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Asymptotic behavior of oscillatory solutions

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1. Introduction

There is an abundance of results in the literature dealing with the oscillation of all solutions of delay differential equations. See, for example, [1]–[6], [8]–[10], and the references cited in [2]. To obtain oscillation results, we usually investigate the asymptotic behavior of the nonoscillatory solutions and then we find conditions on the coefficients and the delays which do not allow such a behavior. By this strategy we are often lead to sufficient conditions for all solutions of certain delay differential equations to oscillate. As a by product, we also learn the asymptotic behavior of the nonoscillatory solutions.

The aim in this paper is to study the asymptotic behavior of the oscillatory solutions of certain delay differential equations of the form

$$x'(t) + p(t)x(t-\tau) + q(t)x(t-\sigma) = 0, \quad t \geqq t_0 \tag{1}$$

and of certain neutral equations of the form

$$(d/dt)[x(t) - px(t-\tau)] + q(t)x(t-\sigma) = 0, \quad t \geqq t_0. \tag{2}$$

Our results, combined with known oscillation results or with known results about the asymptotic behavior of the nonoscillatory solutions of Eqs. (1) and (2), lead to sufficient conditions for the trivial solution of Eqs. (1) and (2) to be asymptotically stable.

In our opinion, the main contribution of this paper is that it shows how oscillation theory may be used, as another tool, in establishing new stability results for differential equations of diverse nature, like Eqs. (1) and (2) above.

Throughout this paper we will assume that the delays τ and σ in Eqs. (1) and (2) are constants and that the coefficients p and q of Eq. (1) and the coefficient q of Eq. (2) are continuous functions for $t \geqq t_0$ while the coefficient p of Eq. (2) is a constant. With the above assumptions, it follows by the method of steps that, if $\varphi \in C[[t_0 - m, t_0], \boldsymbol{R}]$ is a given initial function where $m = \max\{\tau, \sigma\}$, then Eqs. (1) and (2) have a unique solution x valid for $t \geqq t_0$. By a solution x of Eq. (1) we mean a continuous function for $t \geqq t_0 - m$ such that $x(t) = \varphi(t)$ for $t_0 - m \leqq t \leqq t_0$, $x \in C^1[[t_0, \infty), \boldsymbol{R}]$, and x satisfies Eq. (1) for $t \geqq t_0$. On the other hand, by a solution x of the neutral delay differential equation (2) we mean a continuous function for $t \geqq t_0 - m$ such that $x(t) = \varphi(t)$ for $t_0 - m \leqq t \leqq t_0$, $x(t) - px(t-\tau)$ is continuously differentiable for $t \geqq t_0$, and x satisfies Eq. (2) for $t \geqq t_0$.

As usual, a solution of Eqs. (1) or (2) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or negative.

2. Asymptotic behavior of delay equations

In this section we present sufficient conditions for all oscillatory solutions of Eq. (1) to tend to zero as $t \to \infty$.

THEOREM 1. *Consider the delay differential equation*

$$x'(t) + p(t)x(t-\tau) + q(t)x(t-\sigma) = 0, \quad t \geqq t_0 \tag{1}$$

where τ and σ are nonnegative constants and p and q are continuous functions satisfying the conditions

$$p(t) + q(t-\tau+\sigma) \neq 0$$

for τ sufficiently large and

$$2 \limsup_{t\to\infty} \left| \int_{t-\tau}^{t-\sigma} |q(s+\sigma)| ds \right| + \limsup_{t\to\infty} \int_{t-\tau}^{t} |p(s)+q(s-\tau+\sigma)| ds < 1. \tag{3}$$

Then every oscillatory solution of Eq. (1) *tends to zero as $t \to \infty$.*

PROOF. First we will prove that every oscillatory solution of Eq. (1) is bounded. Assume, for the sake of contradiction, that $x(t)$ is an oscillatory solution of Eq. (1) which is unbounded. We will assume $\tau \geqq \sigma$. The case $\tau < \sigma$ is similar. In view of (3), there exists a positive constant Q and a $t_1 \geqq t_0 + \tau$ such that

$$\int_{t-\tau}^{t-\sigma} |q(s+\sigma)| ds \leqq Q < 1 \qquad \text{for} \quad t \geqq t_1,$$

and

$$2Q + \limsup_{t\to\infty} \int_{t-\tau}^{t-\sigma} |p(s)+q(s-\tau+\sigma)| ds < 1. \tag{4}$$

Set

$$z(t) = x(t) + \int_{t-\tau}^{t-\sigma} q(s+\sigma)x(s) ds, \quad t \geqq t_1. \tag{5}$$

Then, for $t \geqq t_1$ we find that

$$|z(t)| \geqq |x(t)| - \int_{t-\tau}^{t-\sigma} |q(s+\sigma)|\,|x(s)|ds$$

$$\geqq |x(t)| - (\max_{t-\tau \leqq s \leqq t-\sigma} |x(s)|)Q.$$

As $x(t)$ is unbounded, there is a $t_2 \geqq t_1$ such that

$$\max_{t-\tau \leqq s \leqq t-\sigma} |x(s)| \leqq \max_{t_1 \leqq s \leqq t} |x(s)|, \quad t \geqq t_2.$$

Then

$$\max_{t_1 \leqq s \leqq t} |z(s)| \geqq (\max_{t_1 \leqq s \leqq t} |x(s)|)(1-Q), \quad t \geqq t_2 \tag{6}$$

which implies that $z(t)$ is also unbounded. From (5) and (1) we find that

$$z'(t) = -\,[p(t)+q(t-\tau+\sigma)]x(t-\tau) \tag{7}$$

which implies that $z'(t)$ is an oscillatory function. This, together with the unboundedness of $z(t)$, implies that there exists a sequence of points $\{\xi_n\}$ such that $\xi_n \geqq t_2$ for $n=1, 2,\ldots$,

$$\lim_{n\to\infty} \xi_n = \infty, \quad z'(\xi_n) = 0 \qquad \text{for} \quad n = 1, 2,\ldots,$$

$$\lim_{n\to\infty} |z(\xi_n)| = \infty,$$

and

$$|z(\xi_n)| = \max_{t_1 \leqq s \leqq \xi_n} |z(s)|, \quad n = 1, 2,\ldots. \tag{8}$$

As $z'(\xi_n)=0$ and $p(\xi_n)+q(\xi_n-\tau+\sigma)\neq 0$ for n large, say $n \geqq n_0$ it follows, from (7), that $x(\xi_n-\tau)=0$ for $n \geqq n_0$. Hence, from (5), we find that

$$z(\xi_n-\tau) = \int_{\xi_n-2\tau}^{\xi_n-\tau-\sigma} q(s+\sigma)x(s)ds, \quad n \geqq n_0. \tag{9}$$

Integrating (7) from $\xi_n-\tau$ to ξ_n and using (9) we obtain, for $n \geqq n_0$,

$$z(\xi_n) = \int_{\xi_n-2\tau}^{\xi_n-\tau-\sigma} q(s+\sigma)x(s)ds - \int_{\xi_n-\tau}^{\xi_n} [p(s)+q(s-\tau+\sigma)]x(s-\tau)ds. \tag{10}$$

Using (6) and (8) we find, from (10),

$$|z(\xi_n)| \leqq \frac{Q}{1-Q}|z(\xi_n)| + \frac{1}{1-Q}|z(\xi_n)| \int_{\xi_n-\tau}^{\xi_n} |p(s) + q(s-\tau+\sigma)|ds \tag{11}$$

or

$$1 \leqq 2Q + \int_{\xi_n-\tau}^{\xi_n} |p(s)+q(s-\tau+\sigma)|ds.$$

This, for n large, contradicts (4) and proves our claim that every oscillatory solution of Eq. (1) is bounded. Next, it remains to show that every bounded and oscillatory solution $x(t)$ of Eq. (1) tends to zero as $t \to \infty$. Otherwise $x(t)$ does not tend to zero as $t \to \infty$. Set $\mu = \limsup_{t\to\infty} |x(t)|$. Then $\mu > 0$ and for any $\varepsilon > 0$, there exists a $t_3 \geqq t_2$ such that

$$|x(t)| < \mu + \varepsilon, \quad t \geqq t_3.$$

From (5) we get

$$|z(t)| \geqq |x(t)| - (\mu + \varepsilon)Q.$$

Thus,

$$\limsup_{t\to\infty} |z(t)| \geqq \mu - (\mu + \varepsilon)Q$$

and because ε is arbitrary,

$$\limsup_{t\to\infty} |z(t)| \geqq \mu(1 - Q) > 0.$$

Set $K = \limsup_{t\to\infty} |z(t)|$. Then $K \geqq \mu(1 - Q) > 0$. From (7), we see that $z'(t)$ oscillates and so there exists a sequence $\{\xi_n\}$ such that

$$\lim_{n\to\infty} \xi_n = \infty, \quad z'(\xi_n) = 0 \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad \lim_{n\to\infty} |z(\xi_n)| = K.$$

As before, (9) and (10) are also valid and so

$$|z(\xi_n)| \leqq (\mu + \varepsilon)Q + (\mu + \varepsilon) \limsup_{t\to\infty} \int_{t-\tau}^{t} |p(s) + q(s - \tau + \sigma)| ds.$$

Hence

$$\mu(1 - Q) \leqq K \leqq (\mu + \varepsilon) [Q + \limsup_{t\to\infty} \int_{t-\tau}^{t} |p(s) + q(s - \tau + \sigma)| ds]$$

which implies that

$$1 \leqq 2Q + \limsup_{t\to\infty} \int_{t-\tau}^{t} |p(s) + q(s - \tau + \sigma)| ds.$$

This contradicts (4) and the proof is complete.

REMARK 1. The conclusion of Theorem 1 remains true if we reverse the roles of p and q in the hypotheses of the theorem. This idea is sometimes useful as can be seen from the following example.

EXAMPLE 1. Consider the delay differential equation

$$x'(t) - e^{-\pi} x(t - \pi) + e^{-\pi/2} x(t - \pi/2) = 0. \tag{12}$$

If we take

$$p = -e^{-\pi}, \quad q = e^{-\pi/2}, \quad \tau = \pi, \quad \text{and} \quad \sigma = \pi/2$$

then condition (3) of Theorem 1 is not satisfied. On the other hand, if we take

$$p = e^{-\pi/2}, \quad q = -e^{-\pi}, \quad \tau = \pi/2, \quad \text{and} \quad \sigma = \pi$$

then the hypotheses of Theorem 1 are satisfied and so every oscillatory solution of Eq. (12) tends to zero as $t \to \infty$. For example, $x(t) = e^{-t} \sin t$ is such a solution.

Consider the delay differential equation

$$x'(t) + px(t-\tau) - qx(t-\sigma) = 0 \tag{13}$$

where p, q, τ, and σ are positive constants. In [2], Arino, Ladas and Sficas have shown that under the hypothesis

$$(\mathrm{H}_1) \qquad 0 < q < p, \quad 0 < \sigma \leqq \tau, \quad \text{and} \quad q(\tau - \sigma) \leqq 1$$

every nonoscillatory solution of Eq. (13) tends to zero as $t \to \infty$. On the other hand, Condition (3), applied to Eq. (13) becomes

$$(\mathrm{H}_2) \qquad 2q|\tau - \sigma| + |p - q|\tau < 1.$$

On the basis of the above discussion and Theorem 1 we obtain the following stability result.

COROLLARY 1. *Assume that p, q, τ, and σ are constants and that Hypotheses* (H_1) *and* (H_2) *are satisfied. Then the trivial solution of Eq.* (13) *is globally uniformly asymptotically stable.*

Assume that the coefficients p and q of Eq. (1) are positive continuous functions satisfying the condition

$$(\mathrm{H}_3) \qquad \liminf_{t \to \infty} (\tau p(t) + \sigma q(t)) > 1/e.$$

Then, it follows by a result of Hunt and Yorke [4] that every solution of Eq. (1) oscillates. On the basis of this and Theorem 1 we have the following stability result.

COROLLARY 2. *Assume that the coefficients p and q of Eq.* (1) *are positive continuous functions satisfying* (H_3) *and*

$$2 \limsup_{t \to \infty} \left| \int_{t-\tau}^{t-\sigma} q(s+\sigma) ds \right| + \limsup_{t \to \infty} \int_{t-\tau}^{t} |p(s) + q(s - \tau + \sigma)| ds < 1.$$

Then the trivial solution of Eq. (1) *is globally asymptotically stable.*

REMARK 2. Corollaries 1 and 2, above, are examples of the two typical applications of Theorem 1 in establishing stability results about the solutions of Eq. (1). In general, when we know the asymptotic behavior of the nonoscillatory solutions of Eq. (1) or when we know that every solution of Eq. (1) oscillates, then Theorem 1 may be used to obtain the stability nature of the trivial solution of Eq. (1).

3. Asymptotic behavior of neutral equations

In this section we present sufficient conditions for all oscillatory solutions of the neutral delay differential equation (2) to tend to zero as $t\to\infty$. Thus, as we explained in Remark 2, when we know that every solution of Eq. (2) oscillates or when we know the asymptotic behavior of the nonoscillatory solutions of Eq. (2), then we will have a stability result for all solutions of Eq. (2).

THEOREM 2. *Consider the neutral delay differential equation*

$$(d/dt)[x(t)-px(t-\tau)] + q(t)x(t-\sigma) = 0, \quad t \geqq t_0 \tag{2}$$

where the delays τ and σ are nonnegative constants, p is a constant, q is a continuous function such that $q(t)\neq 0$ for t sufficiently large, and

$$2|p| + \limsup_{t\to\infty} \int_{t-\sigma}^{t} |q(s)|ds < 1. \tag{14}$$

Then every oscillatory solution of Eq. (2) *tends to zero as $t\to\infty$.*

PROOF. First, we will prove that every oscillatory solution of Eq. (2) is bounded. Otherwise, there exists an oscillatory solution $x(t)$ of Eq. (2) which is unbounded. Set $t_1 = t_0+\tau$ and

$$z(t) = x(t) - px(t-\tau), \quad t \geqq t_1 .$$

Let $t_2 \geqq t_1$ be such that

$$|x(t_2)| \geqq \max_{t_0\leqq t\leqq t_2} |x(t)| .$$

This is possible because $x(t)$ is unbounded. Then for $t\geqq t_2$ we have

$$|z(t)| \geqq |x(t)| - |p|(\max_{t_1\leqq s\leqq t} |x(s-\tau)|) \geqq |x(t)| - |p|(\max_{t_1\leqq s\leqq t} |x(s)|)$$

and so

$$\max_{t_1\leqq s\leqq t} |z(s)| \geqq (\max_{t_1\leqq s\leqq t} |x(s)|)(1-|p|). \tag{15}$$

In particular, (15) implies that $z(t)$ is unbounded. Since

$$z'(t) = - q(t)x(t-\sigma) \tag{16}$$

and $x(t)$ is oscillatory, it follows that $z'(t)$ is also oscillatory. This, together with the fact that $z(t)$ is unbounded, implies that there exists a sequence of points $\{\xi_n\}$ such that

$$\lim_{n\to\infty} \xi_n = \infty, \quad z'(\xi_n) = 0 \qquad \text{for} \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} |z(\xi_n)| = \infty,$$

and

$$(17) \qquad |z(\xi_n)| = \max_{t_1 \leqq s \leqq \xi_n} |z(s)| \qquad \text{for} \quad n = 1, 2, \ldots.$$

From (16) and the fact that $q(t) \neq 0$ for t large we see that $x(\xi_n - \sigma) = 0$ for n large, say $n \geqq n_0$. Thus $z(\xi_n - \sigma) = -px(\xi_n - \sigma - \tau)$ for $n \geqq n_0$. Integrating both sides of (16) from $\xi_n - \sigma$ to ξ_n we obtain

$$(18) \qquad z(\xi_n) = -px(\xi_n - \sigma - \tau) - \int_{\xi_n - \sigma}^{\xi_n} q(s)x(s-\sigma)ds$$

and using (17) and (15) we see that for $n \geqq n_0$,

$$|z(\xi_n)| \leqq |p| \frac{1}{1-|p|} |z(\xi_n)| + \frac{1}{1-|p|} |z(\xi_n)| \int_{\xi_n - \sigma}^{\xi_n} |q(s)| ds.$$

Simplifying this inequality we get, for n large,

$$1 \leqq 2|p| + \int_{\xi_n - \sigma}^{\xi_n} |q(s)| ds$$

which contradicts (14) and proves our claim that every oscillatory solution of Eq. (2) is bounded. Next, it remains to show that every bounded and oscillatory solution of Eq. (2) tends to zero as $t \to \infty$. Assume, for the sake of contradiction, that $x(t)$ does not tend to zero as $t \to \infty$. Set $\mu = \limsup_{t\to\infty} |x(t)|$. Then $\mu > 0$ and for any $\varepsilon > 0$, there exists a $t_3 \geqq t_2$ such that

$$(19) \qquad |x(t)| < \mu + \varepsilon, \quad t \geqq t_3.$$

Then

$$|z(t)| \geqq |x(t)| - (\mu + \varepsilon)|p|$$

and

$$\limsup_{t\to\infty} |z(t)| \geqq \mu - (\mu + \varepsilon)|p|.$$

As ε is arbitrary, it follows that

$$(20) \qquad K \equiv \limsup_{t\to\infty} |z(t)| \geqq \mu(1 - |p|) > 0.$$

From (16), we see that $z'(t)$ oscillates and so there exists a sequence $\{\xi_n\}$ such that

$\lim_{n\to\infty} \xi_n = \infty$, $z'(\xi_n) = 0$ for $n = 1, 2, \ldots$ and $\lim_{n\to\infty} |z(\xi_n)| = K$.

As before, (18) is also true and so using (19) we obtain

$$|z(\xi_n)| \leqq |p|(\mu+\varepsilon) + (\mu+\varepsilon)\int_{\xi_n-\sigma}^{\xi_n} |q(s)|ds.$$

Using (20), we find

$$\mu(1-|p|) \leqq (\mu+\varepsilon)\Big[|p| + \limsup_{t\to\infty}\int_{t-\sigma}^{t} |q(s)|ds\Big]$$

and since ε is arbitrary we get

$$1 \leqq 2|p| + \limsup_{t\to\infty}\int_{t-\sigma}^{t} |q(s)|ds$$

which contradicts (14) and completes the proof of the theorem.

In [6], Ladas and Sficas studied the asymptotic behavior of the nonoscillatory solutions of Eq. (2) and found sufficient conditions for all solutions to oscillate when $0<p<1$ and the coefficient q is a positive function. In [3], Grammatikopoulos, Grove and Ladas examined the same questions for all other values of the real parameter p. Combining Theorem 5 of [6] and Theorem 9 of [3] we obtain the following result about the asymptotic behavior of the nonoscillatory solutions of Eq. (2).

LEMMA 1. *Consider the neutral delay differential equation* (2) *where the delays* τ *and* σ *are nonnegative constants,* p *is a real number and* q *is a nonnegative continuous function. Assume that one of the following conditions hold.*

$$(\mathrm{H}_4) \qquad p \leqq 0$$

or

$$(\mathrm{H}_5) \qquad 0 < p < 1 \quad \text{and} \quad \int_{t_0}^{\infty} q(s)ds = \infty.$$

Then every nonoscillatory solution of Eq. (2) *tends to zero as* $t\to\infty$.

Finally, from Theorem 2 and Lemma 1 we have the following.

COROLLARY 3. *Assume that the hypotheses of Lemma* 1 *are satisfied, that* $q(t)\neq 0$ *for* t *sufficiently large, and that*

$$2|p| + \limsup_{t\to\infty}\int_{t-\sigma}^{t} q(s)ds < 1.$$

Then the trivial solution of Eq. (2) *is globally asymptotically stable.*

REMARK 3. The case $p=0$ of Corollary 3 reduces to Theorem 3 in [7].

EXAMPLE 2. Consider the neutral delay differential equation

$$(d/dt)[x(t)-px(t-\pi)] + Q(2+\cos t)x(t-\pi) = 0, \quad t \geqq 0 \tag{21}$$

where p and Q are constants such that

$$Q > 0 \quad \text{and} \quad 2|p| + Q(\pi+1) < 1.$$

Then

$$\int_0^\infty q(s) = \infty, \int_{t-\pi}^t q(s)ds = 2Q(\pi+\sin t)$$

$$\text{and} \quad \limsup_{t\to\infty} \int_{t-\pi}^t q(s)ds = 2Q(\pi+1).$$

It follows that the hypotheses of Corollary 3 are satisfied and therefore every solution of Eq. (21) tends to zero at $t\to\infty$.

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