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Lie algebras whose inner derivations satisfy certain conditions

Yûji Shimizuike and Shigeaki Tôgô

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Introduction

For a finite-dimensional Lie algebra I. M. Singer ([6]) introduced the condition (A) (§1, Definition 1), A. Jôichi ([4]) introduced the conditions (A_k) and (A_∞) (§1, Definition 2), and the properties of finite-dimensional Lie algebras satisfying these conditions had been investigated by several authors in [4, 6, 8, 9].

For a not necessarily finite-dimensional Lie algebra, we shall define the conditions (A), (A_k) and (A_∞) in the same manner and moreover introduce the condition (B_∞) strengthening the condition (A_∞). The purpose of this paper is mainly to extend the known results on finite-dimensional Lie algebras satisfying these conditions to not necessarily finite-dimensional Lie algebras.

In Section 1, let L be a not necessarily finite-dimensional Lie algebra over a field and let H be an ideal of L. We show that if L satisfies (A_{k+1}) (resp. (A_∞), (B_∞)) then H satisfies (A_k) (resp. (A_∞), (B_∞)) (Proposition 2). More generally we shall give similar results in case that H is a weakly ascendant subalgebra of L (Propositions 5 and 7).

In Section 2, for a Lie algebra L belonging to $\mathrm{L}\mathfrak{N}$ (resp. $\mathfrak{N}_k$) we show that the conditions (A), (B_∞) and "abelian" (resp. (A), (B_∞), (A_∞), $\cdots$, (A_{k+1}), (A_k) and "abelian") are equivalent (Theorem 8). For a Lie algebra L belonging to $\mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ over a field of characteristic 0, we show that the conditions (A), (B_∞) and "abelian" are equivalent (Theorem 9).

In Section 3, for a Lie algebra L belonging to $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ over a field of characteristic 0, we show that L satisfies (A) (resp. (B_∞)) if and only if L is the direct sum of the center and a semisimple ideal S of L satisfying (A) (resp. (B_∞)) (Theorem 11) and that the conditions (A) and (B_∞) are equivalent (Theorem 12). Finally for a Lie algebra L belonging to $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ over an algebraically closed field of characteristic 0, we show that the conditions (A), (B_∞) and "abelian" are equivalent (Theorem 13).

§1.

Throughout this paper Φ is a field of arbitrary characteristic and all Lie algebras are not necessarily finite-dimensional over a field Φ unless otherwise specified.

For a Lie algebra L, by $H \leq L$ (resp. $H \lhd L$) we mean that H is a subalgebra (resp. an ideal) of L. We denote by $\mathfrak{A}$ (resp. $\mathfrak{F}$, $\mathfrak{N}$, $\mathfrak{N}_k$, $\mathrm{E}\mathfrak{A}$) the class of Lie algebras which are abelian (resp. finite-dimensional, nilpotent, nilpotent of class $\leq k$, soluble). For a class $\mathfrak{X}$ of Lie algebras we denote by $\mathrm{L}\mathfrak{X}$ the class of locally $\mathfrak{X}$-algebras.

Let D be a derivation of L and let k be an integer ≥ 2. Then D is called k-nilpotent if $LD^k = 0$ and nil if for each finite-dimensional subspace V of L there exists a positive integer $n = n(V)$ such that $VD^n = 0$.

Now for a finite-dimensional Lie algebra L, the condition (A) was introduced by I. M. Singer ([6]) and the conditions (A_k) and (A_∞) were introduced by A. Jôichi ([4]) as follows.

DEFINITION 1. L is said to satisfy the condition (A) if any pair of elements x, y of L such that $[x, {}_2y] = 0$ satisfies $[x, y] = 0$.

DEFINITION 2. Let k be an integer ≥ 2. L is said to satisfy the condition (A_k) if ad L contains no non-zero k-nilpotent elements and L is said to satisfy the condition (A_∞) if ad L contains no non-zero nilpotent elements.

For a not necessarily finite-dimensional Lie algebra L we define the conditions (A), (A_k) and (A_∞) in the same manner as above. Moreover we introduce the following condition.

DEFINITION 3. We say that L satisfies the condition (B_∞), if ad L contains no non-zero nil elements.

From now on we use the same notation (A) (resp. (A_k), (A_∞), (B_∞)) to express the class of Lie algebras satisfying the condition (A) (resp. (A_k), (A_∞), (B_∞)). For unexplained terminology and notation we refer to [1, 13].

As in [4, Proposition 1], we show

PROPOSITION 1. *Let L be a Lie algebra over a field Φ. Then we have the following implications for L:*

$$(\mathrm{A}) \Rightarrow (\mathrm{B}_\infty) \Rightarrow (\mathrm{A}_\infty) \Rightarrow \cdots \Rightarrow (\mathrm{A}_{k+1}) \Rightarrow (\mathrm{A}_k) \Rightarrow \cdots \Rightarrow (\mathrm{A}_2).$$

Moreover we have $(\mathrm{A}_\infty) = \bigcap_{k \geq 2} (\mathrm{A}_k)$.

PROOF. We only show the implication $(\mathrm{A}) \Rightarrow (\mathrm{B}_\infty)$. Assume that $L \in (\mathrm{A})$ and let $\mathrm{ad}_L\, x$ be a nil element of ad L. Then for any $y \in L$, there exists an integer $k = k(y)$ $(k \geq 2)$ such that $(y)(\mathrm{ad}_L\, x)^k = 0$. Because of $L \in (\mathrm{A})$, $[y, {}_{k-1}x] = 0$. After repeating this procedure $k-2$ times, we have $[y, x] = 0$. Since y is arbitrary, we have $\mathrm{ad}_L\, x = 0$. Therefore $L \in (\mathrm{B}_\infty)$.

EXAMPLES. Let L_0 be the Lie algebra over a field Φ described in terms of a basis x, y, z by the table

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y,$$

and let L be the direct sum of a non-empty set of Lie algebras which are isomorphic to L_0. Then the following statements hold.

(1) In case $\Phi = \boldsymbol{R}$, L belongs to $(\mathrm{A}) \setminus \mathfrak{A}$.

(2) In case $\Phi = \boldsymbol{C}$, L belongs to $(\mathrm{A}_2) \setminus (\mathrm{A}_3)$.

L. A. Simonjan ([5]) and T. Ikeda ([3]) constructed examples of the countable-dimensional Lie algebra M over a field Φ which is non-abelian, locally nilpotent and has no non-zero bounded left Engel elements. Evidently

(3) M belongs to $(\mathrm{A}_\infty) \setminus (\mathrm{B}_\infty)$.

Denoting the center of L by $\zeta(L)$, we have

PROPOSITION 2. *Let L be a Lie algebra over a field Φ and let $H \lhd L$. Then the following statements hold.*

(1) *If $L \in (\mathrm{A}_{k+1})$, then $H \in (\mathrm{A}_k)$. Furthermore if $H \subseteq \zeta(L)$, then $L/H \in (\mathrm{A}_k)$.* $(k = 2, 3, 4, \cdots)$

(2) *If $L \in (\mathrm{A}_\infty)$, then $H \in (\mathrm{A}_\infty)$.*

(3) *If $L \in (\mathrm{B}_\infty)$, then $H \in (\mathrm{B}_\infty)$.*

PROOF. (1) Assume that $L \in (\mathrm{A}_{k+1})$. Let $\mathrm{ad}_H x$ be a k-nilpotent element of $\mathrm{ad}\, H$. Because of $H \lhd L$, $L(\mathrm{ad}_L x)^{k+1} \subseteq H(\mathrm{ad}_H x)^k = 0$ and therefore $\mathrm{ad}_L x$ is $(k+1)$-nilpotent. By assumption we have $\mathrm{ad}_L x = 0$. Then $\mathrm{ad}_H x = 0$ and $H \in (\mathrm{A}_k)$. Furthermore assume that $H \subseteq \zeta(L)$. Let $\bar{x}$ be the element of $\bar{L} = L/H$ corresponding to $x \in L$. Now let $\mathrm{ad}_{\bar{L}} \bar{x}$ be k-nilpotent. Then $L(\mathrm{ad}_L x)^k \subseteq H$. Since $H \subseteq \zeta(L)$, $(\mathrm{ad}_L x)^{k+1} = 0$ and therefore $\mathrm{ad}_{\bar{L}} x = 0$. This implies that $\bar{L} \in (A_k)$.

(2) We omit the proof.

(3) Assume that $L \in (\mathrm{B}_\infty)$ and let $\mathrm{ad}_H x$ be a nil element of $\mathrm{ad}\, H$. Since $H \lhd L$, for each finite-dimensional subspace V of L $[V, x]$ is a finite-dimensional subspace of H. By assumption there exists an integer $k = k(V, x)$ such that $[V, x](\mathrm{ad}_H x)^k = 0$. Therefore $V(\mathrm{ad}_L x)^{k+1} = 0$. Thus $\mathrm{ad}_L x = 0$. It follows that $\mathrm{ad}_H x = 0$ and $H \in (\mathrm{B}_\infty)$.

The following proposition clearly holds.

PROPOSITION 3. *Let L be a direct sum of ideals L_λ $(\lambda \in \Lambda)$. Then $L \in (\mathrm{A})$ (resp. (A_k), (A_∞), (B_∞)) if and only if $L_\lambda \in (\mathrm{A})$ (resp. (A_k), (A_∞), (B_∞)) for all $\lambda \in \Lambda$.*

We shall here discuss the statement of Proposition 2 under a weaker assumption instead of the assumption $H \lhd L$.

DEFINITION 4 ([10]). Let L be a Lie algebra over a field Φ and $H \leq L$. For an ordinal λ, H is said to be a λ-step weakly ascendant subalgebra of L, provided there exists an ascending chain $\{M_\alpha \mid \alpha \leq \lambda\}$ of subspaces of L such that

(1) $M_0 = H$ and $M_\lambda = L$,

(2) $[M_{\alpha+1}, H] \subseteq M_\alpha$ for any ordinal $\alpha < \lambda$,

(3) $M_\beta = \bigcup_{\alpha<\beta} M_\alpha$ for any limit ordinal $\beta \leq \lambda$.

We then write $H \leq^\lambda L$. H is said to be a weakly ascendant subalgebra of L if $H \leq^\lambda L$ for some ordinal λ. We then write H wasc L. Especially H is said to be a weak subideal of L if $\lambda = n < \omega$.

LEMMA 4. *Let L be a Lie algebra over a field Φ, let H* wasc *L and let x be an element of H. If $\mathrm{ad}_H x$ is nil, then so is $\mathrm{ad}_L x$.*

PROOF. Let V be any finite-dimensional subspace of L. By [2, Lemma 2.1], there exists an integer $n = n(V, x)$ such that $[V, {}_n x] \subseteq H$. Then $[V, {}_n x]$ is a finite-dimensional subspace of H. Since $\mathrm{ad}_H x$ is nil, there exists an integer $k = k([V, {}_n x])$ such that $[V, {}_n x](\mathrm{ad}_H x)^k = 0$. Therefore $V(\mathrm{ad}_L x)^{n+k} = 0$. Thus $\mathrm{ad}_L x$ is nil.

PROPOSITION 5. *Let L be a Lie algebra over a field Φ and let H* wasc *L. If $L \in (\mathrm{B}_\infty)$, then $H \in (\mathrm{B}_\infty)$.*

PROOF. Assume that $L \in (\mathrm{B}_\infty)$ and let $\mathrm{ad}_H x$ be a nil element of $\mathrm{ad}\, H$. By Lemma 4 $\mathrm{ad}_L x$ is also nil and therefore $\mathrm{ad}_L x = 0$. In particular $\mathrm{ad}_H x = 0$ and $H \in (\mathrm{B}_\infty)$.

LEMMA 6. *Let L be a Lie algebra over a field Φ. Let $H \leq^n L$ and let x be an element of H. If $\mathrm{ad}_H x$ is m-nilpotent, then $\mathrm{ad}_L x$ is $(m+n)$-nilpotent.*

PROOF. Since $H \leq^n L$, $L(\mathrm{ad}_L x)^n \subseteq H$. By assumption $H(\mathrm{ad}_H x)^m = 0$ and hence $L(\mathrm{ad}_L x)^{m+n} = 0$. Therefore $\mathrm{ad}_L x$ is $(m+n)$-nilpotent.

As a consequence of Proposition 5 and Lemma 6, we have

PROPOSITION 7. *Let L be a Lie algebra over a field Φ and let $H \leq^n L$. Then the following statements hold.*

(1) *If $L \in (\mathrm{A}_{k+n})$, then $H \in (\mathrm{A}_k)$ $(k = 2, 3, 4, \cdots)$.*

(2) *If $L \in (\mathrm{A}_\infty)$, then $H \in (\mathrm{A}_\infty)$.*

(3) *If $L \in (\mathrm{B}_\infty)$, then $H \in (\mathrm{B}_\infty)$.*

§2.

In this section, we study the relationship between the conditions (A), (A_k), (A_∞), (B_∞) and "abelian" in case of locally nilpotent (resp. locally soluble-and-finite) Lie algebras. First we extend a result of A. Jôichi and show the following

THEOREM 8. *Let L be a Lie algebra over a field Φ. Then the following statements hold.*

(1) *If $L \in \mathrm{L}\mathfrak{N}$, then the conditions* (A), (B_∞) *and "abelian" are equivalent for L.*

(2) *If $L \in \mathfrak{N}_k$, then the conditions* (A), (B_∞), $(\mathrm{A}_\infty), \cdots, (\mathrm{A}_{k+1})$, (A_k) *and "abelian" are equivalent for L ($k=2, 3, 4, \cdots$).*

PROOF. (1) It is clear that if $L \in \mathfrak{A}$, then $L \in (\mathrm{A})$. By Proposition 1, if $L \in (\mathrm{A})$, then we have $L \in (\mathrm{B}_\infty)$. Now assume that $L \in \mathrm{L}\mathfrak{N} \cap (\mathrm{B}_\infty)$. For any element x of L and for any finite-dimensional subspace V of L, there exists a subalgebra H of L such that $\{x\} \cup V \subseteq H \in \mathfrak{N} \cap \mathfrak{F}$. Then there exists an integer $n>0$ such that $H^{n+1}=0$. Thus $V(\mathrm{ad}_L x)^n=0$ and $\mathrm{ad}_L x$ is nil. Therefore $\mathrm{ad}_L x=0$. Since x is arbitrary, we have $L \in \mathfrak{A}$.

(2) By Proposition 1, it suffices to prove that if $L \in \mathfrak{N}_k \cap (A_k)$ then $L \in \mathfrak{A}$. Assume that $L \in \mathfrak{N}_k \cap (A_k)$. For any element x of L, $(\mathrm{ad}_L x)^k=0$. That is, $\mathrm{ad}_L x$ is k-nilpotent and by assumption $\mathrm{ad}_L x=0$. Since x is arbitrary, we have $L \in \mathfrak{A}$.

Let L be a Lie algebra over a field Φ. Then by $\rho(L)$ and $\sigma(L)$, we denote the Hirsch-Protkin radical (that is, $\mathrm{L}\mathfrak{N}$-radical) and $\mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$-radical of L respectively.

A Lie algebra L over a field Φ is said to be ideally finite, if any finite subset of L is contained in a finite-dimensional ideal of L. We denote by $\mathrm{L}(\lhd)\mathfrak{F}$ the class of ideally finite Lie algebras ([7]).

THEOREM 9. *Let L be a Lie algebra over a field Φ of characteristic* 0 *and assume that $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$. Then the following statements hold.*

(1) *The conditions* (A), (B_∞) *and "abelian" are equivalent for L.*

(2) *If $\rho(L) \in \mathfrak{N}_k$, then the conditions* (A), (B_∞), $(\mathrm{A}_\infty), \cdots, (\mathrm{A}_{k+2})$, (A_{k+1}) *and "abelian" are equivalent for L ($k=1, 2, 3, \cdots$).*

(3) *In particular if $L \in \mathrm{L}(\lhd)\mathfrak{F}$, then the conditions* (A), (B_∞), (A_∞) *and "abelian" are equivalent for L.*

PROOF. (1) It suffices to prove that if $L \in (\mathrm{B}_\infty)$ then $L \in \mathfrak{A}$. Assume that $L \in (\mathrm{B}_\infty)$. Since $\rho(L) \lhd L$, $\rho(L) \in (\mathrm{B}_\infty)$ by Proposition 2(3) and therefore $\rho(L) \in \mathfrak{A}$ by Theorem 8(1). On the other hand, because of $L \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ we have $[L, L] \subseteq \rho(L)$ by [1, Corollary 13.3.13] and [13, Corollary 8.3.5]. For any $y \in L$ and for any $z \in \rho(L)$ we have $[y, z, z] \in [\rho(L), \rho(L)]=0$. That is, $y(\mathrm{ad}_L z)^2=0$. Since y is arbitrary, $(\mathrm{ad}_L z)^2=0$. Therefore $\mathrm{ad}_L z=0$ because of $L \in (\mathrm{B}_\infty)$. It follows that $[L, \rho(L)]=0$. Now for any $x, y \in L$, we have $[y, x, x] \in [\rho(L), L]=0$ and therefore $y(\mathrm{ad}_L x)^2=0$. Since y is arbitrary, $(\mathrm{ad}_L x)^2=0$. Then $\mathrm{ad}_L x=0$

because of $L\in(\mathrm{B}_\infty)$. Therefore $L\in\mathfrak{A}$.

(2) It suffices to prove that if $L\in(\mathrm{A}_{k+1})$ then $L\in\mathfrak{A}$. Assume that $L\in(\mathrm{A}_{k+1})$. For $k=1$, $\rho(L)\in\mathfrak{N}_1$ by assumption and therefore $\rho(L)\in\mathfrak{A}$. For $k\geq 2$, since $\rho(L)\lhd L$, $\rho(L)\in(\mathrm{A}_k)$ by Proposition 2(1). It follows from Theorem 8(2) that $\rho(L)\in\mathfrak{A}$. Arguing as in the proof of (1), we have $[L, \rho(L)]=0$ and conclude that $L\in\mathfrak{A}$.

(3) Assume that $L\in \mathrm{L}(\lhd)\mathfrak{F}\cap(\mathrm{A}_\infty)$. For any $x\in\rho(L)$, it follows from [12, Lemma 7.3] that $\mathrm{ad}_L x$ is nilpotent. By assumption we have $\mathrm{ad}_L x=0$ and therefore $[L, \rho(L)]=0$. Now as in the proof of (1), we have $L\in\mathfrak{A}$.

§3.

In this section, we investigate the case of serially finite Lie algebras.

DEFINITION 5 ([1. §13.2]). Let L be a Lie algebra over a field Φ and let $H\leq L$. For a totally ordered set Σ, H is said to be a serial subalgebra of type Σ of L, provided there exists a collection $\{\Lambda_\sigma, V_\sigma \mid \sigma\in\Sigma\}$ of subalgebras of L such that

(1) $H\subseteq\Lambda_\sigma$ and $H\subseteq V_\sigma$ for all $\sigma\in\Sigma$,

(2) $\Lambda_\tau\subseteq V_\sigma\subseteq\Lambda_\sigma$ if $\tau<\sigma$,

(3) $L\setminus H=\bigcup_{\sigma\in\Sigma}(\Lambda_\sigma\setminus V_\sigma)$,

(4) $V_\sigma\lhd\Lambda_\sigma$ for all $\sigma\in\Sigma$.

We then write $H\,\mathrm{ser}\,L$. L is said to be serially finite, if any finite subset of L is contained in a finite-dimensional serial subalgebra of L. We denote by $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ the class of serially finite Lie algebras.

A locally finite Lie algebra L is said to be semisimple if $\sigma(L)=0$.

Now we quote the following two results.

THEOREM A ([1, Theorem 13.4.2] and [12, (1.3)]). *Let L be a Lie algebra belonging to $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ over a field Φ of characteristic* 0. *Then L is semisimple if and only if L is a direct sum of finite-dimensional non-abelian simple ideals.*

THEOREM B ([1, Theorem 13.5.7] and [12, (1.5)]). *Let L be a Lie algebra belonging to $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ over a field Φ of characteristic* 0. *Then there exists a semisimple subalgebra S of L such that $L=\sigma(L)+S$ and $\sigma(L)\cap S=0$.*

First we extend a result of M. Sugiura and show the following

PROPOSITION 10. *Let L be a semisimple Lie algebra belonging to $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ over a field Φ of characteristic* 0. *Then the conditions* (A), (B_∞) *and* (A_∞) *are equivalent for L.*

PROOF. By Theorem A, $L=\bigoplus_\lambda S_\lambda$, where each S_λ is a finite-dimensional non-abelian simple ideal of L. By [8, Theorem 1], the conditions (A) and (A_∞) are equivalent for each S_λ and therefore so are the conditions (A), (B_∞) and (A_∞). Now the assertion follows from Proposition 3.

THEOREM 11. *Let L be a Lie algebra belonging to* $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ *over a field* Φ *of characteristic* 0. *Then the following statements hold.*

(1) *$L \in$ (A) (resp. (B_∞)) if and only if $L = \zeta(L) \oplus S$ where S is a semisimple ideal of L belonging to* (A) *(resp.* (B_∞)*).*

(2) *Let $\rho(L) \in \mathfrak{N}_k$. Then $L \in (A_{k+2})$ if and only if $L = \zeta(L) \oplus S$ where S is a semisimple ideal of L belonging to* (A_{k+2}) $(k=0, 1, 2, \cdots)$.

(3) *In particular if $L \in \mathrm{L}(\lhd)\mathfrak{F}$, then the statement* (1) *holds for the condition* (A_∞).

PROOF. Put $R = \sigma(L)$.

(1) We only prove the statement on (B_∞). Assume that $L \in (B_\infty)$. Since $R \lhd L$, we have $R \in (B_\infty)$. Since $R \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$, $R \in \mathfrak{A}$ by Theorem 9 (1). For any $x \in L$ and for any $y \in R$, we have $[x, y, y] \in [R, R] = 0$. Then $x(\mathrm{ad}_L\, y)^2 = 0$. Since x is arbitrary, $(\mathrm{ad}_L\, y)^2 = 0$. Because of $L \in (B_\infty)$, $\mathrm{ad}_L\, y = 0$. This implies that $[L, R] = 0$ and therefore $R = \zeta(L)$. By Theorem B, there exists a semisimple subalgebra S of L such that $L = R + S$ and $R \cap S = 0$. It follows that $S \lhd L$. That is, S is a semisimple ideal of L and $L = R \oplus S$. Because of $L \in (B_\infty)$, we have $S \in (B_\infty)$ by Proposition 3. The converse is clear.

(2) Assume that $L \in (A_{k+2})$. For $k=0$ $[R, R] \subseteq \rho(L) = 0$, that is, $R \in \mathfrak{A}$. For $k \geq 1$ $R \in (A_{k+1})$ by Proposition 2 and since $\rho(R) = \rho(L)$ by [13, Proposition 8.3.3], $R \in \mathfrak{A}$ by Theorem 9(2). Now, the rest of the proof is similar to that of (1).

(3) When $L \in \mathrm{L}(\lhd)\mathfrak{F}$, assume that $L \in (A_\infty)$. Since $R \lhd L$, we have $R \in (A_\infty)$. Since $R \in \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F}) \cap \mathrm{L}(\lhd)\mathfrak{F}$, $R \in \mathfrak{A}$ by Theorem 9(3). Now, as in the proof of (1), we have the assertion of (3).

THEOREM 12. *Let L be a Lie algebra belonging to* $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ *over a field* Φ *of characteristic* 0. *Then the conditions* (A) *and* (B_∞) *are equivalent for L. In particular if $L \in \mathrm{L}(\lhd)\mathfrak{F}$, then the conditions* (A), (B_∞) *and* (A_∞) *are equivalent for L.*

PROOF. Assume that $L \in (B_\infty)$. By Theorem 11(1), there exists a semisimple ideal S of L such that $L = \zeta(L) \oplus S$ and $S \in (B_\infty)$. By Proposition 10 $S \in$ (A). Since $\zeta(L) \in \mathfrak{A} \subseteq$ (A), by Proposition 3 it follows that $L \in$ (A). The converse is evident. In case that $L \in \mathrm{L}(\lhd)\mathfrak{F}$, the assertion follows from Proposition 10 and Theorem 11(3).

Finally we show the following

THEOREM 13. *Let L be a Lie algebra belonging to* $\mathrm{L}(\mathrm{ser})\mathfrak{F}$ *over an alge-*

braically closed field Φ of characteristic 0. *Then the following statements hold.*

(1) *The conditions* (A), (B_∞) *and "abelian" are equivalent for L.*

(2) *If* $\rho(L)\in\mathfrak{N}_k$, *then the conditions* (A), (B_∞), (A_∞),$\cdots$, (A_{k+3}), (A_{k+2}) *and "abelian" are equivalent for L* ($k=1, 2, 3,\cdots$).

(3) *In particular if* $L\in \mathrm{L}(\lhd)\mathfrak{F}$, *then the conditions* (A), (B_∞), (A_∞) *and "abelian" are equivalent for L.*

PROOF. (1) It suffices to prove that if $L\in(B_\infty)$ then $L\in\mathfrak{A}$. Assume that $L\in(B_\infty)$. By Theorem 11(1), there exists a semisimple ideal S of L such that $L=\zeta(L)\oplus S$ and $S\in(B_\infty)$. Then we assert that $S=0$. In fact, if $S\neq 0$, by Theorem A $S=\bigoplus_\lambda S_\lambda$, where each S_λ is a finite-dimensional non-abelian simple ideal of L. By [4, Lemma 3], $S_\lambda\notin(A_3)$ and therefore by Proposition 1 we have $S_\lambda\notin(B_\infty)$. On the other hand, $S_\lambda\in(B_\infty)$ by Proposition 3. Thus we have a contradiction. Therefore $L=\zeta(L)$ and $L\in\mathfrak{A}$.

(2) is similarly proved and (3) follows from (1) and Theorem 12.

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Department of Mathematics,
Hiroshima Institute of Technology