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# Affine semigroups on Banach spaces

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This note treats strongly continuous one-parameter affine semigroups on a Banach space $X$. An affine semigroup decomposes into a linear part and a translation part. These parts are reassembled here as an "augmented" linear semigroup in one higher dimension, and the latter is used to characterize the affine semigroup. Relations among the infinitesimal generators (i.g.) of an affine semigroup, its linear part, and its augmented semigroup are studied. It is shown that the translation part is completely determined from the linear part by an element of $(X \times X)/G(U)$, where $G(U)$ is the graph of the i.g. of the linear part. Also obtained are necessary and sufficient conditions for a curve to be the translation part of some affine semigroup. An application of these conditions is the so-called "screw line" studied by von Neumann and Schoenberg.

Affine concepts are an almost trivial modification of linear ones. The treatment here is intended to aid in the discovery of the correct nonlinear generalizations of familiar linear concepts. Thus affine versions of one-parameter groups, compact semigroups, and analytic semigroups are studied, along with affine cosine functions.

## § 1. Affine semigroups and associated linear semigroups

Except where otherwise noted, Banach spaces are taken to be real.

By a strongly continuous one-parameter affine semigroup on a Banach space $X$, or *affine semigroup* for short, is meant a one-parameter family $\{S(t): t \geq 0\}$ of continuous affine transformations on $X$ with the properties

($s_1$) $S(0)=I$ (the identity operator on $X$), $S(t+s)=S(t)\circ S(s)$ for $s$, $t \geq 0$; and
($s_2$) for each $x$ in $X$, $t \mapsto S(t)x$ is a continuous function from $[0, \infty)$ into $X$.

A *linear semigroup* is defined similarly except that "linear" transformations replace "affine" transformations. Strong continuity for $t \geq 0$ will be assumed in both the linear and affine cases.

Affine semigroups arise naturally in the study of linear differential equations with a nonhomogeneous (constant) term. Applications of the affine theory

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in this vein, for example, the Duhamel Principle ([12], p. 438), asymptotics (cf. [5]), and second order differential equations (see §5 below), although implicit in our discussion, will not appear directly.

Corresponding to any family $\{S(t): t \geq 0\}$ of affine transformations on $X$ are a family $\{T(t): t \geq 0\}$ of linear transformations on $X$ and an $X$-valued function $t \mapsto z(t)$, $t \geq 0$, called respectively the *linear part* and the *translation part* of $\{S(t): t \geq 0\}$. These parts are related to the original family $S(\cdot)$ by the equations

$$\begin{aligned} &z(t) = S(t)0, \\ (1.1) \qquad &T(t)x = S(t)x - S(t)0, \qquad t \geq 0,\ x \text{ in } X, \\ &S(t)x = T(t)x + z(t). \end{aligned}$$

The one-to-one correspondence between families $S(\cdot)$ and pairs $(T(\cdot), z(\cdot))$ leads to a description of an affine semigroup in terms of its linear and translation parts.

PROPOSITION 1.1. *Let $\{S(t): t \geq 0\}$ be a family of affine transformations on $X$ with linear part $\{T(t): t \geq 0\}$ and translation part $\{z(t): t \geq 0\}$. Then $S(\cdot)$ is an affine semigroup on $X$ if and only if $T(\cdot)$ is a linear semigroup on $X$ and $z(\cdot)$ is a continuous map from $[0, \infty)$ into $X$ satisfying*

$$(1.2) \qquad z(t+s) = T(t)z(s) + z(t), \qquad s, t \geq 0.$$

This result follows readily from $(s_1)$, $(s_2)$, and (1.1), and the proof is left to the reader.

A convenient way to establish additional properties of the affine semigroup $S(\cdot)$ is to introduce another linear semigroup.

Let $\hat{X}$ denote the Banach space $X \times R$, and let $\pi_1$ and $\pi_2$ be the canonical projections of $\hat{X}$ onto $X$ and $R$. If $S(\cdot)$ is a family of affine transformations on $X$ with linear part $T(\cdot)$ and translation part $z(\cdot)$, an induced family $\{\hat{T}(t): t \geq 0\}$ of linear transformations on $\hat{X}$ is defined as follows:

$$(1.3) \qquad \hat{T}(t)[x, \xi] = [T(t)x + \xi z(t), \xi], \qquad [x, \xi] \text{ in } \hat{X}.$$

The family $\hat{T}(\cdot)$ satisfies the equation

$$(1.4) \qquad \xi = \pi_2 \circ \hat{T}(t)[x, \xi], \qquad [x, \xi] \text{ in } \hat{X},\ t \geq 0.$$

It is easy to see that any family $\hat{T}(\cdot)$ of linear transformations on $\hat{X}$ satisfying (1.4) induces a family $S(\cdot)$ of affine transformations on $X$ given by

$$(1.5) \qquad S(t)x = \pi_1 \circ \hat{T}(t)[x, 1], \qquad t \geq 0,\ x \text{ in } X,$$

with linear and translation parts

$$(1.6)\qquad \begin{aligned} T(t)x &= \pi_1\circ\hat{T}(t)[x, 0], \\ z(t) &= \pi_1\circ\hat{T}(t)[0, 1]. \end{aligned} \qquad t \geq 0,\ x \text{ in } X$$

PROPOSITION 1.2. *Let $\{S(t)\colon t\geq 0\}$ be a family of affine transformations on $X$ and let $\{\hat{T}(t)\colon t\geq 0\}$ be the associated family of linear transformations on $\hat{X}$. Then $S(\cdot)$ is an affine semigroup on $X$ if and only if $\hat{T}(\cdot)$ is a linear semigroup on $\hat{X}$.*

PROOF. Suppose that $S(\cdot)$ is an affine semigroup. Then $T(\cdot)$ and $z(\cdot)$ have the properties noted in Proposition 1.1. Hence

$$\begin{aligned} \hat{T}(t+s)[x, \xi] &= [T(t+s)x+\xi z(t+s), \xi] \\ &= [T(t)(T(s)x+\xi z(s))+\xi z(t), \xi] \\ &= \hat{T}(t)\circ\hat{T}(s)[x, \xi]. \end{aligned}$$

Strong continuity of $\hat{T}(\cdot)$ follows easily from (1.3) and Proposition 1.1.

Conversely, if $\hat{T}(\cdot)$ is a linear semigroup satisfying (1.4), then $S(\cdot)$ satisfies (1.5) and

$$\begin{aligned} S(t+s)x &= \pi_1\circ\hat{T}(t+s)[x, 1] \\ &= \pi_1\circ\hat{T}(t)\circ\hat{T}(s)[x, 1] \\ &= \pi_1\circ\hat{T}(t)[\pi_1\circ\hat{T}(s)[x, 1], \pi_2\circ\hat{T}(s)[x, 1]] \\ &= \pi_1\circ\hat{T}(t)[S(s)x, 1] \\ &= S(t)\circ S(s)x. \end{aligned}$$

Strong continuity of $S(\cdot)$ is implied by (1.5). ■

The linear semigroup $\{\hat{T}(t)\colon t\geq 0\}$ on $\hat{X}$ is called the *augmented semigroup* associated with $S(\cdot)$.

## §2. The infinitesimal generator

The *infinitesimal generator* of an affine semigroup $S(\cdot)$, hereafter abbreviated as i.g., is the map $A$ defined by

$$(2.1)\qquad Ax = \lim_{h\to 0^+}(1/h)(S(h)x-x).$$

Its domain $D(A)$ consists of the set of all $x$ in $X$ for which the limit in (2.1) exists.

PROPOSITION 2.1. *Let $S(\cdot)$ be an affine semigroup with augmented semigroup $\hat{T}(\cdot)$ and linear part $T(\cdot)$. Then the following relations hold among their respective i.g. $A$, $\hat{U}$, and $U$:*

i) *$D(A)=\{x: [x, 1]$ is in $D(\hat{U})\}$ and*
*$Ax=\pi_1\circ\hat{U}[x, 1]$ for $x$ in $D(A)$;*

ii) *$D(U)=\{x: [x, 0]$ is in $D(\hat{U})\}$ and*
*$Ux=\pi_1\circ\hat{U}[x, 0]$ for $x$ in $D(U)$; and*

iii) *$D(\hat{U})=\{[x, \xi]: \xi\neq 0, (x/\xi)$ is in $D(A)\}\cup\{[x, 0]: x$ is in $D(U)\}$, and for $[x, \xi]$ in $D(\hat{U})$*

$$\hat{U}[x, \xi] = \begin{cases} [\xi A(x/\xi), 0] & \text{if } \xi \neq 0 \\ [U(x), 0] & \text{if } \xi = 0. \end{cases}$$

PROOF. The expression $(1/h)(\hat{T}(h)[x, \xi]-[x, \xi])$ equals

$$[(1/h)(T(h)x-x)+\xi(1/h)z(h), 0]. \tag{2.2}$$

If $\xi=1$, this reduces to $[(1/h)(S(h)x-x), 0]$ since $z(0)=0$. When $h$ goes to $0^+$, the conclusions in i) are obtained. If $\xi=0$, (2.2) reduces to $[(1/h)(T(h)x-x), 0]$. When $h$ goes to $0^+$, ii) results. For other values of $\xi$ linearity of $\hat{U}$ implies that $[x, \xi]$ is in $D(\hat{U})$ if and only if $[x/\xi, 1]$ is $D(\hat{U})$, and by i) $\hat{U}[x, \xi]=\xi\hat{U}[x/\xi, 1]=\xi[A(x/\xi), 0]$. So iii) holds. ■

COROLLARY 2.2. *Let $S(\cdot)$ be an affine semigroup. Let $x$ be an element of $X$ and let $0\leq a<b$. Then $(1/(b-a))\int_a^b S(t)x\,dt$ is a member of $D(A)$, and*

$$A((1/(b-a))\int_a^b S(t)xdt) = (1/(b-a))(S(b)x-S(a)x).$$

PROOF. With $a$, $b$, and $x$ as above, and $\xi$ in $R$, the linear semigroup $\hat{T}(\cdot)$ has the familiar properties

(i) $\int_a^b \hat{T}(t)[x. \xi]dt$ is in $D(\hat{U})$

and

(ii) $\hat{U}\left(\int_a^b \hat{T}(t)[x, \xi]dt\right)=\hat{T}(b)[x, \xi]-\hat{T}(a)[x, \xi]$.

By linearity of $\hat{U}$, $D(\hat{U})$ contains

$$\begin{aligned}
&(1/(b-a))\int_a^b \hat{T}(t)[x, 1]dt \\
&=\int_a^b \hat{T}(t)[x/(b-a), 1/(b-a)]dt \\
&=\int_b^b [T(t)(x/(b-a))+(1/(b-a))z(t), 1/(b-a)]dt \\
&= [(1/(b-a))\int_a^b S(t)xdt, 1],
\end{aligned}$$

and the image of the latter under $\hat{U}$ is $(1/(b-a))(\hat{T}(b)[x, 1]-\hat{T}(a)[x, 1])$. Applying part i) of Proposition 2.1 and the identity (1.6), we obtain the Corollary. ∎

In particular, from Corollary 2.2 with $x=0$, we see that $(1/(b-a))\int_a^b z(t)dt$ is a member of $D(A)$, and $A((1/(b-a))\int_a^b z(t)dt)=(1/(b-a))(z(b)-z(a))$ for $0\leq a<b$.

PROPOSITION 2.3. *Let $S(\cdot)$ be an affine semigroup with linear part $T(\cdot)$, translation part $z(\cdot)$, and i.g. $A$. Then for any pair of elements $v$, $w$ of $X$ satisfying $Av=w$, and for all $t\geq 0$,*

$$z(t) = (I-T(t))v + \int_0^t T(s)wds. \tag{2.3}$$

PROOF. The augmented semigroup $\hat{T}(\cdot)$ of $S(\cdot)$ satisfies the identity

$$\hat{T}(t)[x, \xi] - [x, \xi] = \int_0^t \hat{T}(s)\hat{U}[x, \xi]ds$$

for $[x, \xi]$ in $D(\hat{U})$. By part iii) of Proposition 2.1, with $x=v$ and $\xi=1$, this reduces to

$$\begin{aligned}[T(t)v-v+z(t), 0] &= \int_0^t \hat{T}(s)\hat{U}[v, 1]ds \\ &= \int_0^t \hat{T}(s)[w, 0]ds \\ &= \int_0^t [T(s)w, 0]ds.\end{aligned}$$

Projecting to the first component, we obtain (2.3). ∎

The identity (2.3) was noted by Crandall and Pazy in [1], p. 411.

Let us denote the translation map $x\mapsto x+v$ from $X$ onto $X$ by the symbol $\tau_v$. Note that $S(t)=\tau_{z(t)}\circ T(t)$.

PROPOSITION 2.4. *A map $A$ with domain and range in $X$ is the i.g. of an affine semigroup on $X$ if and only if there exist vectors $v$ and $w$ in $X$ such that $\tau_{-w}\circ A\circ\tau_v$ is the i.g. of a linear semigroup on $X$.*

PROOF. If $A$ is the i.g. of an affine semigroup $S(\cdot)$, let $U$ be the i.g. of its linear part $T(\cdot)$, let $v$ be any member of $D(A)$, and let $w=Av$. Corollary 2.2 guarantees that $v$ and $w$ exist.

For $x$ in $X$ and $h>0$, $(1/h)(S(h)x-x)=(1/h)(S(h)v-v)+(1/h)(T(h)(x-v)-$

$(x-v))$. As $h$ tends to 0, the first term on the right converges to $Av=w$. So the other two terms converge or diverge together, and $x$ is in $D(A)$ if and only if $x-v$ is in $D(U)$. Thus $D(A)=D(U)+v$, and $Ax=w+U(x-v)=\tau_w\circ U\circ\tau_{-v}(x)$ for $x$ in $D(A)$. So $A=\tau_w\circ U\circ\tau_{-v}$ and $U=\tau_{-w}\circ A\circ\tau_v$.

Conversely, suppose that $U=\tau_{-w}\circ A\circ\tau_v$ is the i.g. of a linear semigroup $T(\cdot)$. Then we define $S(\cdot)$ to be the family of affine transformations with linear part $T(\cdot)$ and translation part $z(\cdot)$ given by (2.3). It is easy to see that $z(\cdot)$ satisfies (1.2). Thus by Proposition 1.1, $S(\cdot)$ is an affine semigroup. Let $A'$ be the i.g. of this semigroup. By (2.3) $(1/h)(S(h)v-v)=(1/h)(T(h)v+z(h)-v)=(1/h)\int_0^h T(s)wds$ tends to $w$ as $h$ tends to $0+$. So $v$ is in $D(A')$ and $A'v=w$. It follows from the last paragraph that $\tau_{-w}\circ A'\circ\tau_v=U$ and thus that $A'=A$. ■

COROLLARY 2.5. *If $A$ is the i.g. of an affine semigroup, $A$ generates exactly one such semigroup.*

PROOF. If $A$ generates two affine semigroups $S(\cdot)$ and $S'(\cdot)$, then $A$ has two representations, $A=\tau_w\circ U\circ\tau_{-v}$ and $A=\tau_w\circ U'\circ\tau_{-v}$. Here $U$ and $U'$ are the i.g. of the linear parts of $S(\cdot)$ and $S'(\cdot)$, and as in the proof of Proposition 2.4, $v$ and $w$ can be taken to be the same for both semigroups. Since $D(A)=D(U)+v=D(U')+v$, it follows trivially that $U=U'$. Because the i.g. of a linear semigroup uniquely determines the semigroup, we conclude that $S(\cdot)$ and $S'(\cdot)$ have the same linear part $T(\cdot)$. By (2.3) they also have the same translation part and are identical. ■

## § 3. Relations between the linear and translation parts

Let $T(\cdot)$ be a linear semigroup with i.g. $U$. Let $\mathscr{T}_U$ denote the set of all maps $z(\cdot)$ from $[0, \infty)$ into $X$ that are translation parts of affine semigroups with linear part $T(\cdot)$. By Proposition 1.1 these are just continuous maps satisfying (1.2). Inspection of (1.2) shows that $\mathscr{T}_U$ is a real vector space. It is a Fréchet space under the family of seminorms $\|z(\cdot)\|_K=\sup\{\|z(t)\|: t \text{ is in } K\}$, $K$ a compact subset of $(0, \infty)$. Completeness under these seminorms follows from (1.2).

Motivated by (2.3), we introduce another real vector space $X_U=(X\times X)/G(U)$ where $G(U)$ is the graph of $U$, and a map $\phi$: $X_U\to\mathscr{T}_U$ defined by

$$\phi([v, w]) = (I-T(t))v + \int_0^t T(s)wds \tag{3.1}$$

for $[v, w]$ in $X_U$. Since $U$ is a closed operator, $X_U$ is a Banach space.

PROPOSITION 3.1. *The map $\phi$ is well-defined, and is a continuous linear isomorphism from $X_U$ onto $\mathscr{T}_U$.*

PROOF. To show that $\phi$ is well-defined, observe that if $[v, w]=[v', w']$ then $v'-v$ is in $D(U)$ and $U(v'-v)=w'-w$. It is then a familiar fact from linear semigroup theory that $T(t)(v'-v)-(v'-v)=\int_0^t T(s)U(v'-v)ds=\int_0^t T(s)(w'-w)ds$, from which $\phi([v, w])=\phi([v', w'])$ follows.

That $\phi$ is onto is shown in Proposition 2.3, and linearity is evident.

We next show that $\ker\phi$ is trivial. Suppose $\phi([v, w])$ is the zero function. Then $T(t)v-v=\int_0^t T(s)wds$. Dividing both sides of this equation by $t$ and letting $t$ tend to $0+$, we obtain that $v$ is in $D(U)$ and $Uv=w$. Hence $[v, w]$ is the zero element of $X_U$.

For any compact subset $K$ of $(0, \infty)$,

$$\begin{aligned}\|\phi([v, w])\|_K &= \sup\left\{\left\|(I-T(t))v+\int_0^t T(s)wds\right\|:\ t\in K\right\}\\ &\leq C_K\cdot(\|v\|+\|w\|)\end{aligned}$$

where $C_K$ is a constant whose existence is guaranteed by the strong continuity of $T(\cdot)$. Varying $(v, w)$ in the equivalence class $[v, w]$, we obtain:

$$\begin{aligned}\|\phi([v, w])\|_K &\leq \inf\{C_K\cdot(\|v\|+\|w\|): (v, w)\in[v, w]\}\\ &= C_K\cdot\|[v, w]\|.\end{aligned}$$

This shows that $\phi$ is continuous. ■

The map $\phi$ can even be regarded as an isometric isomorphism if we let it induce a norm on the range space. For $z(\cdot)$ in $\mathscr{T}_U$ let $\|z(\cdot)\|=\inf\{\|x\|+\|Ax\|: x\in D(A)\}$, where $A$ is the i.g. of the unique affine semigroup with translation part $z(\cdot)$ and linear part $T(\cdot)$. With this norm $\mathscr{T}_U$ is a Banach space. Note that $\phi^{-1}(z(\cdot))=$the graph of $A=$a point or equivalence class in $X_U$.

Proposition 3.1 characterizes translation parts corresponding to a given linear part (they come from equivalence classes in $X\times X \bmod G(U)$). It is also possible to obtain a partial characterization of the linear part of an affine semigroup given the translation part.

PROPOSITION 3.2. *Let $z(\cdot)$ be the translation part of an affine semigroup $S(\cdot)$ with linear part $T(\cdot)$, and let $Z$ denote the closed linear span of the range of $z$ in $X$. Then $Z$ is invariant under $S(\cdot)$ and $T(\cdot)$. Furthermore, if $T_1(\cdot)$ and $T_2(\cdot)$ are the linear parts of two affine semigroups with translation part $z(\cdot)$, then $T_1(t)x=T_2(t)x$ for all $t\geq 0$ and all $x$ in $Z$.*

PROOF. This follows from the identity $T(t)z(s)=z(t+s)-z(t)$ and the fact that $S(t)x=T(t)x+z(t)$. ■

PROPOSITION 3.3. *Let $z(\cdot)$ be a map from $[0, \infty)$ into $X$. Then necessary* *and sufficient conditions for* $z(\cdot)$ *to be the translation part of an affine semigroup on the space* $Z$ *defined in Proposition* 3.2 *are that* $z$ *be a continuous map,* $z(0)=0$, *and there exist a constant* $M \geq 1$ *such that*

$$\|\textstyle\sum_{1\leq j\leq n} \lambda_j(z(t_j+t)-z(t))\| \leq M\cdot\|\sum_{1\leq j\leq n}\lambda_j z(t_j)\| \tag{3.2}$$

*whenever* $t$ *is in* $(0, 1]$, $n$ *is a positive integer,* $\lambda_1,\dots,\lambda_n$ *are in* $R$, *and* $t_1,\dots,t_n$ *are nonnegative real numbers.*

PROOF. Let $z(\cdot)$ be the translation part of an affine semigroup on $Z$, and let $T(\cdot)$ be its linear part. By Proposition 1.1 $z(\cdot)$ is continuous. By (1.2) $z(0)=0$ and

$$\begin{aligned}\textstyle\sum_{1\leq j\leq n}\lambda_j(z(t_j+t)-z(t)) &= \textstyle\sum_{1\leq j\leq n}\lambda_j T(t)z(t_j)\\ &= T(t)(\textstyle\sum_{1\leq j\leq n}\lambda_j z(t_j)).\end{aligned}$$

(3.2) now follows from the fact that a strongly continuous linear semigroup $T(\cdot)$ satisfies $\sup\{\|T(t)\|: 0<t\leq 1\}\leq M$ for some constant $M\geq 1$.

Conversely, let $z(\cdot)$ be a continuous map satisfying (3.2). Let $Z_0=$ the linear span of the range of $z$. Define a family of maps $\{T_0(t): t\geq 0\}$ of $Z_0$ into $Z_0$ by

$$T_0(t)(\textstyle\sum_{1\leq j\leq n}\lambda_j z(t_j)) = \sum_{1\leq j\leq n}\lambda_j(z(t_j+t)-z(t)). \tag{3.3}$$

Suppose $\sum_{1\leq j\leq n}\lambda_j z(t_j)=0$. Then by (3.2)

$$\textstyle\sum_{1\leq j\leq n}\lambda_j(z(t_j+t)-z(t)) = 0 \tag{3.4}$$

for $t\leq 1$. If (3.4) holds for all $t$ in $[0, n]$, then for any fixed $t$ the linear combination in (3.4) can be substituted into the right side of (3.2). Since this combination is the zero vector, the vector on the left side of (3.2), with $t$ replaced by some $s$ in $[0, 1]$, is also the zero vector, that is, $0=\sum_{1\leq j\leq n}\lambda_j(z(t_j+t+s)-z(s)) +(\sum_{1\leq j\leq n}(-\lambda_j))(z(t+s)-z(s))=\sum_{1\leq j\leq n}\lambda_j(z(t_j+t+s)-z(t+s))$. Thus (3.4) holds for $t$ in $[0, n+1]$. By induction (3.4) holds for all $t\geq 0$.

This shows that $T_0(t)$ is a well-defined map from $Z_0$ into $Z_0$ for $t\geq 0$. Linearity then follows without difficulty. Moreover, $T_0(t)\circ T_0(s)(\sum_{1\leq j\leq n}\lambda_j z(t_j))= T_0(t)(\sum_{1\leq j\leq n}\lambda_j(z(t_j+s)-z(s))=\sum_{1\leq j\leq n}\lambda_j(z(t_j+s+t)-z(t)) - \sum_{1\leq j\leq n}\lambda_j(z(s+t) - z(t))=\sum_{1\leq j\leq n}\lambda_j(z(t_j+s+t)-z(s+t))=T_0(t+s)(\sum_{1\leq j\leq n}\lambda_j z(t_j))$. So $T_0(\cdot)$ has the semigroup property.

By (3.2) $\|T_0(t)x\|\leq M\|x\|$ for all $x$ in $Z_0$ and $t\leq 1$. Since $T_0(t)$ is a continuous linear map from a dense subset of $Z$ into itself, there is a unique continuous linear extension $T(t)$ from $Z$ into $Z$ with $\|T(t)\|\leq M$ for $t\leq 1$. If $t$ is in $(n, n+1]$ for some integer $n\geq 1$, $T_0(t)=T_0(t-n)\circ T_0(1)^n$. Hence for $t>1$, $T_0(t)$ has a unique continuous linear extension from $Z$ into $Z$, which we also call $T(t)$, and

$\|T(t)\leq M^{n+1}$ for $t\leq n+1$.

By uniqueness of the extension $T(\cdot)$ has the semigroup property. For $x$ in $Z$ we can find a sequence $\{x_n\}$ in $Z_0$ converging to $x$. The functions $t\mapsto T_0(t)x_n = T(t)x_n$ are continuous because of equation (3.3) and continuity of $z(\cdot)$, and these functions converge uniformly on compact subsets of $[0, \infty)$ to the function $t\mapsto T(t)x$. Hence the latter is continuous. So $T(\cdot)$ is a linear semigroup on $Z$.

Since $T(t)z(s) = T_0(t)z(s) = z(t+s) - z(t)$ by (3.3), we may apply Proposition 3.1 to get an affine semigroup $S(\cdot)$ on $Z$ with linear part $T(\cdot)$ and translation part $z(\cdot)$. ■

COROLLARY 3.4. *Let $H$ be a real Hilbert space and let $t\mapsto z(t)$ be a map from $[0, \infty)$ into $H$. Then necessary and sufficient conditions for $z(\cdot)$ to be the translation part of an affine semigroup of isometries on $H$ are that $z$ be a continuous map, $z(0)=0$, and there exist a function $g$: $[0, \infty)\to[0, \infty)$ such that*

$$\|z(t)-z(s)\| = g(|t-s|) \tag{3.5}$$

*for all $t$, $s\geq 0$.*

PROOF. Let $S(\cdot)$ be an affine semigroup of isometries on $H$ with translation part $z(\cdot)$. By Proposition 3.3 $z(\cdot)$ is continuous and $z(0)=0$. For $t\geq 0$, let $g(t) = \|z(t)\| = \|S(t)0\|$. If $t\geq s$, then $\|z(t)-z(s)\| = \|S(t)0 - S(s)0\| = \|S(s)\circ S(t-s)0 - S(s)0\| = \|S(t-s)0 - 0\| = g(t-s)$, and (3.5) is established.

The above proof (i.e., the necessity) is valid if $H$ is an arbitrary Banach space. The Hilbert space structure of $H$ is only needed for the sufficiency argument, which we now give.

Suppose $z(\cdot)$ is continuous, $z(0)=0$, and (3.5) holds. Then a condition stronger than (3.2) holds. Let $\langle\ ,\ \rangle$ denote the inner product in $H$.

$$\begin{aligned}(3.6)\qquad &\|\textstyle\sum_{1\leq j\leq n}\lambda_j(z(t_j+t)-z(t))\|^2\\ &= \textstyle\sum_{1\leq i,j\leq n}\lambda_i\lambda_j\langle z(t_i+t)-z(t),\ z(t_j+t)-z(t)\rangle\\ &= \textstyle\sum_{1\leq i,j\leq n}\lambda_i\lambda_j\langle z(t_i),\ z(t_j)\rangle\\ &= \|\textstyle\sum_{1\leq j\leq n}\lambda_j z(t_j)\|^2.\end{aligned}$$

Here the second equality is valid since $\langle z(t_i+t)-z(t),\ z(t_j+t)-z(t)\rangle = (1/2)\{\|z(t_i+t)-z(t)\|^2 + \|z(t_j+t)-z(t)\|^2 - \|z(t_i+t)-z(t_j+t)\|^2\} = (1/2)\{g(t_i)^2 + g(t_j)^2 - g(|t_i-t_j|)^2\} = (1/2)\{\|z(t_i)\|^2 + \|z(t_j)\|^2 - \|z(t_i)-z(t_j)\|^2\} = \langle z(t_i),\ z(t_j)\rangle$.

Since (3.6) is a special case of (3.2), there is an affine semigroup $S(\cdot)$ on $Z$ with translation part $z(\cdot)$. However, in the notation of the proof of Proposition 3.3, we may also conclude from (3.6) that $\|T_0(t)x\| = \|x\|$ for all $x$ in $Z_0$ and $t\geq 0$. By continuity in $t$, $\|T(t)x\| = \|x\|$ for all $x$ in $Z$ and $t\geq 0$. Hence, each $T(t)$ is a linear isometry of $Z$ into $Z$. So the affine semigroup $S(\cdot)$ consists of affine isometries. If $Z^{\perp}$ denotes the orthogonal complement of $Z$ in $H$, we extend $S(\cdot)$ to $H$ by defining $S'(t)(z+z')=S(t)z+z'$, for $t\geq 0$, $z$ in $Z$, and $z'$ in $Z^{\perp}$. It is routine to verify that $S'(\cdot)$ is an affine semigroup of isometries on $H$ with translation part $z(\cdot)$. ■

Corollary 3.4 is a variation of a result due to von Neumann and Schoenberg [9], given in a group setting rather than a semigroup setting. In the present context their problem might be posed as follows. Let $g$ be a continuous function from $[0, \infty)$ into $[0, \infty)$ with $g(0)=0$. Regard the half-line $[0, \infty)$ as a metric space $M$ under the metric $d(t, s)=|t-s|$. If we replace this metric by a new metric-like function $d_g(t, s)=g(d(t, s))$, the half-line is denoted by $g(M)$ and is called the *metric transform* of $M$. The problem is to determine those functions $g$ for which the metric transform $g(M)$ is isometrically embeddable in a real Hilbert space.

The functions $g$ are called *screw functions* and the embedded metric transforms might be called *screw half-lines*. By Corollary 3.4 screw functions are of the form $g(t)=\|z(t)\|$ and screw half-lines are subsets of $H$ of the form $\{x+z(t): t\geq 0\}$ where $x$ is any fixed element of $H$ and $z(\cdot)$ is the translation part of an affine semigroup of isometries of $H$. With the aid of (2.3), the screw functions and screw half-lines can be expressed in terms of linear semigroups of isometries.

If $\{S(t): t\geq 0\}$ is a strongly continuous semigroup of *surjective* isometries on a Banach space $X$, by a classical result of Mazur and Ulam [7] $S(t)$ is automatically affine for each $t\geq 0$. Then $S(\cdot)$ can be extended to an affine group on $X$, and the translation part of this affine group satisfies (3.5) for $t$, $s$ in $R$.

## § 4. Some classes of affine semigroups

In this section we state generation theorems for a few major classes of strongly continuous affine semigroups. We emphasize criteria that do not appeal to the corresponding linear objects, although the proofs may make such appeals. Here is an example.

PROPOSITION 4.1. *A map $A$ with domain and range in $X$ is the i.g. of a (strongly continuous) affine semigroup on $X$ if and only if*

i) *$D(A)$ is a dense subset of $X$, closed under affine combinations;*

ii) *$A(\lambda x+(1-\lambda)y)=\lambda Ax+(1-\lambda)Ay$ for $x$ and $y$ in $D(A)$;*

iii) *$G(A)=\{(x, Ax): x$ is in $D(A)\}$ is a closed subset of $X\times X$; and*

iv) *there exist constants $\omega\geq 0$ and $M\geq 0$ such that for all $\lambda>\omega$ and $n=1, 2,\dots$, the map $x\mapsto x-(Ax)/\lambda$ is one-to-one from $D(A)$ onto $X$ and $\|(I-A/\lambda)^{-n}\|_{\mathrm{Lip}}\leq M(1-\omega/\lambda)^{-n}$.*

PROOF. Suppose $A$ is an i.g. By Proposition 2.4 $A=\tau_w\circ U\circ\tau_{-v}$ for vectors

$v$ and $w$ in $X$ and a linear operator $U$ that is the i.g. of a linear semigroup. By the Generation Theorem for strongly continuous linear semigroups ([3], p. 20) $U$ is a linear operator, $D(U)$ is a dense linear subspace of $X$, $G(U)$ is closed in $X \times X$, and there exist constants $\omega$, $M \geq 0$ such that for all $\lambda > \omega$ and $n = 1, 2, \ldots$, the map $x \mapsto x - (Ux)/\lambda$ is one-to-one from $D(U)$ onto $X$ and $\|(I - U/\lambda)^{-n}\| \leq M(1-\omega/\lambda)^{-n}$. It follows that $A$ satisfies i), ii) and iii) with $D(A) = D(U) + v$.

Let $x = y - (Ay)/\lambda$ for $y$ in $D(A)$. Then

$$x = y - (\tau_w \circ U \circ \tau_{-v} y)/\lambda = y - (U(y-v) + w)/\lambda$$

and

$$x + w/\lambda - v = (I - U/\lambda)(y - v). \tag{4.1}$$

For $\lambda > \omega$ and $x$ in $X$ there is exactly one $y$ in $D(A)$ such that (4.1) holds. Applying the inverse $(I - U/\lambda)^{-1}$ to (4.1) and solving for $y$, we obtain:

$$(I - A/\lambda)^{-1} x = (I - U/\lambda)^{-1}(x - v + (w/\lambda)) + v, \tag{4.2}$$

and by an elementary induction

$$(I - A/\lambda)^{-n} x = (I - U/\lambda)^{-n}(x - v) + \textstyle\sum_{1 \leq j \leq n} (I - U/\lambda)^{-j}(w/\lambda) + v \tag{4.3}$$

for $n \geq 1$. Then $\|(I - A/\lambda)^{-n} x - (I - A/\lambda)^{-n} x'\| = \|(I - U/\lambda)^{-n}(x - x')\| \leq M(1 - \omega/\lambda)^{-n} \|x - x'\|$ for all $\lambda > \omega$ and $x$, $x'$ in $X$. Hence iv) is true.

Conversely, suppose that $A$ satisfies i) through iv). Let $(v, w)$ be an arbitrary member of $G(A)$. Then it is a simple exercise to verify, by reversing the above steps, that the operator $U$ defined by $U = \tau_{-w} \circ A \circ \tau_v$ satisfies the conditions of the Generation Theorem and is the i.g. of a linear semigroup. By Proposition 2.4, $A$ is the i.g. of an affine semigroup. ∎

To treat other classes we pause for definitions.

An *affine group* on the Banach space $X$ is a one-parameter family $\{G(t)\colon t \in R\}$ of continuous affine transformations on $X$ with the properties

($g_1$) $G(0) = I$ and $G(t+s) = G(t) \circ G(s)$ for $s$, $t$ in $R$; and

($g_2$) for each $x$ in $X$, $t \mapsto G(t)x$ is a continuous function from $R$ into $X$.

A *compact affine semigroup* (cf. [10], p. 48) is an affine semigroup $S(\cdot)$ with the property that the closure of $S(t)(B)$ is compact for any bounded set $B$ and any $t > 0$.

It is convenient (but not necessary) in the following instance to take the underlying Banach space to be complex. Let $C$ denote the set of complex numbers. An *analytic affine semigroup* (cf. [2], p. 80; [3], p. 33; [10], p. 60) is an affine semigroup $S(\cdot)$ with the property that for some positive number $\delta \leq \pi/2$ there exists a family $\{S(\lambda)\colon \lambda \in \Delta\}$ of continuous affine transformations on

$X$ with $\Delta=\{0\} \cup \{\lambda: \lambda \in C, \lambda \neq 0, \text{ and } |\arg \lambda| < \delta\}$ such that

($a_1$) the family $\{S(\lambda): \lambda \in \Delta\}$ extends the original semigroup and satisfies $S(\lambda_1+\lambda_2)=S(\lambda_1)\circ S(\lambda_2)$ for $\lambda_1$, $\lambda_2$ in $\Delta$;

($a_2$) for each $x$ in $X$, $\lambda \mapsto S(\lambda)x$ is a continuous function from $\Delta$ into $X$; and

($a_3$) for each $x$ in $X$ and $h$ in $X^*$, $\lambda \mapsto h(S(\lambda)x)$ is an analytic function from $\Delta \setminus \{0\}$ into $C$.

The semigroup property and the affine character of the family $\{S(\lambda): \lambda \in \Delta\}$ are consequences of the analyticity: if we assume that each $S(\lambda)$ is a continuous map from $X$ into $X$, coinciding with a member of the affine semigroup for $\lambda = t \geq 0$, then ($a_3$) guarantees that ($a_1$) holds and that $S(\lambda)$ is affine for each $\lambda$ in $\Delta$.

Each affine concept above has a linear counterpart whose definition is identical except for substitution of "linear" transformation for "affine" transformation.

The linear part and the translation part of one of the above affine families are defined as in (1.1), with $t \in R$ for the group case.

PROPOSITION 4.2.

i) *A family* $\{G(t): t \in R\}$ *of affine transformations is an affine group if and only if its linear part is a linear group and its translation part is a continuous map from* $R$ *into* $X$ *satisfying* (1.2) *for* $t$, $s$ *in* $R$;

ii) *an affine semigroup is compact if and only if its linear part is compact; and*

iii) *an affine semigroup is analytic if and only if its linear part is analytic.*

PROOF. The proof of i) is similar to that of Proposition 1.1, and we omit it.

The proof of ii) reduces to the observation that for any bounded set $B$ and any $t>0$, the closure of $S(t)(B)$ is just the translate of the closure of $T(t)(B)$ by the vector $z(t)$. If one closure is compact, so is the other.

If the affine semigroup $S(\cdot)$ is analytic with analytic extension $\{S(\lambda): \lambda \in \Delta\}$, then $T(\lambda)x = S(\lambda)x - S(\lambda)0$ for $x$ in $X$ and $\lambda$ in $\Delta$ defines a family $\{T(\lambda): \lambda \in \Delta\}$ of continuous linear transformations that is an analytic extension of the linear part $T(\cdot)$ of $S(\cdot)$. Conversely, if such an analytic extension exists for the linear part $T(\cdot)$ of an affine semigroup $S(\cdot)$, then equation (2.3) describing the translation part $z(\cdot)$ of $S(\cdot)$ can be used to define $z(\lambda)$ for $\lambda$ in $\Delta$ if $t$ is replaced by $\lambda$. Then we define $S(\lambda)$ for $\lambda$ in $\Delta$ by $S(\lambda)x = T(\lambda)x + z(\lambda)$ for $x$ in $X$. From (2.3) and the analyticity of $\{T(\lambda): \lambda \in \Delta\}$, the map $\lambda \mapsto z(\lambda)$ is continuous on $\Delta$ and $\lambda \mapsto h(z(\lambda))$ is analytic on $\Delta \setminus \{0\}$ for each fixed $h$ in $X^*$. It follows that $\{S(\lambda): \lambda \in \Delta\}$ is an analytic extension of $S(\cdot)$ as required. ■

The translation part of the extension of an analytic affine semigroup satisfies

$$z(\lambda_1+\lambda_2) = T(\lambda_1)z(\lambda_2) + z(\lambda_1) \qquad \text{for} \quad \lambda_1, \lambda_2 \quad \text{in} \quad \Delta.$$

The above-mentioned affine families may also be augmented in the manner of (1.3) to give linear families. It is easily seen that $\{G(t)\colon t \in R\}$ is an affine group if and only if its augmented version is a linear group, and that the affine semigroup $\{S(t)\colon t \geq 0\}$ is compact or analytic if and only if the associated augmented linear semigroup $\{\hat{T}(t)\colon t \geq 0\}$ is likewise. (In the analytic case, work in the Banach space $X \times C$.) We omit details.

We now give generation criteria for the various families. The i.g. of an affine (or linear) group is defined by an equation similar to (2.1) with "$h \to 0^+$" replaced by "$h \to 0$" so that the limit is two-sided.

PROPOSITION 4.3. *A map $A$ with domain and range in $X$ is the i.g. of an affine group on $X$ if and only if it satisfies conditions* i), ii), *and* iii) *of Proposition* 4.1 *and*

iv)′ *there exist constants $\omega \geq 0$ and $M \geq 0$ such that for all real $\lambda$ with $|\lambda| > \omega$ and $n = 1, 2, \ldots$, the map $x \mapsto x - (Ax)/\lambda$ is one-to-one from $D(A)$ onto $X$ and $\|(I - A/\lambda)^{-n}\|_{\mathrm{Lip}} \leq M(1 - \omega/|\lambda|)^{-n}$.*

PROPOSITION 4.4. *An affine semigroup $\{S(t)\colon t \geq 0\}$ is compact if and only if the family of functions $\{t \mapsto S(t)x\colon \|x\| \leq 1\}$ from* $(0, \infty)$ *into $X$ is equicontinuous on each compact subinterval of* $(0, \infty)$ *and the i.g. $A$ has the property that the closure of $(I - A/\lambda)^{-1}(B)$ is compact for each bounded set $B$ and some $\lambda > \omega$ with $\omega$ as in* (4.1iv).

PROPOSITION 4.5. *An affine semigroup $\{S(t)\colon t \geq 0\}$ on a complex Banach space $X$ is analytic if and only if its i.g. $A$ is such that there exist constants $C > 0$ and $\omega_1 > \omega$ (with $\omega$ as in* (4.1iv)) *such that the map $x \mapsto x - (Ax)/\lambda$ is one-to-one from $D(A)$ onto $X$ and $\|(I - A/\lambda)^{-1}\|_{\mathrm{Lip}} \leq C|\lambda|/|\mathrm{Im}\,\lambda|$ for all complex $\lambda$ with* $\mathrm{Re}\,\lambda > \omega_1$ *and* $\mathrm{Im}\,\lambda \neq 0$.

PROOF OUTLINES FOR 4.3, 4.4, AND 4.5. With regard to 4.3, the identity $A = \tau_w \circ U \circ \tau_{-v}$, the equations (4.2) and (4.3), and the Generation Theorem for linear groups ([3], p. 22) may be applied in the same manner as in Proposition 4.1.

As to 4.4, by the proof of Theorem 3.3 of [10], p. 49, necessary and sufficient conditions for the linear part $T(\cdot)$ of $S(\cdot)$ to be compact are that $t \mapsto T(t)$ be continuous in the uniform operator topology for $t > 0$ and that $(\lambda I - U)^{-1}$ be compact for some $\lambda > \omega$ where $U$ is the i.g. of $T(\cdot)$. It is a simple matter to verify that equicontinuity of the family $\{t \mapsto S(t)x\colon \|x\| \leq 1\}$ on compact subintervals of $(0, \infty)$ corresponds to continuity of $t \mapsto z(t)$ and to continuity in the uniform operator topology of $t \mapsto T(t)$ for $t > 0$. The condition on $U$ translates into a condition on $A$ in the manner of Proposition 4.1.

In the case of 4.5, we employ the analyticity criterion in part b) of Theorem 5.2 of [10], p. 61, with adjustments because the linear part $T(\cdot)$ may not be uniformly bounded and 0 may not be in the resolvent set of $U$, the i.g. of $T(\cdot)$. The condition for analyticity of $T(\cdot)$ is that $\|(\lambda I-U)^{-1}\| \leq C/|\mathrm{Im}\,\lambda|$ for $\mathrm{Im}\,\lambda \neq 0$, $\mathrm{Re}\,\lambda > \omega_1 > \omega$ (where $\omega$ is the constant appearing in (4.1iv), the criteria in (4.1iv) being valid for real or complex Banach spaces-cf. [11], p. 20). Proposition 4.2(iii) then permits us to proceed in the manner of Proposition 4.1, showing that the condition on $U$ corresponds to a similar one on $A$. ■

For other analyticity criteria see section 1.5 of [3], section 2.5 of [10], and [8].

## §5. Affine cosine functions

Now let us consider affine cosine functions (cf. Chapter 2 of [2], §2.8 of [3], and [4]).

An *affine cosine function* is a family $\{K(t): t \in R\}$ of continuous affine transformations on $X$ satisfying

($c_1$) $K(0)=I$, $K(t+s)+K(t-s)=2K(t)\circ K(s)$ for $t$, $s$ in $R$; and

($c_2$) for each $x$ in $X$, $t \mapsto K(t)x$ is a continuous function from $R$ into $X$.

A *linear cosine function* is defined similarly.

PROPOSITION 5.1. *Let $\{K(t): t \in R\}$ be a family of affine transformations with linear part $\{C(t): t \in R\}$ and translation part $\{z(t): t \in R\}$ as in* (1.1), *and let $\{\hat{C}(t): t \in R\}$ be the associated augumented linear family as in* (1.3). *Then the following statements are equivalent*:

i) *$K(\cdot)$ is an affine cosine function*;

ii) *$C(\cdot)$ is a linear cosine function and $z(\cdot)$ is a continuous function from $R$ into $X$ satisfying*

$$z(t+s) + z(t-s) = 2C(t)z(s) + 2z(t) \quad \textit{for} \quad t, s \textit{ in } R; \tag{5.1}$$

iii) *$\hat{C}(\cdot)$ is a linear cosine function.*

The proof of this result is straightforward, and is left to the reader.

(5.1) implies that $z(0)=0$ and $z(-t)=z(t)$ for all $t$.

By the *infinitesimal generator*, or i.g., of an affine cosine function $K(\cdot)$, we mean the map $A$ defined by

$$Ax = \lim_{h \to 0} (2/h^2)(K(h)x - x). \tag{5.2}$$

The domain of $A$, $D(A)$, is the set of all $x$ for which the limit in (5.2) exists. The i.g. of a linear cosine function is defined similarly.

With the aid of the known properties of linear cosine functions, in the manner of §2 above we can establish the following properties for an affine cosine function $K(\cdot)$:

i) the i.g. of $K(\cdot)$, $C(\cdot)$, $\hat{C}(\cdot)$, denoted by $A$, $U$, and $\hat{U}$ respectively, satisfy i), ii), and iii) of Proposition 2.1;

ii) $(2/t^2)\int_0^t (t-s)K(s)xds$ is in $D(A)$ for $t\neq 0$ and $x$ in $X$, and its image under $A$ is $(2/t^2)(K(t)x-x)$;

iii) $A=\tau_w \circ U \circ \tau_{-v}$ where $v$ and $w$ are arbitrary vectors satisfying $Av=w$;

iv) the translation part $z(\cdot)$ of $K(\cdot)$ is given by

$$(5.3)\qquad \begin{aligned} z(t) &= v - C(t)v + \int_0^t (t-s)C(s)wds \\ &= v - C(t)v + \int_0^t S(s)wds \end{aligned}$$

for $t$ in $R$, where $v$ and $w$ are as in (iii) and $S(\cdot)$ is the linear sine function associated with $C(\cdot)$;

v) every affine cosine function with linear part $C(\cdot)$ is obtained from a translation part of the form (5.3) where $(v, w)$ is an arbitrary vector in $X\times X$, any two such pairs in the same equivalence class mod $G(U)$ giving the same function $z(\cdot)$; and

vi) there exists a constant $\omega>0$ such that for $\lambda>\omega$ the maps $x\mapsto x-(Ax)/\lambda^2$ and $x\mapsto x-(Ux)/\lambda^2$ are one-to-one from $D(A)$ resp. $D(U)$ onto $X$, with inverses infinitely differentiable as functions of $\lambda$ satisfying for $m=1, 2,\ldots$

$$(5.4)\qquad \|d^m/d\lambda^m(\lambda^{-1}(I-A/\lambda^2)^{-1})\|_{\mathrm{Lip}} = \|d^m/d\lambda^m(\lambda^{-1}(I-U/\lambda^2)^{-1})\|\,.$$

Using (5.4), we can prove a generation theorem for affine cosine functions like 8.3 in [3], p. 119, but we omit the statement of this theorem.

As is well known (cf. [3], p. 120, for example), given a linear cosine function $C(\cdot)$, the formula

$$(5.5)\qquad T(t)x = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} \exp(-s^2/4t)C(s)xds,$$

for $t>0$, defines a linear semigroup $T(\cdot)$ (which is in fact an analytic semigroup). Moreover, the infinitesimal generator of $C(\cdot)$ is the i.g. of $T(\cdot)$, i.e., $C''(0)=T'(0)=U$. This is also valid in the affine case.

Proposition 5.2. *Let $\{K(t)\colon t\in R\}$ be an affine cosine function with i.g. $A$. Then the family $\{S(t)\colon t\geq 0\}$ of functions on $X$ defined by*

$$(5.6)\qquad \begin{cases} S(t)x = (4\pi t)^{-1/2}\int_{-\infty}^{\infty} \exp(-s^2/4t)K(s)xds & (t>0) \\ S(0)x = x \end{cases}$$

*for $x$ in $X$ is an affine semigroup with i.g. $A$.*

PROOF. The semigroup property of $T(\cdot)$ defined in (5.5) leads to the calculus identity

$$(5.7)\qquad (4\pi t)^{-1/2}(4\pi\tau)^{-1/2}\int_{-\infty}^{\infty}\exp[-(s+u)^2/16t]\exp[-(s-u)^2/16\tau]du$$
$$= (4\pi(t+\tau))^{-1/2}\exp[-s^2/4(t+\tau)].$$

Given an affine cosine function $K(\cdot)$, it is a standard result (see Theorem 3.1 of [2], p. 33) that the associated augmented linear cosine function $\hat{C}(\cdot)$ satisfies $\|\hat{C}(t)\| \leq Me^{\omega|t|}$ for suitable positive constants $M$ and $\omega$ and all $t$ in $R$. Hence, as in (1.5), $\|A(t)x\| = \|\pi_1\circ\hat{C}(t)[x, 1]\| \leq Me^{\omega|t|}(\|x\|+1)$. This bound shows that the integral in (5.6) is finite and $S(\cdot)$ is well-defined.

Then for $x$ in $X$ and $t>0$, $\tau>0$,

$$S(t)\circ S(\tau)x = (4\pi t)^{-1/2}(4\pi\tau)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\exp(-s^2/4t)\exp(-u^2/4\tau)$$
$$K(s)K(u)x\}duds$$
$$= (4\pi t)^{-1/2}(4\pi\tau)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{\exp(-s^2/4t)\exp(-u^2/4\tau)$$
$$(1/2)(K(s+u)x+K(s-u)x)\}duds$$
$$= (4\pi(t+\tau))^{-1/2}\int_{-\infty}^{\infty}\exp[-s^2/4(t+\tau)]K(s)xds = S(t+\tau)x.$$

Here we have used the cosine function identity ($c_1$), a change of variables, and the identity (5.7).

Additional identities from calculus may be used to establish that $t\mapsto S(t)x$ is strongly continuous at $t=0^+$ for all $x$ and that $A'(x)=A(x)$ for all $x$ in $D(A)$ where $A'$ is the i.g. of $S(\cdot)$. By the text adjacent to (5.4) and by Proposition 4.1, we know that the maps $x\mapsto x-(Ax)/\lambda$ and $x\mapsto x-(A'x)/\lambda$ are one-to-one and onto $X$ for $\lambda$ sufficiently large. If $y=x-(A'x)/\lambda$ for some $x$ in $D(A')$ and some $\lambda$ as above, then $y=x_1-(Ax_1)/\lambda$ for some $x_1$ in $D(A)$. Since $A'$ extends $A$, one-to-oneness implies that $x_1=x$ and $Ax=A'x$. Hence, $A=A'$. ■

If we denote the translation parts of $S(\cdot)$ and $K(\cdot)$ in Proposition 5.2 by $w(\cdot)$ and $z(\cdot)$ and set $x=0$ in (5.6), we get:

$$w(t) = (4\pi t)^{-1/2}\int_{-\infty}^{\infty}\exp(-s^2/4t)z(s)ds \quad (t>0).$$

Since $K(\cdot)$ is an even function, (5.6) may also be rewritten for $t>0$ as

$$S(t)x = (\pi t)^{-1/2}\int_0^{\infty}\exp(-s^2/4t)K(s)xds$$
$$= (\pi t)^{-1/2}\mathscr{L}(f)(1/4t)$$

where $\mathscr{L}$ denotes the Laplace transform and $f$ is the function given by $f(r)=r^{-1/2}K(r^{1/2})x$ for $r>0$. Thus there is a one-to-one correspondence between the functions $K(\cdot)x$ and $S(\cdot)x$ for $x$ in $X$.

The affine semigroup $S(\cdot)$ obtained in Proposition 5.2 is also analytic (assume the Banach space is complex). Its linear part is analytic by Theorem 8.7 of [3], p. 120, since it can be defined by (5.5) with $C(\cdot)$ the linear part of $K(\cdot)$, and then Proposition 4.2(iii) is applicable.

Under certain circumstances Kisyński [6] has shown how a linear cosine function $C(\cdot)$ gives rise to a linear group $T(\cdot)$ on a suitable "energy norm" product space $X_{en}\times X$ where $X_{en}$ is a Banach space intermediate between $X$ and $D(U)$ and $U$ is the i.g. of $C(\cdot)$. The methods of this note then indicate how affine groups might be constructed on this same product space with linear part $T(\cdot)$. The translation parts of such affine groups have formulas similar to (5.3) with $(v, w)$ in $X_{en}\times X$.

See [4], [6], and [11] for further discussion of these product spaces.

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