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On the maximum principle for a class of linear parabolic differential systems

Yoshinori ISOBE

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§1. Introduction

Consider second order linear parabolic systems of the form

$$(1.1)\quad D_t\boldsymbol{u}(x, t) = \sum_{j,k=1}^{n} A_{jk}(x, t)D_{x_j}D_{x_k}\boldsymbol{u}(x, t) + \sum_{j=1}^{n} B_j(x, t)D_{x_j}\boldsymbol{u}(x, t)$$

in $\mathbf{R}^n \times (0, T]$, $T>0$, where $D_t=\partial/\partial t$, $D_{x_j}=\partial/\partial x_j$, $1\leq j\leq n$, $A_{jk}(x, t)$ and $B_j(x, t)$ $(1\leq j, k\leq n)$ are $m\times m$ (possibly complex-valued) matrix functions in $\mathbf{R}^n\times[0, T]$, and $\boldsymbol{u}(x, t)$ is an m-vector function in $\mathbf{R}^n\times[0, T]$.

Suppose that the following hypotheses are satisfied:

(A_1) There exists a constant $\delta>0$ such that

$$(1.2)\quad \mathscr{Re}\left(\sum_{j,k=1}^{n} A_{jk}(x, t)\sigma_j\sigma_k\zeta, \zeta\right) \geq \delta|\sigma|^2|\zeta|^2$$

for all $\sigma=(\sigma_1,\dots, \sigma_n)\in\mathbf{R}^n$, $\zeta=\mathrm{col}(\zeta_1,\dots, \zeta_m)\in\mathbf{C}^m$ and $(x, t)\in\mathbf{R}^n\times[0, T]$, where $(\ ,\ )$ denotes the inner product in $\mathbf{C}^m$ and $|\ \ |$ denotes the Euclidean length of a vector in $\mathbf{R}^n$ or $\mathbf{C}^m$.

(A_2) $A_{jk}(x, t)$ and $B_j(x, t)$ $(1\leq j, k\leq n)$ are bounded and continuous in $\mathbf{R}^n\times[0, T]$ and satisfy uniform Hölder conditions (exponent $\alpha\in(0, 1]$) with respect to x.

Then, the Cauchy problem for (1.1) with the initial condition

$$(1.3)\quad \lim_{t\downarrow 0}\boldsymbol{u}(x, t) = \boldsymbol{u}_0(x), \quad x\in\mathbf{R}^n,$$

has a bounded solution (in the classical sense) $\boldsymbol{u}(x, t)$ in $\mathbf{R}^n\times[0, T]$ provided $\boldsymbol{u}_0(x)$ is bounded and continuous in $\mathbf{R}^n$. And if (A_3), which is referred to afterward in Theorem, is assumed, then $\boldsymbol{u}(x, t)$ is unique and represented in the form

$$(1.4)\quad \boldsymbol{u}(x, t) = \int_{\mathbf{R}^n} Z(t, 0, x, \xi)\boldsymbol{u}_0(\xi)d\xi,$$

where $Z(t, \tau, x, \xi)$ is a fundamental matrix of (1.1) (see Section 2).

We denote by $\boldsymbol{U}$ the set of all bounded complex-valued solutions $\boldsymbol{u}(x, t)$ of (1.1) defined in $\mathbf{R}^n\times[0, T]$, and define

$$(1.5) \qquad \mathscr{K}(\mathbf{R}^n \times [0, T]) = \sup_{u \in U} \frac{\|u\|_{C(\mathbf{R}^n \times [0,T])}}{\|u\|_{C(\mathbf{R}^n \times \{0\})}},$$

where $\|u\|_{C(D)} = \sup\{|u(x, t)|; (x, t) \in D\}$. The quantity $\mathscr{K}(\mathbf{R}^n \times [0, T])$ is finite, since (1.4) implies the existence of a constant $K > 0$, independent of u, such that

$$|u(x, t)| \leq K \sup_{x \in \mathbf{R}^n} |u(x, 0)|, \quad (x, t) \in \mathbf{R}^n \times [0, T].$$

(It is trivial that $\mathscr{K}(\mathbf{R}^n \times [0, T]) \geq 1$.) We say that the *maximum principle* holds for system (1.1) if $\mathscr{K}(\mathbf{R}^n \times [0, T]) = 1$.

In an interesting paper [1] Maz'ya and Kresin have considered the parabolic system with constant coefficients

$$(1.6) \qquad D_t u(x, t) = \sum_{j,k=1}^n A_{jk} D_{x_j} D_{x_k} u(x, t),$$

and have shown that the maximum principle holds for (1.6) if and only if all the A_{jk} are scalar matrices in the sense that $A_{jk} = a_{jk} E_m$ with $a_{jk} \in \mathbf{R}$, where E_m denotes the m-dimensional unit matrix.

The purpose of this paper is to extend this result of Maz'ya and Kresin to more general parabolic systems of the form (1.1). More precisely, we prove the following theorem in Section 3.

THEOREM. *In addition to* (A_1) *suppose that*

(A_3) $A_{jk}(x, t)$ $(1 \leq j, k \leq n)$ *have second derivatives with respect to* x, $B_j(x, t)$ $(1 \leq j \leq n)$ *have first derivatives with respect to* x, *and these derivatives together with* $A_{jk}(x, t)$ *and* $B_j(x, t)$ *are bounded continuous functions of* (x, t) *in* $\mathbf{R}^n \times [0, T]$, *and are uniformly Hölder continuous functions (exponent* α*) of* x.
Then, the maximum principle holds for (1.1) *if and only if*

$$A_{jk}(x, t) = a_{jk}(x, t) E_m, \quad B_j(x, t) = b_j(x, t) E_m, \quad 1 \leq j, k \leq n,$$

where $a_{jk}(x, t)$ *and* $b_j(x, t)$ *are real-valued scalar functions with the same regularities in* $\mathbf{R}^n \times [0, T]$ *as* $A_{jk}(x, t)$ *and* $B_j(x, t)$*, respectively.*

To prove this theorem the method used by Maz'ya and Kresin [1] is closely followed. In particular, a crucial role is played by an explicit representation of $\mathscr{K}(\mathbf{R}^n \times [0, T])$ in terms of a fundamental matrix $Z(t, \tau, x, \xi)$ of (1.1). Section 2 summarizes basic results concerning fundamental matrices which are needed in the development of Section 3.

§2. Fundamental matrices

In this preparatory section we state bacic results concerning fundamental matrices (or fundamental solutions) for parabolic systems of the form (1.1).

By a *fundamental matrix* of (1.1) we mean an $m\times m$ matrix function $Z(t, \tau, x, \xi)$ defined for $(x, t)\in \mathbf{R}^n\times(0, T]$, $(\xi, \tau)\in\mathbf{R}^n\times[0, T)$, $t>\tau$, which as a function of $(x, t)\in\mathbf{R}^n\times(\tau, T]$ satisfies (1.1) and is such that

$$\lim_{t\downarrow\tau}\int_{\mathbf{R}^n} Z(t, \tau, x, \xi)\boldsymbol{u}_0(\xi)d\xi = \boldsymbol{u}_0(x), \quad x\in\mathbf{R}^n,$$

for any continuous bounded function $\boldsymbol{u}_0(x)$ in $\mathbf{R}^n$.

1) We start with the simple case where $A_{jk}(x, t)=A_{jk}(t)$ and $B_j(x, t)\equiv 0$ $(1\le j, k\le n)$, i.e. the system

$$D_t\boldsymbol{u}(x, t) = \textstyle\sum_{j,k=1}^n A_{jk}(t)D_{x_j}D_{x_k}\boldsymbol{u}(x, t). \tag{2.1}$$

In this case one obtains a fundamental matrix of (2.1) of the form $Z(t, \tau, x, \xi)= G(t, \tau, x-\xi)$ with $G(t, \tau, x)$ given by

$$G(t, \tau, x) = (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i(x,\sigma)}Q(t, \tau, \sigma)d\sigma.$$

Here $Q(t, \tau, \sigma)$ is the solution of the initial value problem

$$\begin{aligned}&D_tQ(t, \tau, \sigma) = [-\textstyle\sum_{j,k=1}^n A_{jk}(t)\sigma_j\sigma_k]Q(t, \tau, \sigma), \quad 0\le\tau<t\le T,\\ &Q(\tau, \tau, \sigma) = E_m.\end{aligned}$$

Note that if $A_{jk}(t)$ are real-valued, then so is the fundamental matrix $G(t, \tau, x)$ of (2.1). In fact, $Q(t, \tau, \sigma)$ is clearly real-valued, and since $Q(t, \tau, \sigma)=Q(t, \tau, -\sigma)$ we have

$$\begin{aligned}\overline{G(t, \tau, x)} &= (2\pi)^{-n}\int_{\mathbf{R}^n}\overline{e^{i(x,\sigma)}Q(t, \tau, \sigma)}d\sigma\\ &= (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i(x,-\sigma)}Q(t, \tau, -\sigma)d\sigma\\ &= (2\pi)^{-n}\int_{\mathbf{R}^n} e^{i(x,\sigma)}Q(t, \tau, \sigma)d\sigma\\ &= G(t, \tau, x).\end{aligned}$$

Furthermore it can be shown that $G(t, \tau, x)$ is an entire function of $(x_1/(t-\tau)^{1/2},\dots, x_n/(t-\tau)^{1/2})$ and satisfies the inequalities

$$|D_x^\ell G(t, \tau, x+iv)| \le C_\ell(t-\tau)^{-(n+\ell)/2}\exp\{-c|x|^2+F|v|^2/(t-\tau)\}$$

for $x\in\mathbf{R}^n$, $v\in\mathbf{R}^n$, $0\le\tau<t\le T$ and $\ell=0, 1, 2,\dots$, where $|A|$ denotes the norm of an $m\times m$ matrix A as a linear mapping i.e. $|A|=\sup_{\xi\in\mathbf{C}^m,|\xi|=1}|A\xi|$, $D_x^\ell= D_{x_1}^{\ell_1}\cdots D_{x_n}^{\ell_n}$, $\ell=\ell_1+\cdots+\ell_n$, and C_ℓ, c, F are positive constants depending only on n, m, δ, T and the coefficients $A_{jk}(t)$ $(1\le j, k\le n)$. For the proof of

the above results we refer to Èidel'man [2] (Chapter 1, §2).

2) Next we consider the general system (1.1) for which hypotheses (A_1) and (A_2) are satisfied. Then, there exists a fundamental matrix $Z(t, \tau, x, \xi)$ of (1.1), which is constructed in the following manner. First, let $G_0(t, \tau, x, y)$ denote a fundamental matrix of the system

$$D_t \boldsymbol{u}(x, t) = \sum_{j,k=1}^n A_{jk}(y, t) D_{x_j} D_{x_k} \boldsymbol{u}(x, t),$$

$y \in \mathbf{R}^n$ being regarded as a parameter, define $K_p(t, \tau, x, \xi)$, $p=1, 2,\dots,$ by the formulas

$$\begin{aligned} K_1(t, \tau, x, \xi) &= \sum_{j,k=1}^n \{A_{jk}(x, t) - A_{jk}(\xi, t)\} D_{x_j} D_{x_k} G_0(t, \tau, x-\xi, \xi) \\ &\quad + \sum_{j=1}^n B_j(x, t) D_{x_j} G_0(t, \tau, x-\xi, \xi), \\ K_p(t, \tau, x, \xi) &= \int_\tau^t \left[\int_{\mathbf{R}^n} K_1(t, \beta, x, y) K_{p-1}(\beta, \tau, y, \xi) dy \right] d\beta, \ p = 2, 3,\dots, \end{aligned}$$

and put

$$\Phi(t, \tau, x, \xi) = \sum_{p=1}^\infty K_p(t, \tau, x, \xi).$$

The desired fundamental matrix $Z(t, \tau, x, \xi)$ of (1.1) is then given by

$$\begin{aligned} (2.2) \qquad Z(t, \tau, x, \xi) &= G_0(t, \tau, x-\xi, \xi) \\ &\quad + \int_\tau^t \left[\int_{\mathbf{R}^n} G_0(t, \beta, x-y, y) \Phi(\beta, \tau, y, \xi) dy \right] d\beta. \end{aligned}$$

We can verify that $Z(t, \tau, x, \xi)$ is well defined, is indeed a fundamental matrix of (1.1) and satisfies the inequalities

$$(2.3) \qquad |D_x^\ell Z(t, \tau, x, \xi)| \le C(t-\tau)^{-(n+\ell)/2} \exp\{-c|x-\xi|^2/(t-\tau)\},$$

for $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$, $0 \le \tau < t \le T$, and $\ell = 0, 1, 2$, where C and c are positive constants depending only on n, m, δ, T and the coefficients $A_{jk}(x, t)$ and $B_j(x, t)$ $(1 \le j, k \le n)$. In particular, for any continuous bounded function $\boldsymbol{u}_0(x)$ in $\mathbf{R}^n$, the function

$$(2.4) \qquad \boldsymbol{u}(x, t) = \int_{\mathbf{R}^n} Z(t, \tau, x, \xi) \boldsymbol{u}_0(\xi) d\xi$$

represents a bounded solution of Cauchy problem for (1.1) in $\mathbf{R}^n \times (\tau, T]$ with the initial condition

$$(2.5) \qquad \lim_{t \downarrow \tau} \boldsymbol{u}(x, t) = \boldsymbol{u}_0(x), \quad x \in \mathbf{R}^n.$$

From the above construction we see that a fundamental matrix $Z(t, \tau, x, \xi)$ is real-valued provided all the coefficients $A_{jk}(x, t)$ and $B_j(x, t)$ $(1 \leq j, k \leq n)$ are real-valued.

3) Let us consider the system (1.1) for which (A_1) and (A_3) are satisfied. Then, we can define the *adjoint system*:

$$(2.6) \qquad D_{(-\tau)}\boldsymbol{v}(\xi, \tau) = \sum_{j,k=1}^{n} D_{(-\xi_j)}D_{(-\xi_k)}[{}^tA_{jk}(\xi, \tau)\boldsymbol{v}(\xi, \tau)] + \sum_{j=1}^{n} D_{(-\xi_j)}[{}^tB_j(\xi, \tau)\boldsymbol{v}(\xi, \tau)],$$

where ${}^tA_{jk}$ and ${}^tB_j$ denote the transpose of A_{jk} and B_j, respectively. Proceeding as in the preceding subsection, we can prove the existence of a fundamental matrix of (2.6), that is, an $m \times m$ matrix function $Z^*(t, \tau, x, \xi)$ defined for $(x, t) \in \mathbf{R}^n \times (0, T]$, $(\xi, \tau) \in \mathbf{R}^n \times [0, T)$, $t > \tau$, satisfying (2.6) as a function of (ξ, τ) $(\xi \in \mathbf{R}^n$, $0 \leq \tau < t \leq T)$, and satisfying the relation

$$\lim_{\tau \uparrow t} \int_{\mathbf{R}^n} Z^*(t, \tau, x, \xi)\boldsymbol{v}_0(x)dx = \boldsymbol{v}_0(\xi), \quad \xi \in \mathbf{R}^n,$$

for any continuous bounded function $\boldsymbol{v}_0(\xi)$ in $\mathbf{R}^n$. An exact analogue of the estimate (2.3) is shown to hold for the fundamental matrix $Z^*(t, \tau, x, \xi)$ of (2.6) and its first and second derivatives with respect to ξ. The following relations holding for $x, y \in \mathbf{R}^n$, $0 \leq \tau < \beta < t \leq T$, are needed in the next section:

$$(2.7) \qquad Z(t, \tau, x, y) = {}^tZ^*(t, \tau, x, y),$$

$$(2.8) \qquad Z(t, \tau, x, y) = \int_{\mathbf{R}^n} Z(t, \beta, x, \xi)Z(\beta, \tau, \xi, y)d\xi.$$

The detailed proofs of the facts stated in 2) and 3) can be found in Èidel'man [2] (Chapter 1, §3).

§3. Maximum principle

1) We first restrict our attention to the case where all the coefficients $A_{jk}(x, t)$ and $B_j(x, t)$ in (1.1) are real-valued. We denote by $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T])$ the quantity $\mathscr{K}(\mathbf{R}^n \times [0, T])$ defined by (1.5), where the supremum is taken over the set of real-valued, bounded, continuous solutions $\boldsymbol{u}(x, t)$ of (1.1). The following result gives a characterization of $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T])$.

THEOREM 3.1. *Suppose that* (A_1) *and* (A_3) *are satisfied. Then,*

$$(3.1) \qquad \mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T]) = \sup_{\boldsymbol{z} \in \mathbf{R}^m, |\boldsymbol{z}|=1} \sup_{x \in \mathbf{R}^n, 0 < t \leq T} \int_{\mathbf{R}^n} |{}^tZ(t, 0, x, \xi)\boldsymbol{z}|d\xi.$$

PROOF. Let (x, t) be fixed in $\mathbf{R}^n \times (0, T]$. Consider the linear mapping which assigns to each bounded continuous function $\boldsymbol{u}_0(x)$ in $\mathbf{R}^n$ the value at (x, t) of the solution $\boldsymbol{u}(x, t)$ of the Cauchy problem

$$D_t\boldsymbol{u}(x, t) = \textstyle\sum_{j,k=1}^n A_{jk}(x, t)D_{x_j}D_{x_k}\boldsymbol{u}(x, t) + \sum_{j=1}^n B_j(x, t)D_{x_j}\boldsymbol{u}(x, t),$$
$$(x, t)\in \mathbf{R}^n \times (0, T],$$

$$\lim_{t\downarrow 0} \boldsymbol{u}(x, t) = \boldsymbol{u}_0(x), \quad x\in \mathbf{R}^n.$$

The norm of this mapping is computed as follows:

$$\begin{aligned}\|\boldsymbol{u}(x, t)\| &= \sup_{\|\boldsymbol{u}_0\|\le 1} \left|\int_{\mathbf{R}^n} Z(t, 0, x, \xi)\boldsymbol{u}_0(\xi)d\xi\right| \\ &= \sup_{\|\boldsymbol{u}_0\|\le 1} \sup_{\boldsymbol{z}\in\mathbf{R}^m, |\boldsymbol{z}|=1}\left(\boldsymbol{z}, \int_{\mathbf{R}^n} Z(t, 0, x, \xi)\boldsymbol{u}_0(\xi)\, d\xi\right) \\ &= \sup_{|\boldsymbol{z}|=1} \sup_{\|\boldsymbol{u}_0\|\le 1}\left(\boldsymbol{z}, \int_{\mathbf{R}^n} Z(t, 0, x, \xi)\boldsymbol{u}_0(\xi)\, d\xi\right) \\ &= \sup_{|\boldsymbol{z}|=1} \sup_{\|\boldsymbol{u}_0\|\le 1} \int_{\mathbf{R}^n} ({}^tZ(t, 0, x, \xi)\, \boldsymbol{z}, \boldsymbol{u}_0(\xi))d\xi.\end{aligned}$$

If we define $N_{(x,t)}(\boldsymbol{z})=\{\xi\in\mathbf{R}^n\colon {}^tZ(t, 0, x, \xi)\boldsymbol{z}=\mathbf{0}\}$, then,

$$\|\boldsymbol{u}(x, t)\| = \sup_{|\boldsymbol{z}|=1} \sup_{\|\boldsymbol{u}_0\|\le 1} \int_{\mathbf{R}^n-N_{(x,t)}(\boldsymbol{z})} ({}^tZ(t, 0, x, \xi)\, \boldsymbol{z}, \boldsymbol{u}_0(\xi))d\xi,$$

and since the interior supremum of this integral is attained by

$\boldsymbol{u}_0(\xi)={}^tZ(t, 0, x, \xi)\boldsymbol{z}/|{}^tZ(t, 0, x, \xi)\boldsymbol{z}|$, we obtain

$$\begin{aligned}\|\boldsymbol{u}(x, t)\| &= \sup_{|\boldsymbol{z}|=1}\int_{\mathbf{R}^n-N_{(x,t)}(\boldsymbol{z})} |{}^tZ(t, 0, x, \xi)\boldsymbol{z}|d\xi \\ &= \sup_{|\boldsymbol{z}|=1}\int_{\mathbf{R}^n} |{}^tZ(t, 0, x, \xi)\boldsymbol{z}|d\xi.\end{aligned}$$

Since $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n\times[0, T])=\sup_{x\in\mathbf{R}^n, 0<t\le T} \|\boldsymbol{u}(x, t)\|$, the conclusion readily follows.

Q.E.D.

The real-valued version of the theorem stated in the introduction can be proved with the help of Theorem 3.1.

THEOREM 3.2. *Suppose that* (A_1) *and* (A_3) *are satisfied. Then,* $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n\times[0, T])=1$ *if and only if*

$$A_{jk}(x, t) = a_{jk}(x, t)E_m, \quad B_j(x, t) = b_j(x, t)E_m, \quad 1\le j, k\le n,$$

where $a_{jk}(x, t)$ *and* $b_j(x, t)$ *are scalar functions with the same regularities as* $A_{jk}(x, t)$ *and* $B_j(x, t)$, *respectively.*

PROOF. The proof of the "if" part is easy. In fact, let $\boldsymbol{u}(x, t)$ be the unique bounded solution of the Cauchy problem

$$D_t \boldsymbol{u}(x, t) = \sum_{j,k=1}^n a_{jk}(x, t) D_{x_j} D_{x_k} \boldsymbol{u}(x, t) + \sum_{j=1}^n b_j(x, t) D_{x_j} \boldsymbol{u}(x, t), \quad (x, t) \in \mathbf{R}^n \times (0, T],$$

$$\lim_{t \downarrow 0} \boldsymbol{u}(x, t) = \boldsymbol{f}(x), \quad x \in \mathbf{R}^n,$$

where $\boldsymbol{f}(x)$ is a given bounded continuous function in $\mathbf{R}^n$, and define $w(x, t) = (\boldsymbol{u}(x, t), \boldsymbol{q})$, where $\boldsymbol{q} \in \mathbf{R}^m$ is a fixed vector with $|\boldsymbol{q}| = 1$. Then, $w(x, t)$ is a solution of the Cauchy problem

$$D_t w(x, t) = \sum_{j,k=1}^n a_{jk}(x, t) D_{x_j} D_{x_k} w(x, t) + \sum_{j=1}^n b_j(x, t) D_{x_j} w(x, t), \quad (x, t) \in \mathbf{R}^n \times (0, T].$$

$$\lim_{t \downarrow 0} w(x, t) = (\boldsymbol{f}(x), \boldsymbol{q}), \quad x \in \mathbf{R}^n,$$

and so by the well-known maximum principle ([3], §1, Theorem 10) we see that

$$|w(x, t)| \le \sup_{x \in \mathbf{R}^n} |(\boldsymbol{f}(x), \boldsymbol{q})| \le \sup_{x \in \mathbf{R}^n} |\boldsymbol{f}(x)|, \quad (x, t) \in \mathbf{R}^n \times [0, T].$$

It follows that $|\boldsymbol{u}(x, t)| \le \sup_{x \in \mathbf{R}^n} |\boldsymbol{f}(x)|$, $(x, t) \in \mathbf{R}^n \times [0, T]$, which implies $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T]) = 1$.

The proof of the "only if" part proceeds as follows. We begin by showing that $\mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T]) = 1$ ensures that

$$(3.2) \qquad {}^tZ(t, 0, x, \xi)\boldsymbol{z} = |{}^tZ(t, 0, x, \xi)\boldsymbol{z}|\boldsymbol{z}$$

for all $\boldsymbol{z} \in \mathbf{R}^m$, $|\boldsymbol{z}| = 1$, and (t, x, ξ) with $x, \xi \in \mathbf{R}^n$ and $0 < t \le T$. In fact, if

$$(3.3) \qquad |{}^tZ(t_0, 0, x_0, \xi_0)\boldsymbol{z}_0|\boldsymbol{z}_0 \ne {}^tZ(t_0, 0, x_0, \xi_0)\boldsymbol{z}_0$$

for some $\boldsymbol{z}_0 \in \mathbf{R}^m$, $|\boldsymbol{z}_0| = 1$, x_0, $\xi_0 \in \mathbf{R}^n$ and $0 < t_0 \le T$, then in view of the proof of the preceeding theorem and with the use of (3.3) we see that

$$\begin{aligned}
1 &= \mathscr{K}_{\mathbf{R}}(\mathbf{R}^n \times [0, T]) \ge \sup_{\|\boldsymbol{u}_0\| \le 1} \int_{\mathbf{R}^n} ({}^tZ(t_0, 0, x_0, \xi)\boldsymbol{z}_0, \boldsymbol{u}_0(\xi)) d\xi \\
&= \int_{\mathbf{R}^n - N_{(x_0, t_0)}(\boldsymbol{z}_0)} ({}^tZ(t_0, 0, x_0, \xi)\boldsymbol{z}_0, {}^tZ(t_0, 0, x_0, \xi)\boldsymbol{z}_0 / |{}^tZ(t_0, 0, x_0, \xi)\boldsymbol{z}_0|) d\xi \\
&> \int_{\mathbf{R}^n} ({}^tZ(t_0, 0, x_0, \xi)\boldsymbol{z}_0, \boldsymbol{z}_0) d\xi \\
&= \int_{\mathbf{R}^n} (\boldsymbol{z}_0, Z(t_0, 0, x_0, \xi)\boldsymbol{z}_0) d\xi \\
&= (\boldsymbol{z}_0, E_m \boldsymbol{z}_0) = 1.
\end{aligned}$$

Here we used the continuity of ${}^tZ(t_0, 0, x_0, \xi)$ in ξ and the fact that $\int_{\mathbf{R}^n} Z(t_0, 0, x_0, \xi) d\xi = E_m$. This contradiction verifies the truth of (3.2).

Let $Z^{(\ell s)}(t, \tau, x, \xi)$ $(1 \le \ell, s \le m)$ denote the (ℓ, s) element of the fundamental matrix $Z(t, \tau, x, \xi)$ of (1.1). Putting

$$z = z_1 = \mathrm{col}(1, 0,\dots, 0),\dots, z = z_m = \mathrm{col}(0,\dots, 0, 1)$$

successively in (3.2), we find

$$Z^{(\ell s)}(t, 0, x, \xi) = 0 \quad \text{for} \quad \ell \neq s, \quad x, \xi \in \mathbf{R}^n, \quad 0 < t \leq T.$$

Put $z = \mathrm{col}\, m^{-1/2}(1,\dots, 1)$ in (3.2). Then, for every ℓ, $1 \leq \ell \leq m$,

$$\{(Z^{(11)}(t, 0, x, \xi))^2 + \cdots + (Z^{(mm)}(t, 0, x, \xi))^2\}^{1/2} = m^{1/2} Z^{(\ell\ell)}(t, 0, x, \xi),$$

so that

$$Z^{(11)}(t, 0, x, \xi) = \cdots = Z^{(mm)}(t, 0, x, \xi) \quad \text{for} \quad x, \xi \in \mathbf{R}^n, \quad 0 < t \leq T.$$

It follows that there exists a scalar function $z(t, 0, x, \xi)$ such that

$$Z(t, 0, x, \xi) = z(t, 0, x, \xi)E_m \quad \text{for} \quad x, \xi \in \mathbf{R}^n, \quad 0 < t \leq T. \tag{3.4}$$

Applying (2.8) and using (3.4), we have

$$\int_{\mathbf{R}^n} Z^{(\ell s)}(t, \tau, x, \xi) z(\tau, 0, \xi, y) d\xi = 0 \quad \text{for} \quad \ell \neq s, \quad 1 \leq \ell, s \leq m,$$

and

$$\int_{\mathbf{R}^n} Z^{(\ell\ell)}(t, \tau, x, \xi) z(\tau, 0, \xi, y) d\xi = z(t, 0, x, y) \quad \text{for} \quad \ell, 1 \leq \ell \leq m,$$

for all (t, x, y) with $x, y \in \mathbf{R}^n$ and $0 < \tau < t \leq T$, whence we obtain for any bounded continuous function $\psi(x)$ in $\mathbf{R}^n$

$$\int_{\mathbf{R}^n} Z^{(\ell s)}(t, \tau, x, \xi)\left(\int_{\mathbf{R}^n} z(\tau, 0, \xi, y)\psi(y) dy\right) d\xi = 0 \quad \text{for} \quad \ell \neq s, \quad 1 \leq \ell, s \leq m,$$

and

$$\int_{\mathbf{R}^n} Z^{(\ell\ell)}(t, \tau, x, \xi)\left(\int_{\mathbf{R}^n} z(\tau, 0, \xi, y)\psi(y) dy\right) d\xi = \int_{\mathbf{R}^n} z(t, 0, x, y)\psi(y) dy$$

for ℓ, $1 \leq \ell \leq m$. Since $\psi(x)$ is arbitrary, we see that

$$Z^{(\ell s)}(t, \tau, x, \xi) = 0 \quad \text{for} \quad \ell \neq s,$$
$$Z^{(11)}(t, \tau, x, \xi) = \cdots = Z^{(mm)}(t, \tau, x, \xi),$$

and hence there exists a scalar function $z(t, \tau, x, \xi)$ such that

$$Z(t, \tau, x, \xi) = z(t, \tau, x, \xi) E_m \tag{3.5}$$

for all $(x, t) \in \mathbf{R}^n \times (0, T]$, $(\xi, \tau) \in \mathbf{R}^n \times [0, T)$, $t > \tau$. Furthermore, in view of (2.7), we get

$$Z^*(t, \tau, x, \xi) = {}^t Z(t, \tau, x, \xi) = z(t, \tau, x, \xi) E_m. \tag{3.6}$$

Let $\psi_0(x)$ be a bounded continuous function with compact support in $\mathbf{R}^n$, and define

$$u_0(\xi, \tau) = \int_{\mathbf{R}^n} z(t, \tau, x, \xi) \psi_0(x) dx, \quad (\xi, \tau) \in \mathbf{R}^n \times [0, t).$$

Put $\boldsymbol{h}_c(\xi, \tau) = u_0(\xi, \tau)\boldsymbol{c}$, where $\boldsymbol{c} = \mathrm{col}(c_1, \ldots, c_m) \in \mathbf{R}^m$. Then, $\boldsymbol{h}_c(\xi, \tau)$ satisfies

$$\begin{aligned} L^*(\boldsymbol{h}_c(\xi, \tau)) = D_{(-\tau)} \boldsymbol{h}_c(\xi, \tau) &- \textstyle\sum_{j,k=1}^n D_{(-\xi_j)} D_{(-\xi_k)} [{}^t A_{jk}(\xi, \tau) \boldsymbol{h}_c(\xi, \tau)] \\ &- \textstyle\sum_{j=1}^n D_{(-\xi_j)} [{}^t B_j(\xi, \tau) \boldsymbol{h}_c(\xi, \tau)] = \mathbf{0}, \quad \xi \in \mathbf{R}^n, \quad 0 \le \tau < t \le T, \end{aligned} \tag{3.7}$$

and $\lim_{\tau \uparrow t} \boldsymbol{h}_c(\xi, \tau) = \psi_0(\xi)\boldsymbol{c}$, $\xi \in \mathbf{R}^n$.
Substituting

$$\boldsymbol{c}_1 = \mathrm{col}(1, 0, \ldots, 0), \ldots, \boldsymbol{c}_m = \mathrm{col}(0, \ldots, 0, 1)$$

in (3.7) yields m^2 Cauchy problems

$$\begin{aligned} &L^*_{(\ell s)}(u_0(\xi, \tau)) = \delta_{\ell s} D_{(-\tau)} u_0(\xi, \tau) - \textstyle\sum_{j,k=1}^n D_{(-\xi_j)} D_{(-\xi_k)} [A_{jk}^{(\ell s)}(\xi, \tau) u_0(\xi, \tau)] \\ &\quad - \textstyle\sum_{j=1}^n D_{(-\xi_j)} [B_j^{(\ell s)}(\xi, \tau) u_0(\xi, \tau)] = 0, \quad \xi \in \mathbf{R}^n, \quad 0 \le \tau < t \le T, \\ &\lim_{\tau \uparrow t} u_0(\xi, \tau) = \psi_0(\xi), \quad \xi \in \mathbf{R}^n, \quad 1 \le \ell, s \le m, \end{aligned}$$

where $\delta_{\ell s}$ is Kronecker's symbol.

Because of (A_1), $L^*_{(\ell\ell)}$ $(1 \le \ell \le m)$ are backward parabolic. Using (2.3), we see that

$$|u_0(\xi, \tau)| \le C_1(t, \tau) \exp\{-C_2(t, \tau)|\xi|^2\}, \quad \xi \in \mathbf{R}^n, \quad 0 \le \tau < t \le T, \tag{3.8}$$

for some positive constasts $C_1(t, \tau)$ and $C_2(t, \tau)$. Since (3.8) implies $u_0(\xi, \tau) \in L^q_\xi(\mathbf{R}^n)$ $(1 \le q \le \infty)$ for $\tau < t$, we can take Fourier transforms of $L^*_{(\ell s)}(u_0(\xi, \tau)) = 0$ with respect to ξ.

Let $\ell \ne s$, $1 \le \ell, s \le m$. Then, the Fourier transforms satisfy

$$\textstyle\sum_{j,k=1}^n \mathscr{F}[A_{jk}^{(\ell s)} u_0] \sigma_j \sigma_k - i \sum_{j=1}^n \mathscr{F}[B_j^{(\ell s)} u_0] \sigma_j = 0 \tag{3.9}$$

for $\tau < t$ and $\ell \ne s$, $1 \le \ell, s \le m$. In view of (3.8) it can be shown that the Fourier transforms $\mathscr{F}[A_{jk}^{(\ell s)} u_0]$, $\mathscr{F}[B_j^{(\ell s)} u_0]$ are entire functions of σ. Let $J^{(\ell s)}(\sigma, \tau)$ denote the left-hand side of (3.9). Then, each $J^{(\ell s)}(\sigma, \tau)$ is represented by a power series in σ:

$$J^{(\ell s)}(\sigma, \tau) = \sum_{p_1,\dots,p_n\geq 0} h^{(\ell s)}_{p_1\cdots p_n}(\tau)\sigma_1^{p_1}\cdots\sigma_n^{p_n}$$

with all the coefficients $h^{(\ell s)}_{p_1\cdots p_n}(\tau)=0$. Since

$$0 = h^{(\ell s)}_{10\cdots 0}(\tau) = -i\int_{\mathbf{R}^n} B_1^{(\ell s)}(\xi, \tau)u_0(\xi, \tau)d\xi$$

and ψ_0 is arbitrary, we have $B_1^{(\ell s)}(\xi, \tau)=0$ for all $\ell\neq s$, $\xi\in\mathbf{R}^n$ and $0\leq\tau<t\leq T$. Likewise, we obtain $B_j^{(\ell s)}(\xi, \tau)=0$ for all $1\leq j\leq n$, $\ell\neq s$, $\xi\in\mathbf{R}^n$ and $0\leq\tau<t\leq T$. Similarly, since

$$0 = h^{(\ell s)}_{110\cdots 0}(\tau) = \int_{\mathbf{R}^n} A_{11}^{(\ell s)}(\xi, \tau)u_0(\xi, \tau)d\xi,$$

it follows that $A_{11}^{(\ell s)}(\xi, \tau)=0$ for $\ell\neq s$, $\xi\in\mathbf{R}^n$ and $0\leq\tau<t\leq T$. Furthermore, $A_{jk}^{(\ell s)}(\xi, \tau)=0$ for all j, k $(1\leq j,\ k\leq n)$, $\ell\neq s$, $\xi\in\mathbf{R}^n$ and $0\leq\tau<t\leq T$.

Next we fix ℓ and s, $1\leq\ell, s\leq m$, and take the Fourier transform of $L^*_{(\ell\ell)}(u_0(\xi, \tau))-L^*_{(ss)}(u_0(\xi, \tau))=0$. Arguing as above, we conclude that

$$A_{jk}^{(11)}(\xi, \tau) = \cdots = A_{jk}^{(mm)}(\xi, \tau), \quad B_j^{(11)}(\xi, \tau) = \cdots = B_j^{(mm)}(\xi, \tau)$$

for all j, k $(1\leq j,\ k\leq n)$, $\xi\in\mathbf{R}^n$ and $0\leq\tau<t\leq T$. This completes the proof.

Q. E. D.

2) Finally we deal with the general case of (1.1) where $A_{jk}(x, t)$ and $B_j(x, t)$ $(1\leq j,\ k\leq n)$ are complex matrices. Let $R_{jk}(x, t)=\mathscr{R}e A_{jk}(x, t)$, $H_{jk}(x, t)=\mathscr{I}m A_{jk}(x, t)$, $S_j(x, t)=\mathscr{R}e B_j(x, t)$ and $U_j(x, t)=\mathscr{I}m B_j(x, t)$, that is,

$$A_{jk}(x, t) = R_{jk}(x, t) + iH_{jk}(x, t)$$
$$B_j(x, t) = S_j(x, t) + iU_j(x, t), \quad 1\leq j,\ k\leq n.$$

Define the $2m\times 2m$ real matrices $K_{jk}(x, t)$ and $M_j(x, t)$ by

$$K_{jk}(x, t) = \begin{pmatrix} R_{jk}(x, t) & -H_{jk}(x, t) \\ H_{jk}(x, t) & R_{jk}(x, t) \end{pmatrix},$$
$$M_j(x, t) = \begin{pmatrix} S_j(x, t) & -U_j(x, t) \\ U_j(x,t) & S_j(x, t) \end{pmatrix}, \quad 1\leq j,\ k\leq n.$$

Let $\boldsymbol{u}(x, t)$ be a bounded continuous solution of (1.1) and denote by $\boldsymbol{v}(x, t)$ and $\boldsymbol{w}(x, t)$ the real and imaginary parts of $\boldsymbol{u}(x, t)$, respectively. If we define $\boldsymbol{y}(x, t)=\begin{pmatrix}\boldsymbol{v}(x, t)\\ \boldsymbol{w}(x, t)\end{pmatrix}$, then $\boldsymbol{y}(x, t)$ is a bounded continuous solution of the system

(3.10) $\quad D_t\boldsymbol{y}(x, t) = \sum_{j,k=1}^n K_{jk}(x, t)D_{x_j}D_{x_k}\boldsymbol{y}(x, t) + \sum_{j=1}^n M_j(x, t)D_{x_j}\boldsymbol{y}(x, t).$

As is easily verified, for any $\sigma=(\sigma_1,\dots,\sigma_n)\in\mathbf{R}^n$ and $\eta=\mathrm{col}(\eta_1,\dots,\eta_{2m})\in\mathbf{C}^{2m}$,

$$\mathscr{R}e\,(\textstyle\sum_{j,k=1}^n K_{jk}(x,t)\sigma_j\sigma_k\eta,\ \eta) = \sum_{\ell=1}^2 \mathscr{R}e\,(\sum_{j,k=1}^n A_{jk}(x,t)\sigma_j\sigma_k\zeta_\ell,\ \zeta_\ell),$$

where $\zeta_1=\mathrm{col}(\mathscr{R}e\,\eta_1,\dots,\mathscr{R}e\,\eta_m)+i\,\mathrm{col}(\mathscr{R}e\,\eta_{m+1},\dots,\mathscr{R}e\,\eta_{2m})$ and $\zeta_2=\mathrm{col}(\mathscr{I}m\,\eta_1,\dots,\mathscr{I}m\,\eta_m)+i\,\mathrm{col}(\mathscr{I}m\,\eta_{m+1},\dots,\mathscr{I}m\,\eta_{2m})$, and so the new system (3.10) is parabolic because (A_1) is assumed for the original system (1.1) and $|\eta|^2=\sum_{\ell=1}^2|\zeta_\ell|^2$. Since the coefficients of (3.10) are real-valued, there exists a real-valued fundamental matrix $Y(t,\tau,x,\xi)$ of (3.10), and entirely the same arguments as in the preceding subsection are applied to the system (3.10). Thus, we have the following results.

THEOREM 3.3. *Suppose that* (A_1) *and* (A_3) *hold. Then,*

$$(3.11)\qquad \mathscr{K}(\mathbf{R}^n\times[0,T])=\sup_{z\in\mathbf{R}^{2m},|z|=1,x\in\mathbf{R}^n,0<t\le T}\int_{\mathbf{R}^n}|{}^tY(t,0,x,\xi)z|\,d\xi.$$

THEOREM 3.4. *Suppose that* (A_1) *and* (A_3) *hold. Then,* $\mathscr{K}(\mathbf{R}^n\times[0,T])=1$ *if and only if*

$$(3.12)\qquad A_{jk}(x,t)=a_{jk}(x,t)E_m,\quad B_j(x,t)=b_j(x,t)E_m,\quad 1\le j,k\le n,$$

where $a_{jk}(x,t)$ *and* $b_j(x,t)$ *are real-valued scalar functions in* $\mathbf{R}^n\times[0,T]$ *with the same regularities as* $A_{jk}(x,t)$ *and* $B_j(x,t)$*, respectively.*

PROOF OF THEOREM 3.4. As in the proof of Theorem 3.2, using (3.11), we can show that $\mathscr{K}(\mathbf{R}^n\times[0,T])=1$ if and only if

$$(3.13)\qquad K_{jk}(x,t)=a_{jk}(x,t)E_{2m},\quad M_j(x,t)=b_j(x,t)E_{2m},\quad 1\le j,k\le n,$$

where $a_{jk}(x,t)$ and $b_j(x,t)$ are real-valued scalar functions in $\mathbf{R}^n\times[0,T]$. It is easy to see that (3.13) is equivalent ot (3.12). Q.E.D.

We conclude by referring to a paper by Otsuka [4] in which a characterization of the positivity of fundamental matrices of parabolic systems is obtained.

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Department of Mathematics,
Faculty of Science,
Hiroshima University