product measure $\mu \times P$ is T_X-invariant (see Lemma 3.1). One of the deterministic version lemmas which we will prove asserts that if a function $\Phi \in L^1(\mu \times P)$ satisfies $\Phi \circ T_X = \lambda \Phi$ for some $\lambda \in \boldsymbol{C}$, then there is a function $\phi \in L^1(\mu)$ such that $\Phi = \phi$ $\mu \times P$-a.e. From this lemma we can obtain some interesting phenomena which reflect the difference between the ergodic behaviors of a random

*) Supported partially by the YUKAWA foundation.

dynamical system and those of a single transformation (see Section 4).

In Section 2, we shall give some definitions and facts which will be used later. In Section 3, we shall prove the deterministic version lemmas and in the final section, we shall give three applications of them. The first application is a generalization of the result in [9], the second one is concerned with the asymptotic properties of random rotations on a circle, and the third one is concerned with the characteristic exponents of random products of matrices.

The author would like to express his thanks to Professors H. Totoki and H. Ishitani for their encouragements.

2. Preliminaries

First of all we will define some concepts. Let $(M, \mathscr{B})$ be a standard measurable space and let T be a measurable transformation on M. Let $Y_x = \{Y_n(\omega)x\}_{n=0}^{\infty}$ be a time homogeneous Markov process on a probability space $(\Omega, \mathscr{F}, P)$ with state space M and starting from $x \in M$. We denote by $q(y, A)$ $(y \in M, A \in \mathscr{B})$ the transition probabilities of Y_x. In what follows the terminology Markov process is used for the family $Y = \{Y_x\}_{x \in M}$. The Markov operator $\mathscr{Q}$ associated with Y is defined by

$$(\mathscr{Q}\phi)(x) = \int \phi(y) q(x, dy) \tag{2.1}$$

for any bounded $\mathscr{B}$-measurable function ϕ. The following definitions are well-known but we summarize them for convenience.

DEFINITION 2.1 (c.f. [11]). Let T be a transformation and Y be a Markov process as above.

(T.1) A probability measure μ is called *T-invariant* if $\mu(A) = \mu(T^{-1}A)$ for any $A \in \mathscr{B}$.

(T.2) A T-invariant measure μ is called *T-ergodic* if $T^{-1}A = A$ implies $\mu(A) = 0$ or 1 for $A \in \mathscr{B}$.

(T.3) A T-invariant measure μ is called *T-weakly mixing* if there exists a subset $\boldsymbol{J}$ of $\boldsymbol{N}$ such that $\#(\boldsymbol{J} \cap \{1, 2, \ldots, n\})/n \to 0$ $(n \to \infty)$ and

$$\int (\phi \circ T^n)(x) \psi(x) \mu(dx) \longrightarrow \int \phi(x) \mu(dx) \int \psi(x) \mu(dx) \quad (n \to \infty),\ n \notin \boldsymbol{J}$$

for any $\phi, \psi \in L^2(\mu)$.

(T.4) A T-invariant measure μ is called *T-strongly mixing* if

$$\int (\phi \circ T^n)(x) \psi(x) \mu(dx) \longrightarrow \int \phi(x) \mu(dx) \int \psi(x) \mu(dx) \quad (n \to \infty)$$

for any $\phi, \psi \in L^2(\mu)$.

(T.5) A T-invariant measure μ is called *T-exact* if $\bigcap_{n=0}^{\infty} T^{-n}\mathscr{B}$ is trivial with respect to μ.

(Y.1) A probability measure μ is called *Y-invariant* if

$$\int q(x, A)\mu(dx) = \mu(A) \quad \text{for any} \quad A \in \mathscr{B}.$$

(Y.2) A Y-invariant measure μ is called *Y-ergodic* if $\mu(A)>0$ and $q(x, A)=1$ μ-a.e. $x \in A$ imply $\mu(A)=1$, for $A \in \mathscr{B}$.

(Y.3) A Y-invariant measure μ is called *Y-weakly mixing* if there exists a subset $\boldsymbol{J}$ of $\boldsymbol{N}$ such that $\sharp(\boldsymbol{J} \cap \{1, 2,\dots, n\})/n \to 0$ $(n \to \infty)$ and

$$\int (\mathscr{Q}^n\phi)(x)\psi(x)\mu(dx) \longrightarrow \int \phi(x)\mu(dx) \int \psi(x)\mu(dx) \quad (n \to \infty), \quad n \notin \boldsymbol{J}$$

for any ϕ, $\psi \in L^2(\mu)$.

(Y.4) A Y-invariant measure μ is called *Y-strongly mixing* if

$$\int (\mathscr{Q}^n\phi)(x)\psi(x)\mu(dx) \longrightarrow \int \phi(x)\mu(dx) \int \psi(x)\mu(dx) \quad (n \to \infty)$$

for any ϕ, $\psi \in L^2(\mu)$.

(Y.5) A Y-invariant measure μ is called *Y-uniformly mixing* if

$$\int |((\mathscr{Q}')^n\phi)(x) - \int \phi(x)\mu(dx)|\mu(dx) \longrightarrow 0 \quad (n \to \infty)$$

for any $\phi \in L^1(\mu)$, where $\mathscr{Q}'$ denotes the dual operator of $\mathscr{Q}$.

REMARK 2.1. The operator $\mathscr{Q}'$ is defined as the dual operator of $\mathscr{Q}$ in $L^2(\mu)$. But in this case it is uniquely extended to the operator on $L^\infty(\mu)$. This is the reason why we call $\mathscr{Q}'$ the dual operator of $\mathscr{Q}$ without designating its domain.

If μ is a T-invariant measure, we often call the triple (M, μ, T) a dynamical system. Then it is clear that the operator U_T: $\phi \to \phi \circ T$ is an isometry on $L^2(\mu)$. Next we define:

DEFINITION 2.2 ([6]). Let T be a μ-nonsingular transformation. We define an operator $\mathscr{L} = \mathscr{L}_{T,\mu}$: $L^1(\mu) \to L^1(\mu)$ by

$$\mathscr{L}\phi = \frac{d}{d\mu} \int_{T^{-1}(\cdot)} \phi d\mu \qquad \text{for} \quad \phi \in L^1(\mu), \tag{2.2}$$

and we call it the *Perron-Frobenius operator of T with respect to* μ.

REMARK 2.2. In the case when μ is T-invariant, $\mathscr{L}$ is just the dual operator of U_T.

If a Markov process Y has an invariant measure μ we can consider a dynamical system on $\Sigma = M \times M \times \cdots = \{\underline{x} = (x_0, x_1, \ldots): x_k \in M \text{ for any } k \geqq 0\}$ as follows: We induce a probability measure P_μ on Σ which is uniquely determined by

$$(2.3) \qquad P_\mu([A_0 \times A_1 \times \cdots \times A_n]) = \int_{A_0} \mu(dx_0) \int_{A_1} q(x_0, dx_1) \cdots \int_{A_n} q(x_{n-1}, dx_n)$$

for any $n \geqq 0$ and $A_0, A_1, \ldots, A_n \in \mathscr{B}$ where $[A_0 \times A_1 \times \cdots \times A_n] = \{\underline{x} \in \sum ; x_k \in A_k$ for $0 \leqq k \leqq n\}$. Then it is easy to see that the shift transformation τ: $\underline{x} \to \tau \underline{x}$ with $(\tau \underline{x})_i = (\underline{x})_{i+1}$ preserves P_μ, that is, P_μ is τ-invariant.

DEFINITION 2.3. The shift dynamical system $(\sum, P_\mu, \tau)$ is called the *Markov transformation* induced by a Markov process Y and its invariant measure μ.

The following propositions will be used frequently in this paper. The proofs are found in [6, Proposition 2.2 and Proposition 3.2]. So we omit them.

PROPOSITION 2.1. *Let (M, μ, T) be a dynamical system and $\mathscr{L} = \mathscr{L}_{T,\mu}$ be the Perron-Frobenius operator of T with respect to μ. Then we have*

$$(2.4) \qquad E_\mu[\phi \mid T^{-n}\mathscr{B}] = U_T^n \mathscr{L}^n \phi = (\mathscr{L}^n \phi) \circ T^n$$

for any $\phi \in L^1(\mu)$, where $E_\mu[\phi \mid T^{-n}\mathscr{B}]$ denotes the conditional expectation of ϕ given $T^{-n}\mathscr{B}$ with respect to μ.

PROPOSITION 2.2. *We use the same notation as in Proposition 2.1. μ is T-exact if and only if*

$$\int |(\mathscr{L}^n \phi)(x) - \int \phi(x) \mu(dx)| \mu(dx) \longrightarrow 0 \quad (n \to \infty)$$

for any $\phi \in L^1(\mu)$.

PROPOSITION 2.3. *Assume that μ is T-weakly mixing. Then, μ is T-exact if and only if $\{\mathscr{L}^n \phi\}_{n=0}^\infty$ is relatively compact in $L^1(\mu)$ for any $\phi \in L^1(\mu)$.*

PROPOSITION 2.4. *Assume that μ is T-invariant. Then for $\phi \in L^1(\mu)$ and $\lambda \in \boldsymbol{C}$, the following are equivalent:*

(1) $U_T \phi = \lambda \phi$.

(2) $\lambda \mathscr{L}_{T,\mu} \phi = \phi$, *and* $|\lambda| = 1$.

3. Deterministic version lemmas

As we mentioned in Section 1, we consider a random dynamical system $X = (\{X_n\}_{n=0}^\infty, f, \xi)$ on a probability space $(\Omega, \mathscr{F}, P)$ with state space $(M, \mathscr{B}_M)$ and parameter space $(S, \mathscr{B}_S)$. Recall that we assume that S-valued random variables ξ_n's are independent and $\mathscr{F}=\mathscr{F}(\xi_1, \xi_2, \ldots)$. Under the assumption, the random orbit $X_x=\{X_n(\omega)x\}_{n=0}^{\infty}$ $(x \in M)$ becomes a Markov process starting from $x \in M$ and with transition probability

$$q(x, A) = \int I_A(X_1(\omega)x)P(d\omega) = \int I_A(f_{\xi(\omega)}x)P(d\omega). \tag{3.1}$$

Let $T_X\colon M \times \Omega \to M \times \Omega;\ (x, \omega) \to (X_1(\omega)x, \sigma\omega)$ be the skew product transformation induced by the random dynamical system X.

First we prove:

LEMMA 3.1. *Let μ be a probability measure on the state space $(M, \mathscr{B}_M)$ of X. Then, μ is X-invariant if and only if $\mu \times P$ is T_X-invariant.*

PROOF. Write T for T_X. Assume that μ is X-invariant. Let $A \in \mathscr{B}_M$ and $\Gamma \in \mathscr{F}$. Then we have

$$\begin{aligned}(\mu \times P)(T^{-1}(A \times \Gamma)) &= \iint I_{A \times \Gamma}(T(x, \omega))\mu(dx)P(d\omega) \\ &= \iint I_A(X_1(\omega)x) I_{\sigma^{-1}\Gamma}(\omega)\mu(dx)P(d\omega) \\ &= \int \mu(dx) \int P(d\omega) I_A(X_1(\omega)x) I_{\sigma^{-1}\Gamma}(\omega) \\ &= \int \mu(dx) P(\sigma^{-1}\Gamma) \int P(d\omega) I_A(X_1(\omega)x)\end{aligned}$$

since $\sigma^{-1}\Gamma \in \mathscr{F}(\xi_2, \xi_3, \ldots)$ and $I_A(X_1(\omega)x)$ is $\mathscr{F}(\xi_1)$-measurable. Thus we have

$$\begin{aligned}(\mu \times P)(T^{-1}(A \times \Gamma)) &= P(\Gamma) \int \mu(dx) \int I_A(X_1(\omega)x)P(d\omega) \\ &= P(\Gamma) \int \mu(dx) q(x, A) \\ &= \mu(A)P(\Gamma)\end{aligned}$$

from the assumption. Therefore $\mu \times P$ is T-invariant. Obviously if $\mu \times P$ is T-invariant, we have

$$\begin{aligned}\mu(A) = (\mu \times P)(A \times \Omega) &= (\mu \times P)(T^{-1}(A \times \Omega)) \\ &= \iint I_A(X_1(\omega)x)\mu(dx)P(d\omega) \\ &= \int q(x, A)\mu(dx).\end{aligned}$$

Thus μ is X-invariant.

From now on, unless otherwise stated X denotes a random dynamical system, T denotes the skew product transformation T_X induced by X.

LEMMA 3.2. *Assume that T is nonsingular with respect to a product measure $\mu\times P$. $\mathscr{L}$ denotes the Perron-Frobenius operator of T with respect to $\mu\times P$. For a function $\phi\in L^1(\mu)$ and for a bounded $\overbrace{\mathscr{B}_S\times\mathscr{B}_S\times\cdots\times\mathscr{B}_S}^{k}$-measurable function ψ, put*

$$\Phi(x,\omega)=\phi(x)\psi(\xi_1(\omega),\dots,\xi_k(\omega)). \tag{3.2}$$

Then $\mathscr{L}^n\Phi$ has a version in $L^1(\mu)$ for $n\geqq k$. In particular, $\mathscr{L}|_{L^1(\mu)}=\mathscr{Q}'$ where $\mathscr{Q}'$ is the dual operator of the Markov operator $\mathscr{Q}$ associated with X.

PROOF. Let $A\in\mathscr{B}_M$ and $\Gamma\in\mathscr{F}$. Then we have

$$\begin{aligned}
&\int_{A\times\Gamma}(\mathscr{L}^n\Phi)(x,\omega)(\mu\times P)(d(x,\omega))\\
&=\int I_{A\times\Gamma}(T^n(x,\omega))\Phi(x,\omega)(\mu\times P)(d(x,\omega))\\
&=\int\int\mu(dx)P(d\omega)I_A(X_n(\omega)x)I_{\sigma^{-n}\Gamma}(\omega)\phi(x)\psi(\xi_1(\omega),\dots,\xi_k(\omega))\\
&=\int\mu(dx)\phi(x)\int P(d\omega)I_{\sigma^{-n}\Gamma}(\omega)I_A(X_n(\omega)x)\psi(\xi_1(\omega),\dots,\xi_k(\omega))\\
&=P(\Gamma)\int\mu(dx)\phi(x)\int P(d\omega)I_A(X_n(\omega)x)\psi(\xi_1(\omega),\dots,\xi_k(\omega))
\end{aligned}$$

since $\sigma^{-n}\Gamma\in\mathscr{F}(\xi_{n+1},\xi_{n+2},\dots)$ and $I_A(X_n(\omega)x)\psi(\xi_1(\omega),\dots,\xi_k(\omega))$ is $\mathscr{F}(\xi_1,\xi_2,\dots,\xi_n)$-measurable if $n\geqq k$. Therefore we have

$$\int_{A\times\Gamma}(\mathscr{L}^n\Phi)(x,\omega)(\mu\times P)(d(x,\omega))=\mu_n(A)P(\Gamma)$$

where $\mu_n(A)=\int\mu(dx)\phi(x)\int P(d\omega)I_A(X_n(\omega)x)\psi(\xi_1(\omega),\dots,\xi_k(\omega))$. It is easy to see that μ_n is a μ-absolutely continuous σ-additive set function. Thus there exists a function $\phi_n\in L^1(\mu)$ such that $\mu_n(A)=\int_A\phi_n(x)\mu(dx)$. Therefore we conclude that $(\mathscr{L}^n\Phi)(x,\omega)=\phi_n(x)$ $\mu\times P$-a.e. Next for any $A\in\mathscr{B}_M$ we have

$$\begin{aligned}
\int_A(\mathscr{Q}'\phi)(x)\mu(dx)&=\int(\mathscr{Q}I_A)(x)\phi(x)\mu(dx)\\
&=\int\Big(\int I_A(X_1(\omega)x)P(d\omega)\Big)\phi(x)\mu(dx)
\end{aligned}$$

$$= \int I_{A\times\Omega}(T(x, \omega))\phi(x)(\mu\times P)(d(x, \omega))$$

$$= \int I_A(x)(\mathscr{L}\phi)(x)(\mu\times P)(d(x, \omega))$$

$$= \int_A (\mathscr{L}\phi)(x)\mu(dx).$$

Here we have used the first assertion of the lemma. Thus $\mathscr{L}\phi = \mathscr{Q}'\phi$ $\mu\times P$-a.e.

Now we can give one of the main theorems:

THEOREM 3.1 (A deterministic version lemma for eigen-functions of the Perron-Frobenius operator). *Assume that the skew product transformation T is nonsingular with respect to a product measure $\mu\times P$. Let $\mathscr{L}$ be the Perron-Frobenius operator of T with respect to $\mu\times P$. If a function $\Phi\in L^1(\mu\times P)$ satisfies $\lambda\mathscr{L}\Phi=\Phi$ for some $\lambda\in\boldsymbol{C}$ with $|\lambda|=1$, then there exists a function $\phi\in L^1(\mu)$ such that $\Phi(x, \omega)=\phi(x)$ $\mu\times P$-a.e.*

PROOF. First we show that

$$\lim_{n\to\infty} \inf_{\phi\in L^1(\mu)} \|\mathscr{L}^n\Psi - \phi\|_{L^1(\mu\times P)} = 0 \tag{3.3}$$

for any $\Psi\in L^1(\mu\times P)$. Since the linear hull of such elements that have the form as (3.2) is dense in $L^1(\mu\times P)$, we can see that for any $\Psi\in L^1(\mu\times P)$ and for any $\varepsilon>0$, there exists an element $\Psi_\varepsilon\in L^1(\mu\times P)$ such that $\mathscr{L}^n\Psi_\varepsilon$ has a version in $L^1(\mu)$ for sufficiently large n. Therefore we have

$$\begin{aligned}
&\inf_\phi \|\mathscr{L}^n\Psi - \phi\|_{L^1(\mu\times P)} \\
&\leqq \|\mathscr{L}^n\Psi - \mathscr{L}^n\Psi_\varepsilon\|_{L^1(\mu\times P)} + \inf_\phi \|\mathscr{L}^n\Psi_\varepsilon - \phi\|_{L^1(\mu\times P)} \\
&\leqq \varepsilon
\end{aligned}$$

for sufficiently large n, since the operator norm of $\mathscr{L}$ on $L^1(\mu\times P)$ is 1. Thus we obtain (3.3).

The theorem is easily proved if we apply (3.3) to Φ.

From Theorem 3.1, we obtain the following:

COROLLARY 3.1. *Under the same assumption as in Theorem 3.1 any $\mu\times P$-absolutely continuous T-invariant measure has the form $\mu_1\times P$, where μ_1 is a μ-absolutely continuous X-invariant measure.*

PROOF. Let Φ be the Radon-Nikodym derivative of a $\mu\times P$-absolutely continuous T-invariant measure Q. Then we have $\mathscr{L}\Phi=\Phi$. Therefore Φ has a deterministic version $\phi\in L^1(\mu)$ in virtue of Theorem 3.1. Putting $\mu_1=\phi\mu$, we conclude that $Q=\mu_1\times P$ and μ_1 is X-invariant from Lemma 3.1.

The following theorem is also a corollary of Theorem 3.1.

THEOREM 3.2 (A deterministic version lemma for eigenfunctions of U_T). *Assume that μ is X-invariant measure. If a function $\Phi\in L^1(\mu\times P)$ satisfies $U_T\Phi=\Phi\circ T=\lambda\Phi$ for some $\lambda\in\boldsymbol{C}$ then there exists a function $\phi\in L^1(\mu)$ such that $\Phi(x, \omega)=\phi(x)$ $\mu\times P$-a.e. In particular, there is a $\Gamma\in\mathscr{F}$ with $P(\Gamma)=1$ such that $\omega\in\Gamma$ implies that $\phi(f_{\xi(\omega)}x)=\lambda\phi(x)$ μ-a.e.*

PROOF. This is an easy consequence of Proposition 2.4 and Theorem 3.1.

In the rest of this section we give two corollaries of the above theorem.

COROLLARY 3.2 (The random ergodic theorem [4]). *Assume that for each $s\in S$, μ is f_s-invariant and $\phi\in L^1(\mu)$. Then, there is a $\Gamma\in\mathscr{F}$ with $P(\Gamma)=1$ such that for any sample $\omega\in\Gamma$, $(1/n)\sum_{k=0}^{n-1}\phi(X_k(\omega)x)$ converges μ-a.e. and in $L^1(\mu)$. Moreover the limit function has a version in $L^1(\mu)$.*

PROOF. Put $\Phi(x, \omega)=\phi(x)$. Then $(1/n)\sum_{k=0}^{n-1}\phi(X_k(\omega)x)=(1/n)\sum_{k=0}^{n-1}(\Phi\circ T^k)(x, \omega)$ converges $\mu\times P$-a.e. and in $L^1(\mu\times P)$ from the Birkhoff ergodic theorem since $\mu\times P$ is T-invariant. The limit function Φ^* is T-invariant. Therefore it has a version in $L^1(\mu)$ in virtue of Theorem 3.2.

COROLLARY 3.3. *Assume that for each $s\in S$, μ is f_s-invariant. Then,*

(1) *$P\{\omega;\ \mu$ is $f_{\xi(\omega)}$-ergodic$\}>0$ implies that $\mu\times P$ is T-ergodic.*

(2) *$P\{\omega;\ \mu$ is $f_{\xi(\omega)}$-weakly mixing$\}>0$ implies that $\mu\times P$ is T-weakly mixing.*

PROOF. (1) Assume that $\Gamma\in\mathscr{B}_M\times\mathscr{F}$ with $(\mu\times P)(\Gamma)>0$ satisfies $T^{-1}\Gamma=\Gamma$, i.e., $I_\Gamma\circ T=I_\Gamma$. Then from Theorem 3.2, there is a measurable set $A\in\mathscr{B}_M$ such that $I_\Gamma(x, \omega)=I_A(x)$ $\mu\times P$-a.e. and $I_A\circ X_1(\omega)=I_A$ in $L^1(\mu)$. From the assumption we have $\mu(A)=1$. Therefore $(\mu\times P)(\Gamma)=1$. Hence $\mu\times P$ is T-ergodic.

(2) We use the fact that a dynamical system is weakly mixing if and only if it is ergodic and it has no eigenvalue except 1. For the proof see [11, p. 48] in the invertible case and consider the natural extension in general case. Let $\Phi\circ T=\lambda\Phi$ for some $\lambda\in\boldsymbol{C}$. In virtue of Theorem 3.2, there exists an element ϕ in $L^1(\mu)$ such that $\Phi=\phi$ in $L^1(\mu\times P)$ and $\phi\circ X_1(\omega)=\lambda\phi$ in $L^1(\mu)$ for a.e. ω. Therefore λ has to be 1, from the assumption. Hence $\mu\times P$ is T-weakly mixing.

REMARK 3.1. In [6] we considered a more general situation than Corollary 3.3 to prove the spectral decomposition of the random iteration of one-dimensional transformations. Corollary 3.3 asserts that we can expect the weakly mixing property of T if the family $\{f_s\}_{s\in S}$ consists of transformations having distinct spectral types one another.

4. Applications

In this section we give some applications.

4.1. Ergodic properties of Markov transformations. Let X be a random dynamical system as in the previous section and let μ be an X-invariant measure. As before we denote by $\mathscr{Q}$ and $\mathscr{Q}'$ the Markov operator and its dual operator associated with X, respectively. Recall that the Markov transformation $(\sum, P_\mu, \tau)$ is, the shift dynamical system on $\sum=\{\underline{x}=(x_0, x_1,\dots); x_k \in M, k=0, 1,\dots\}$, where P_μ is the probability measure satisfying the equation (2.3). The following theorem is a generalization of the results in [4] and [9].

THEOREM 4.1. *In the following*, (a), (b) *and* (c) *are equivalent*:

(1) (a) *μ is X-ergodic.*
  (b) *$\mu \times P$ is T-ergodic.*
  (c) *P_μ is τ-ergodic.*

(2) (a) *μ is X-weakly mixing.*
  (b) *$\mu \times P$ is T-weakly mixing.*
  (c) *P_μ is τ-weakly mixing.*

(3) (a) *μ is X-strongly mixing.*
  (b) *$\mu \times P$ is T-strongly mixing.*
  (c) *P_μ is τ-strongly mixing.*

(4) (a) *μ is X-uniformly mixing.*
  (b) *$\mu \times P$ is T-exact.*
  (c) *P_μ is τ-exact.*

PROOF. Denote by $\mathscr{L}_T$ and $\mathscr{L}_\tau$ the Perron-Frobenius operators of T and τ with respect to $\mu \times P$ and P_μ respectively. Before proving the theorem, we claim: First, the assertion (c) follows directly from (b), since the dynamical system $(\sum, P_\mu, \tau)$ is a factor of the dynamical system $(M \times \Omega, \mu \times P, T)$. Precisely, if we define a map $\pi\colon M \times \Omega \to \sum$ by $\pi(x, \omega)=(x, X_1(\omega)x, X_2(\omega)x,\dots)$ then $\pi \circ T = \tau \circ \pi$ and $\pi^*(\mu \times P)=P_\mu$. Secondly, for $\phi \in L^1(\mu)$ define a function $\underline{\phi}$ on $\sum$ by $\underline{\phi}(\underline{x})=\phi(x_0)$ for $\underline{x} \in \sum$. Then $(\mathscr{L}_\tau)\underline{\phi}(\underline{x})=\underline{(\mathscr{Q}'\phi)}(\underline{x})$. In fact, putting $J(x, \omega)=I_{A_0}(x)I_{A_1}(X_1(\omega)x)\cdots I_{A_k}(X_k(\omega)x)$ for $A_0, A_1,\dots, A_k \in \mathscr{B}_M$, we have

$$\int I_{[A_0\times A_1\times\cdots\times A_k]}(x)(\mathscr{L}_\tau\underline{\phi})(\underline{x})\,P_\mu(d\underline{x})$$

$$=\int I_{[A_0\times A_1\times\cdots\times A_k]}(\tau\underline{x})\underline{\phi}(\underline{x})\,P_\mu(d\underline{x})$$

$$=\int I_{[M\times A_0\times A_1\times\cdots\times A_k]}(\underline{x})\underline{\phi}(\underline{x})\,P_\mu(d\underline{x})$$

$$= \int I_{A_0}(X_1(\omega)x) I_{A_1}(X_2(\omega)x) \cdots I_{A_k}(X_{k+1}(\omega)x)\phi(x)(\mu \times P)(d(x, \omega))$$

$$= \int J(T(x, \omega))\phi(x)(\mu \times P)(d(x, \omega))$$

$$= \int J(x, \omega)(\mathscr{L}_T\phi)(x)(\mu \times P)(d(x, \omega))$$

$$= \int J(x, \omega)(\mathscr{Q}'\phi)(x)(\mu \times P)(d(x, \omega))$$

$$= \int_{[A_0 \times A_1 \times \cdots \times A_k]} \underline{(\mathscr{Q}'\phi)}(\underline{x}) P_\mu(d\underline{x}),$$

since $\mathscr{L}_T\phi = \mathscr{Q}'\phi$ in virtue of Lemma 3.2.

Now we prove the theorem.

(1) Assume (a) and $T^{-1}\Gamma = \Gamma$ for some $\Gamma \in \mathscr{B}_M \times \mathscr{F}$ with $(\mu \times P)(\Gamma) > 0$. Then by Theorem 3.2, there is a measurable set $A \in \mathscr{B}_M$ such that $I_A = I_\Gamma$ in $L^1(\mu \times P)$ and $I_A \circ X_1(\omega) = I_A$ in $L^1(\mu)$ for P-a.e. ω. Thus

$$q(x, A) = \int I_A(X_1(\omega)x)P(d\omega) = I_A(x) \quad \mu\text{-a.e.}$$

Therefore $\mu(A) = 1$ from (a). Hence $(\mu \times P)(\Gamma) = 1$. Next we prove (a) under the assumption that (c) holds. If $\mathscr{Q}I_A = I_A$ for some $A \in \mathscr{B}_M$ with $\mu(A) > 0$, we have $\mathscr{Q}'I_A = I_A$. In fact, from the definition, we have

$$\int I_A(x)\ (\mathscr{Q}'I_A)(x)\mu(dx) = \int (\mathscr{Q}I_A)(x)I_A(x)\mu(dx)$$

$$= \int I_A(x)\mu(dx)$$

$$= \mu(A).$$

Since $\mathscr{Q}'$ is a positive operator and $\mathscr{Q}'1 = 1$, $(\mathscr{Q}'I_A)(x) = I_A(x)$ μ-a.e. $x \in A$. Thus $(\mathscr{Q}'I_A)(x) \geqq I_A(x)$ μ-a.e. On the other hand, $\mathscr{Q}'$ preserves the value of the integration with respect to μ. Therefore $(\mathscr{Q}'I_A)(x) = I_A(x)$ μ-a.e. By the second claim we have $(\mathscr{L}_\tau I_{[A]})(\underline{x}) = I_{[A]}(\underline{x})$ P_μ-a.e. This implies that $I_{[A]}(\tau\underline{x}) = I_{[A]}(\underline{x})$ P_μ-a.e. in virtue of Proposition 2.4. Hence $\mu(A) = P_\mu([A]) = 1$ from (c). This completes the proof of (1).

(2) The proof of (2) is quite similar to that of (3), so we omit the proof of (2).

(3) Assume (a). For any $\Psi \in L^2(\mu \times P)$ which has the form $\Psi(x, \omega) = \phi(x)\psi(\xi_1(\omega), \ldots, \xi_k(\omega))$ as in (3.2) and for any $\Phi \in L^2(\mu \times P)$ we have, for any $n \geqq k$,

$$\int \Phi(T^n(x,\omega))\Psi(x,\omega)(\mu\times P)(d(x,\omega))$$
$$=\int \Phi(x,\omega)(\mathscr{L}_T^n\Psi)(x,\omega)(\mu\times P)(d(x,\omega))$$
$$=\int \Phi(x,\omega)((\mathscr{Q}')^{n-k}\chi)(x)(\mu\times P)(d(x,\omega))$$
$$=\int P(d\omega)\int \Phi(x,\omega)((\mathscr{Q}')^{n-k}\chi)(x)\mu(dx)$$

where χ denotes the deterministic version of $\mathscr{L}_T^k\Psi$ in Lemma 3.2. From the assumption we have

$$\int P(d\omega)\int \Phi(x,\omega)((\mathscr{Q}')^n\chi)(x)\mu(dx)$$
$$\to \int P(d\omega)\int \Phi(x,\omega)\mu(dx)\int \chi(x)\mu(dx) \quad (n\to\infty)$$
$$=\int \Phi(x,\omega)(\mu\times P)(d(x,\omega))\int \Psi(x,\omega)(\mu\times P)(d(x,\omega))$$

since $\mathscr{L}_T$ preserves the value of the integration. Since the linear hull of such Ψ's is dense in $L^2(\mu\times P)$, we obtain (b). Now we assume (c). For any $\phi, \psi \in L^2(\mu)$, in virtue of the second claim as above, we have

$$\int (\mathscr{Q}^n\phi)(x)\psi(x)\mu(dx)$$
$$=\int \phi(x)((\mathscr{Q}')^n\psi)(x)\mu(dx)$$
$$=\int \underline{\phi}(\underline{x})(\mathscr{L}_\tau^n\underline{\psi})(\underline{x})P_\mu(d\underline{x})$$
$$=\int \underline{\phi}(\tau^n\underline{x})\underline{\psi}(\underline{x})P_\mu(d\underline{x})$$
$$\to \int \underline{\phi}(\underline{x})P_\mu(d\underline{x})\int \underline{\psi}(\underline{x})P_\mu(d\underline{x}) \quad (n\to\infty)$$
$$=\int \phi(x)\mu(dx)\int \psi(x)\mu(dx).$$

(4) Assume (a). In virtue of Proposition 2.2, we have to prove

$$\int \left|(\mathscr{L}_T^n\Psi)(x,\omega)-\int \Psi(x,\omega)(\mu\times P)(d(x,\omega))\right|(\mu\times P)(d(x,\omega)) \longrightarrow 0 \quad (n\to\infty)$$

for any $\Psi\in L^1(\mu\times P)$. For any Ψ which has the form as in (3.2), we have

$$\int |(\mathscr{L}_T^n \Psi)(x, \omega) - \int \Psi(x, \omega)(\mu \times P)(d(x, \omega))| (\mu \times P)(d(x, \omega))$$

$$= \int |((\mathscr{Q}')^{n-k}\chi)(x) - \int \chi(x)\mu(dx)|\mu(dx) \longrightarrow 0 \quad (n \to \infty)$$

by the assumption, where χ is the deterministic version of $\mathscr{L}_T^k \Psi$. Since the linear hull of such Ψ's is dense in $L^1(\mu \times P)$, we have (b). Next we assume (c). For any $\phi \in L^1(\mu)$ we have

$$\int \left| ((\mathscr{Q}')^n \phi)(x) - \int \phi(x)\mu(dx) \right| \mu(dx)$$

$$= \int \left| \underline{((\mathscr{Q}')^n \phi)}(\underline{x}) - \int \underline{\phi}(\underline{x}) P_\mu(d\underline{x}) \right| P_\mu(d\underline{x})$$

$$= \int \left| (\mathscr{L}_\tau^n \underline{\phi})(\underline{x}) - \int \underline{\phi}(\underline{x}) P_\mu(d\underline{x}) \right| P_\mu(d\underline{x}) \longrightarrow 0 \quad (n \to \infty).$$

Here we used the second claim.

REMARK 4.1. In the proof of Theorem 4.1 we have shown that $\mathscr{Q} I_A = I_A$ implies $\mathscr{Q}' I_A = I_A$. Furthermore, we can prove the following: Assume that an operator $\mathscr{R}$ on $L^\infty(\mu)$ satisfies:

(1) $\mathscr{R}\varphi \geqq 0$ μ-a.e. if $\varphi \geqq 0$ μ-a.e.

(2) $\mathscr{R}1 = 1$ μ-a.e.

(3) $\int \mathscr{R}\varphi d\mu = \int \varphi d\mu$ for any $\varphi \in L^\infty(\mu)$.

Then, it can be uniquely extended to the operator on $L^1(\mu)$ and its dual operator $\mathscr{R}'$ also satisfies (1), (2) and (3), and $\mathscr{R}\varphi = \varphi$ if and only if $\mathscr{R}'\varphi = \varphi$ for $\varphi \in L^1(\mu)$.

4.2. The radom ratation on a circle. Here we consider the following case: $S = M = S^1 = \{x \in \boldsymbol{C};\ |x| = 1\}$: the unit circle in the plane, $\mathscr{B}_S = \mathscr{B}_M = \mathscr{B}(S^1)$: the topological Borel field of S^1, $f: S \times M \to M$; $(s, x) \to f_s x = sx$. The probability space $(\Omega, \mathscr{F}, P)$ is defined as an infinite product measure space $(S^1, \mathscr{B}(S^1), \nu) \times (S^1, \mathscr{B}(S^1), \nu) \times \cdots$, where ν is a probability distribution on S^1. In this case, the coordinate functions $\xi_n(\omega) = \omega_n$ are regarded as the S-valued independent random variables, where $\omega = (\omega_1, \omega_2, \dots)$ and the random dynamical system $X = (\{X_n\}_{n=0}^\infty, f, \xi_1)$ is given by $X_n(\omega)x = \omega_n \omega_{n-1} \cdots \omega_1 x$. Since the Haar measure m on S^1 is f_s-invariant for each s, it is also X-invariant. Denote by $(\sum, P_m, \tau)$ the Markov transformation induced by X and m. Then we have:

THEOREM 4.2. (1) *P_m is not τ-egrodic if and only if there is an integer $p > 0$ such that $\nu\{1, \lambda, \dots, \lambda^{p-1}\} = 1$ for some p-th root λ of* 1.

(2) *P_m is not τ-weakly mixing but τ-ergodic if and only if there is an integer $p > 0$ such that $\nu\{\kappa, \kappa\lambda, \dots, \kappa\lambda^{p-1}\} = 1$ for some $\kappa \in S^1$ which is not a root of* 1 *and $\lambda \in S^1$ which is a p-th root of* 1.

(3) *P_m is τ-exact if and only if it is τ-weakly mixing. In particular, in the case except* (1) *and* (2), *P_m is τ-exact.*

PROOF. Before proving the theorem we notice that we may assume that if $\phi \in L^2(m\times P)$ satisfies $\phi\circ T=\lambda\phi$ in $L^2(m\times P)$ for some $\lambda\in \boldsymbol{C}$ then $\phi\in L^2(m)$ and $\phi\circ f_s=\lambda\phi$ in $L^2(m)$ for ν-a.e.$s\in S^1$, in virtue of the deterministic version lemma (Theorem 3.2).

Proof of (1). Assume that P_m is not τ-ergodic. From Theorem 4.1, $m\times P$ is not T-ergodic. Thus there is a non-constant function $\phi\in L^2(m)$ with $\phi\circ f_s=\phi$ in $L^2(m)$, ν-a.e. s. In $L^2(m)$, ϕ is uniquely represented as $\phi(x)=\sum_{n\in \boldsymbol{Z}}\phi_n x^n$, where $\phi_n\in \boldsymbol{C}$ and $\sum_{n\in \boldsymbol{Z}}|\phi_n|^2<\infty$. Let $p\neq 0$ be the integer which has the smallest modulus among all n with $\phi_n\neq 0$. From the uniqueness of the representation, we have $s^p=1$ ν-a.e. s. Thus we have $\nu\{1, \lambda,\dots, \lambda^{|p|-1}\}=1$ for some $|p|$-th root λ of 1. The converse direction is obvious.

Proof of (2). Assume that P_m is not τ-weakly mixing but τ-ergodic. From Theorem 4.1, $m\times P$ is not T-weakly mixing but T-ergodic. Therefore, there exists a non-constant function $\phi\in L^2(m)$ and $\gamma\in \boldsymbol{C}$ with $\gamma\neq 1$ such that $\phi\circ f_s=\gamma\phi$ in $L^2(m)$ for ν-a.e. $s\in S^1$. As before ϕ is represented as $\phi(x)=\sum_{n\in \boldsymbol{Z}}\phi_n x^n$. From the uniqueness of the representation, we have $\gamma=s^p$ ν-a.e. $s\in S^1$. Thus there exist $\kappa\in S^1$ which is independent of s and an integer q which may depend on s such that $s=\kappa\lambda^q$ for ν-a.e. s, where λ is a $|p|$-th root of 1. Therefore $\nu\{\kappa, \kappa\lambda,\dots, \kappa\lambda^{|p|-1}\}=1$. Since $m\times P$ is T-ergodic, κ is not a root of 1 in virtue of the statement (1). The converse direction is obvious.

Proof of (3). In virtue of Proposition 2.3 and (4) in Theorem 4.1, it suffices to show that $\{(\mathscr{Q}')^n\phi\}_{n=0}^{\infty}$ is relatively compact in $L^1(m)$ for any $\phi\in L^1(m)$. Recall that $(\mathscr{Q}'\phi)(x)=(\mathscr{L}_T\phi)(x)=\int(\mathscr{L}_{f_s}\phi)(x)\nu(ds)=\int\phi(s^{-1}x)\nu(ds)$ in virtue of Lemma 3.2. Since the map f_s is an isometry on S^1 for each s, $\{(\mathscr{Q}')^n\phi\}_{n=0}^{\infty}$ is relatively compact in $C(S^1)$ for any $\phi\in C(S^1)$ in virtue of the Ascoli-Arzela theorem. Thus $\{(\mathscr{Q}')^n\phi\}_{n=0}^{\infty}$ is relatively compact in $L^1(m)$ for any $\phi\in C(S^1)$. Noticing that the operator norm of $\mathscr{Q}'$ on $L^1(m)$ is 1 and $C(S^1)$ is dense in $L^1(m)$, we can see that $\{(\mathscr{Q}')^n\phi\}_{n=0}^{\infty}$ is relatively compact in $L^1(m)$ for any $\phi\in L^1(m)$.

REMARK 4.2. Theorem 4.2 illustrates the difference between the random iteration and the iterations of a single transformation. In fact, consider a rotation f_s: $S^1\to S^1$; $x\to sx$. Then,

(1) m is f_s-ergodic if and only if s is not a root of 1,

(2) m is not f_s-weakly mixing for any $s\in S^1$.

4.3. Random products of matrices. Let $X=(\{X_n\}_{n=0}^{\infty}, f, \xi)$ be a random dynamical system and let μ be an X-invariant measure. Consider a measurable map $D(\cdot,\cdot)$: $S\times M\to M_k$ where M_k stands for the space of all $k\times k$ real matrices. Write

$$D^n(x, \omega) = D(\xi_n(\omega), X_{n-1}(\omega)x)\cdots D(\xi_1(\omega), x).$$

Then we have:

THEOREM 4.3 (see [8] and [10]). *Assume that*

$$\int \log^+ \|D(s, x)\| \nu(ds)\mu(dx) < \infty$$

where ν *is the distribution of* ξ *and* $\log^+ x = \max(\log x, 0)$. *Then, there exist* $\mathscr{B}_M$*-measurable functions* $\lambda_1(\cdot), \lambda_2(\cdot),\dots, \lambda_k(\cdot), m_1(\cdot), m_2(\cdot),\dots, m_k(\cdot), s(\cdot)$ *and a measurable set* $\Gamma \in \mathscr{B}_M \times \mathscr{F}$ *such that the following hold*:

(1) $m_1(\cdot), m_2(\cdot),\dots, m_k(\cdot)$ *and* $s(\cdot)$ *are integer valued and* $0 < s(x) \leqq k$, $m_i(x) > 0$ $(i \leqq s(x))$, $m_i(x) = 0$ $(i > s(x))$ and $\sum_{i=1}^k m_i(x) = k$.

(2) $-\infty \leqq \lambda_1(x) = \cdots = \lambda_{m_1(x)}(x) < \lambda_{m_1(x)+1}(x) = \cdots = \lambda_{m_1(x)+m_2(x)}(x) < \cdots < \lambda_{m_1(x)+\cdots+m_{s(x)-1}(x)+1}(x) = \cdots = \lambda_k(x) < \infty$ *and* $\lambda_i^+(\cdot) \in L^1(\mu)$.

Write $\lambda^{(1)}(x) < \cdots < \lambda^{(s(x))}(x)$ *for the distinct values of* $\lambda_i(x)$'s.

(3) $(\mu \times P)(\Gamma) = 1$, *and* $T_X \Gamma \subset \Gamma$.

(4) *If* $(x, \omega) \in \Gamma$, *then the limit*

$$\lim_{n\to\infty} (D^n(x, \omega)^t D^n(x, \omega))^{1/2n} = \Lambda(x, \omega)$$

exists and its eigenvalues are just $\exp(\lambda^{(1)}(x)),\dots, \exp(\lambda^{(s(x))}(x))$.

(5) *Denote the corresponding eigenspaces by* $U^{(1)}_{(x,\omega)},\dots, U^{(s(x))}_{(x,\omega)}$. *Then* $\dim U^{(i)}_{(x,\omega)} = m_i(x)$.

(6) *Write* $V^{(0)}_{(x,\omega)} = \{0\}$, $V^{(i)}_{(x,\omega)} = U^{(1)}_{(x,\omega)} + \cdots + U^{(i)}_{(x,\omega)}$. *Then,*

$$\lim_{n\to\infty} 1/n \log \|D^n(x, \omega)v\| = \lambda^{(i)}(x) \quad \textit{if} \quad v \in V^{(i)}_{(x,\omega)} \setminus V^{(i-1)}_{(x,\omega)}.$$

PROOF. Substituting T, $\mu \times P$ and $D(\xi(\omega), x)$ for τ, ρ and $T(x)$ respectively in Theorem 1.6 in [10], we can prove the assertions except for that λ_i, m_i and s have deterministic versions. Since they are T_X-invariant, we complete the proof in virtue of Theorem 3.2.

REMARK 4.3. From Theorem 4.3, we can define the Lyapunov exponents of differentiable random dynamical systems if we consider the case when M is a compact smooth manifold, S is the space of all C^1-differentiable maps and $f_s x = s(x)$ (see [1], [8], and [10]).

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