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Valuations of a quasi-pythagorean field

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In [3], B. Jacob constructed valuations of a formally real pythagorean field and used them to clarify the structure of such a field. We show in this paper that his method is applicable to a quasi-pythagorean field.

All fields are assumed to be formally real.

§1. Valuations

Let F be a (formally real) field and T be a fan of F with $[\dot{F}: \dot{T}] \geqq 4$. We denote by T^2 the set $\{x^2; x \in T\}$ and by $[\alpha]$ the class of $\alpha \in \dot{F}$ in $\dot{F}/\dot{T}$.

Let $R(T)$ be the subgroup $\{[\beta] \in \dot{F}/\dot{T}; T^2 - \beta^2 T^2$ represents non trivial elements of $\dot{R}/\pm\dot{T}\}$. Then as shown in [3], $R(T) = \{\pm 1\}$ or $R(T) = \{\pm 1, \pm[\alpha]\}$ for some $\alpha \in \dot{F}$, where we denote $[1]$ by 1 and $[-\alpha]$ by $-[\alpha]$. For a subgroup $\hat{R}$ of $\dot{F}/\dot{T}$ containing $R(T)$, we define:

$$O_1(T, \hat{R}) = \{x \in \dot{F}; [x] \notin \hat{R} \text{ and } [1+x] = 1\} \cup \{0\},$$

$$O_2(T, \hat{R}) = \{x \in \dot{F}; [x] \in \hat{R} \text{ and } xO_1(T, \hat{R}) \subseteq O_1(T, \hat{R})\}, \text{ and}$$

$$O(T, \hat{R}) = O_1(T, \hat{R}) \cup O_2(T, \hat{R}).$$

Then we have

THEOREM 1.1. *$O(T, \hat{R})$ is a valuation ring of F which is fully compatible with T, that is, $1 + M \subseteq T$ for the maximal ideal M of $O(T, \hat{R})$. If $\hat{R}$ equals $R(T)$, then for the image $\bar{T}$ of $T \cap O(T, \hat{R})$ in the residue field $\bar{F}$, we have $[\bar{F}^{\cdot}: \bar{T}^{\cdot}] \leqq 4$.*

This theorem was proved in [3] with the assumption that F is pythagorean. But the assumption may be removed (compare [6], Theorem 3.3).

Now we generalize Theorem 1 of [3] as follows.

THEOREM 1.2. *$O(T, \hat{R})$ is fully compatible with a preordering S of F if and only if $[1-t] \in \hat{R}$ for all $t \in \dot{T} \setminus \dot{S}$.*

PROOF. Suppose that $O(T, \hat{R})$ is fully compatible with S, but $[1-t] \notin \hat{R}$ for some $t \in \dot{T} \setminus \dot{S}$. Then $1-t$ is not a unit, for every unit is an element of $\hat{R}$. If we have $t \in O(T, \hat{R})$, then $t = 1-(1-t) \in 1 + M \subseteq S$ which is a contradiction. So we have $t \notin O(T, \hat{R})$ and $\mathrm{ord}(t) = \mathrm{ord}(1-t) < 0$. Hence $t^{-1} - 1 = t^{-1}(1-t)$ is a unit. But we have $[t^{-1}-1]=[t^{-1}]\ [1-t]\notin\hat{R}$ which is a contradiction.

Conversely suppose that $[1-t]\in\hat{R}$ for all $t\in\dot{T}\setminus\dot{S}$. Then we have $[t_1-t_2]\in\hat{R}$ for all t_1, $t_2\in\dot{T}$ with $t_1\dot{S}\neq t_2\dot{S}$. Suppose that $x=1+m\notin\dot{S}$ for some $m\in M$. It follows from $x\dot{S}\neq\dot{S}$ that $[m]=[x-1]\in\hat{R}$. For any $y\in O_1(T,\hat{R})$ we have $[x+y/2]=[x]=1$, because x is a unit contained in T. Thus we have $(x+y/2)\dot{S}=x\dot{S}$ and similarly $(1-y/2)\dot{S}=\dot{S}$. So we have $[x+y/2-(1-y/2)]\in\hat{R}$, that is, $[m+y]\in\hat{R}$. Since $y\notin\hat{R}$, it follows that $[m+y]=[m]$. This means $[1+m^{-1}y]=1$ whence $m^{-1}\in O_2(T,\hat{R})\subseteq O(T,\hat{R})$, a contradiction. Q.E.D.

§2. The case of a quasi-pythagorean field

From now on we always assume that F is a quasi-pythagorean field. In other words we assume that Kaplansky's radical $R(F):=\{a\in\dot{F};\ D_F<1,\ -a>=\dot{F}\}$ coincides with $D_F(2)$. Then we know $R(F)\cup\{0\}$ is the weak preordering $\sum F^2$ which we denote by S in the rest of this paper. We denote by $(X_F,\ \dot{F}/\dot{S})$ the space of orderings of F. We refer to [5] for spaces of orderings, especially for the group extension of a space and the direct sum of spaces.

THEOREM 2.1. *Let F be a quasi-pythagorean field and $(X_F,\dot{F}/\dot{S})=(X',G')\times H$ be a proper* (i.e., $H\neq 1$) *group extension of a space (X',G'), which itself is not a proper group extension. Suppose that S is not a trivial fan. Then there is a valuation v on F which satisfies the following conditions*:

(i) *v is fully compatible with S,*

(ii) *$(X_{\bar{F}},\ \bar{F}^{\cdot}/\bar{S}^{\cdot})\sim(X',G')$ and $\Gamma/\Gamma^2\cong H$,*

where $\bar{F}$ and Γ are the residue field and the value group of v respectively and $\sim$ denotes an equivalence of spaces.

PROOF. If we replace $\alpha\dot{F}^2$ for $\alpha\in\dot{F}$ by $\alpha\dot{S}$, then all the arguments in [3] are valid. So we see that for a minimal fan T of (X',G') which is (regarded as a fan of F) different from $\dot{S}$, we may set $G'=\hat{R}$ in Theorem 1.1. Thus we have the valuation ring $O(T,G')$. We show that the valuation v which corresponds to $O(T,G')$ satisfies the conditions stated in the theorem. For $t\in\dot{T}\setminus\dot{S}$ we see that $(1-\alpha)\dot{S}\in G'$, for otherwise there would be an ordering of F in which $\alpha<0$ and $1-\alpha<0$. So $O(T,G')$ is fully compatible with S by Theorem 1.2. It is easily seen that $\bar{F}^{\cdot}/\bar{S}^{\cdot}$ is isomorphic to a subgroup of G'. Since we suppose (X',G') is not a proper group extension and $(X_F,\dot{F}/\dot{S})\sim(X_{\bar{F}},\bar{F}^{\cdot}/\bar{S}^{\cdot})\times\Gamma/\Gamma^2$ by Corollary 3.11 of [4], we have $(X',G')\sim(X_{\bar{F}},\ \bar{F}^{\cdot}/\bar{S}^{\cdot})$ and $H\cong\Gamma/\Gamma^2$. Q.E.D.

COROLLARY 2.2. *In the situation of Theorem 2.1, a 2-henselization $\tilde{F}$ of F with respect to v is a pythagorean field and we have $(X_F,\dot{F}/\dot{S})\sim(X_{\tilde{F}},\ \tilde{F}^{\cdot}/\tilde{F}^{\cdot 2})$.*

PROOF. By Theorem 2.1, $(X_F,\dot{F}/\dot{S})\sim(X_{\bar{F}},\bar{F}^{\cdot}/\bar{F}^{\cdot 2})\times\Gamma/\Gamma^2$. Since $\Gamma/\Gamma^2\cong H\neq 1$,

$\bar{F}$ is pythagorean by [2], Proposition 1.3. So $\tilde{F}$ is also pythagorean by [4], Theorem 3.16. As $\tilde{F}$ is an immediate extension of F, we have $(X_F, \dot{F}/\dot{S}) \sim (X_{\tilde{F}}, \tilde{F}^{\cdot}/\tilde{F}^{\cdot 2})$.

Q.E.D.

Now we consider the case where $(X_F, \dot{F}/\dot{S})$ has a finite chain length so that it is a direct sum of elementary indecomposable spaces. Thus $(X_F, \dot{F}/\dot{S}) = (X_1, G_1) \oplus \cdots \oplus (X_m, G_m)$, where (X_i, G_i) is one element space or a proper group extension of some space (X_i', G_i').

THEOREM 2.3. *In the above situation, we have* $(X_i, G_i) \sim (X_{F_i}, \dot{F}_i/\dot{F}_i^2)$ *for some pythagorean field* F_i *contained in the maximal* 2*-extension* $F(2)$ *of* F.

PROOF. Fix i for which $G_i \neq 1$, so that (X_i, G_i) is a group extension of (X_i', G_i'). Then (X_i, G_i) contains a fan which we denote by T_i. If we replace $\alpha\dot{F}^2$ by $\alpha\dot{S}$ in the proof of Theorem 4 of [3], we see that $\hat{T} = T_i \oplus (\oplus_{j\neq i} G_j)$ may be regarded as a fan of F and that $R(\hat{T}_i) \subseteq G_i' \oplus (\oplus_{j\neq i} G_j)$. Thus for $\hat{R} = G_i \oplus (\oplus_{j\neq i} G_j)$ we obtain a valuation ring $O(\hat{T}_i, \hat{R}_i)$ by Theorem 1.1. Let F_i be a 2-henselization of F with respect to the valuation v_i corresponding to $O(\hat{T}_i, \hat{R}_i)$. We show that $\dot{F}_i/\dot{S}_i \cong G_i$ where S_i denotes the weak preordering of F_i. Let φ be the homomorphism which makes the following diagram commutative (where the maps other than φ are obvious ones):

$$\begin{array}{ccc} \dot{F}/\dot{S} & \longrightarrow & \oplus_i G_i \\ \big\downarrow & & \big\downarrow \varphi \\ \dot{F}_i/\dot{S}_i & \longrightarrow & \bar{F}^{\cdot}/\bar{S}^{\cdot} \times \Gamma/\Gamma^2 \end{array} .$$

Then we see, by following the proof of Theorem 3 of [3], that $\oplus_{j\neq i} G_j \subseteq \operatorname{Ker}\varphi$ and that the restriction of φ to G_i is injective. Thus $\dot{F}_i/\dot{S}_i \cong G_i$. From this it follows that $(X_{\bar{F}i}, \bar{F}_i/\bar{S}_i) \sim (X_i', G_i')$ and that F_i is pythagorean as in the proof of Corollary 2.2. So we have $\dot{S}_i = \dot{F}_i^2$ and $(X_{F_i}, \dot{F}_i/\dot{F}_i^2) \sim (X_i, G_i)$. If $G_i = 1$, then we may take an euclidean closure of F for F_i.

Q.E.D.

Now we apply above theorem to the problem treated in [1].

THEOREM 2.4 *Let* F *be a quasi-pythagorean field for which the chain length of* X_F *is finite. Then the canonical homomorphisms* $h_n\colon k_nF \to H^n(F, 2)$ *are injective for all* n.

PROOF. We may assume that $(X_F, \dot{F}/\dot{S}) = (X_1, G_1) \oplus \cdots \oplus (X_m, G_m)$ in the notation before Theorem 2.3 (cf. [5]). Now consider the following commutative diagram:

$$\begin{array}{ccc} k_nF & \xrightarrow{h_n(F)} & H^n(F, 2) \\ \varphi\Big\downarrow & & \Big\downarrow\psi \\ \oplus_i k_nF_i & \xrightarrow{\oplus h_n(F_i)} & \oplus_i H^n(F_i, 2) \end{array}$$

where F_i are pythagorean fields obtained in Theorem 2.3, and φ, ψ are natural homomorphisms. We showed in [1], Theorem 1.5 that $k_nF \cong I^nF/I^{n+1}F$ for $n \geqq 2$, and $I^nF/I^{n+1}F \cong \oplus_i I^nF_i/I^{n+1}F_i$ by the structure of X_F. Thus φ is an isomorphism. Since $h_n(F_i)$ is an isomorphism by [3], Theorem 6, we see that $h_n(F)$ is injective (and ψ is surjective) for $n \geqq 2$. $h_1(F)$ is an isomorphism for any field F. Q.E.D.

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