HIROSHIMA MATH. J.
18 (1988), 541–597

Riesz potentials, higher Riesz transforms and Beppo Levi spaces

Takahide KUROKAWA

(Received December 23, 1987)

Table of contents

§1. Introduction

Let R^n be the n-dimensional Euclidean space and f be a continuous function on R^n with compact support. For a positive integer l with $2l<n$, a solution of the equation

(1.1) $$\Delta^l u = c_{l,n} f$$

is given by

$$U_{2l}^f(x) = \int |x-y|^{2l-n} f(y) dy,$$

where $c_{l,n} = (2l-n)(2l-2-n)\cdots(2-n)2^l(l-1)!\pi^{n/2}/\Gamma(n/2)$. The function U_m^f is called the Riesz potential of order m of f. In particular, U_2^f is the Newton potential of f. Naturally, the following problem arises: Find a representation of a solution of the equation (1.1) for any positive integer l and any L^p-function f. We note here that for an L^p-function f, U_m^f does not necessarily exist in case $m-(n/p)\geqq 0$.

Let m be a positive integer and $p>1$. We denote by $\mathscr{L}_m^p$ the space of all distributions u such that $D^\alpha u \in L^p$ for any $|\alpha| = m$. If $m-(n/p)<0$, then $u\in\mathscr{L}_m^p$ can be represented as follows ([12]):

(1.2) $$u(x) = \sum_{|\gamma|\leqq m-1} a_\gamma x^\gamma + U_m^f(x), \quad f\in L^p.$$

We are also concerned with the following problem: For any positive integer m and p

>1, represent $u \in \mathscr{L}_m^p$ in a form like (1.2).

To answer the above problems, we need to introduce potentials of order m of L^p-functions for any positive integer m and any $p>1$. For a positive integer m, the Riesz kernel of order m is given by

$$\kappa_m(x) = \begin{cases} |x|^{m-n}, & m<n \quad \text{or} \quad m \geqq n, \quad m-n \quad \text{odd}, \\ (\delta_{m,n} - \log|x|)|x|^{m-n}, & m \geqq n, \quad m-n \quad \text{even}, \end{cases}$$

where $\delta_{m,n}$ are suitable constants (see §3). Further, for an integer $k \leqq m-1$ we consider the kernel

$$\kappa_{m,k}(x, y) = \begin{cases} \kappa_m(x-y) - \sum_{|\gamma| \leq k} (x^\gamma/\gamma!) D^\gamma \kappa_m(-y), & 0 \leqq k \leqq m-1, \\ \kappa_m(x-y), & k \leqq -1 \end{cases}$$

and potentials

$$U_{m,k}^f(x) = \int \kappa_{m,k}(x, y) f(y) dy.$$

The kernels $\kappa_{2,k}(x, y)$ appeared in the context of integral representations of subharmonic functions ([9], (17]). In §3, we show the existence and integral estimates of $U_{m,k}^f$ for $f \in L^p$ and $k=[m-(n/p)]$ (the integral part of $m-(n/p)$) in case $m-(n/p) \neq 0, 1, \cdots, m-1$. An inequality by G. O. Okikiolu [16] plays a central role in the study of the integral estimates. We also discuss the case $m-(n/p)=0, 1, \cdots, m-1$. In case $m-(n/p)<0$, the integral estimates are given in S. L. Sobolev [22], E.M. Stein and G. Weiss [25] and D. R. Adams [1]. E. Sawyer [20; Proposition 3.2] gives a weighted norm inequality for $U_{m,k}^f$ with $f \in L^2$ under several conditions. In §4, we introduce higher Riesz transforms of $f \in L^p$, by which we can express partial derivatives of order m of $U_{m,k}^f$. This is a generalization of M. Ohtsuka [15; Theorem 9.6] and Y. Mizuta [14; Theorem 5.1]. In §5, we give a representation of a primitive of higher order, as an analogue of the integral representation given by Yu. G. Reshetnyak [18; Lemma 6.2]. We establish reltionship between the primitive of order m and the potential of order m. By using this relationship we make some improvements of integral estimates for $U_{m,k}^f$. In §6, we investigate the Beppo Levi space $\mathscr{L}_m^p$, and give potential and integral representations of Beppo Levi functions for arbitrary positive integer m and $p>1$. Note that M. Ohtsuka [15; Theorem 9.11] and Y. Mizuta [14; Theorem 5.2] give an expression of Beppo Levi functions by U_m^f under some conditions on m and p. We also investigate characterizations of the closure L_m^p *of* $\mathscr{D}$ in $\mathscr{L}_m^p$, where $\mathscr{D}$ is the class of all infinitely differentiable functions with compact support. These results include a characterization by P. I. Lizorkin [12; Theorem 4]. As a consequence, we give a representation of a solution of the equation (1.1) for any positive integer l and any L^p-function f. In §7, using potential representations of Beppo Levi functions, we establish embedding theorems and interpolation theorems for the spaces L_m^p. As corollaries we obtain some extensions of Sobolev's embedding theorem [2; Theorem 5.4].

§2. Notation and preliminaries

We use R^n to denote the n-dimensional Euclidean space ($n \geqq 2$) and for each point $x=(x_1,\cdots, x_n)$ we write $|x|=(x_1{}^2+\cdots+x_n^2)^{1/2}$. For a positive number r, we write $B_r=\{x; |x|<r\}$ and $S_r=\{x; |x|=r\}$. We denote by $\omega_n(r)$ and $\sigma_n(\mathrm{r})$ the volume of B_r and the surface area of S_r, respectively. We simply write $\omega_n(1)=\omega_n$ and $\sigma_n(1)=\sigma_n$. For a nonnegative integer k, C^k stands for the space of all k times continuously differentiable functions on R^n, and C^∞ denotes the space of all infinitely differentiable functions on R^n. According to L. Schwartz [20], $\mathscr{E}$ denotes the Fréchet space consisting of all C^∞-functions on R^n, $\mathscr{S}$ stands for the Fréchet space consisting of all C^∞-functions rapidly decreasing at infinity and $\mathscr{D}$ denotes the LF-space consisting of all C^∞-functions with compact support. The symbols $\mathscr{E}'$, $\mathscr{S}'$ and $\mathscr{D}'$ stand for the topological duals of $\mathscr{E}$, $\mathscr{S}$ and $\mathscr{D}$, respectively. We use the symbol $<u, \phi>$ for the canonical bilinear form on $\mathscr{E}'\times\mathscr{E}$ or $\mathscr{S}'\times\mathscr{S}$ or $\mathscr{D}'\times\mathscr{D}$. The inclusion relations $\mathscr{E}'\subset\mathscr{S}'\subset\mathscr{D}'$ hold. We call an element of $\mathscr{D}'$ a distribution.

If $\alpha=(\alpha_1,\cdots,\alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index and ednote by x^α the monomial $x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, which has degree $|\alpha|=\sum_{j=1}^n\alpha_j$. If α and β are two multi-indices, we write $\alpha\geqq\beta$ provided $\alpha_j\geqq\beta_j$ for $1\leqq \mathrm{j}\leqq \mathrm{n}$. We also write

$$\alpha! = \alpha_1!\cdots\alpha_n!, \quad \alpha+\beta=(\alpha_1+\beta_1,\cdots, \alpha_n+\beta_n)$$

and if $\alpha\geqq\beta$

$$\binom{\alpha}{\beta}=\binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n}, \quad \alpha-\beta=(\alpha_1-\beta_1,\cdots, \alpha_n-\beta_n).$$

For $1\leqq j\leqq n$, e_j stands for the multi-index $(0,\cdots,\overset{j}{1},\cdots,0)$. If $D_j=\partial/\partial x_j$ for $1\leqq j\leqq n$, then

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}$$

gives a differential operator of order $|\alpha|$. The Laplace operator on R^n is

$$\Delta=\sum_{j=1}^n(\partial^2/\partial x_j^2)=\sum_{j=1}^n D^{2e_j}=\sum_{j=1}^n D_j^2$$

and its iterations are denoted by Δ^l, $l=1, 2,\cdots$. The Leibniz formula

$$D^\alpha(fg)\,(x)=\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}D^\beta f(x)D^{\alpha-\beta}g(x)$$

is valid for functions f and g which are $|\alpha|$ time continuously differentiable.

Throughout this paper let $1<p<\infty$ and $(1/p)+(1/p')=1$. As usual we denote by L^p the class of all measurable functions for which

$$\|f\|_p=\left(\int|f(x)|^p dx\right)^{1/p}<\infty.$$

Moreover, for a measurable set $E\subset R^n$, we put

$$L^p(E)=\{f\in L^p; f(x)=0 \quad \text{for} \quad x\notin E\}.$$

For a function f on R^n and a set $E\subset R^n$, we write

$$f|_E(x)=\begin{cases} f(x), & x\in E,\\ 0, & x\notin E.\end{cases}$$

For measurable functions f and g, $f*g$ stands for the convolution product of f and g, that is,

$$f*g(x)=\int f(x-y)g(y)dy$$

if the integral exists for almost every x. For distributions u and v, we denote their convolution also by $u*v$, if it exists.

If f is an integrable function, then the Fourier transform of f is the function $\mathscr{F}f=\hat{f}$ defined by

$$\mathscr{F}f(x)=\int e^{-2\pi i x\cdot y}f(y)dy$$

for all $x\in R^n$ where $x\cdot y=\sum_{j=1}^n x_j y_j$. Moreover, we define the Fourier transform $\mathscr{F}u$ of $u\in\mathscr{S}'$ to be the element of $\mathscr{S}'$ whose value at $\phi\in\mathscr{S}$ is

$$<\mathscr{F}u, \phi> = <u, \mathscr{F}\phi>.$$

For $u\in\mathscr{S}'$ and a multi-index α, the following formula holds:

$$\mathscr{F}(D^\alpha u)=(2\pi i x)^\alpha\mathscr{F}u.$$

For a real number r, $[r]$ denotes the integral part of r.

Throughout this paper, we use the symbol C for generic positive constant whose value may be different at each occurrence, even on the same line.

§3. Integral inequalities and Riesz potentials of order (m, k)

3.1. Kernels of order (m, k)

We begin with some observations on Taylor's theorem. Let $u \in C^\infty$. For a nonnegative integer k, by Taylor's theorem we have

$$(3.1)\quad u(x) = \sum_{|\gamma| \leqq k} (x^\gamma/\gamma!) D^\gamma u(0) + (k+1) \sum_{|\gamma| = k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} (x')^\gamma D^\gamma u(tx') dt$$

where $x' = x/|x|$ $(x \neq 0)$. We put

$$u_k(x) = \begin{cases} (k+1) \sum_{|\gamma| = k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} (x')^\gamma D^\gamma u(tx') dt, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We note that

$$(3.2)\qquad D^\beta u_k(0) = 0 \quad \text{for} \quad |\beta| \leqq k,$$

$$(3.3)\qquad D^\alpha u_k(x) = D^\alpha u(x) \quad \text{for} \quad |\alpha| \geqq k+1.$$

For a multi-index δ with $|\delta| \leqq k$, applying (3.1) to $D^\delta u_k$ and using (3.2) and (3.3), we obtain

$$(3.4)\qquad \begin{aligned} D^\delta u_k(x) &= \sum_{k-|\delta|+1 \leqq |\gamma| \leqq k} (x^\gamma/\gamma!) D^{\gamma+\delta} u(0) \\ &\quad + (k+1) \sum_{|\gamma| = k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!} (x')^\gamma D^{\gamma+\delta} u(tx') dt. \end{aligned}$$

Let m be a positive integer, and K_m be a homogeneous function of degree $m-n$ which is infinitely differentiable in $R^n - \{0\}$. For a multi-index α, $D^\alpha K_m(x)$ is a homogeneous function of degree $m-n-|\alpha|$, and hence

$$(3.5)\qquad |D^\alpha K_m(x)| \leqq C|x|^{m-n-|\alpha|}.$$

Furthermore, in case $m \geqq n$, let $L_m(x) = P_m(x) \log|x|$ where $P_m(x)$ is a homogeneous polynomial of degree $m-n$. For a multi-index α, we see that

$$D^\alpha L_m(x) = H_\alpha(x) \log|x| + h_\alpha(x),$$

where $H_\alpha(x)$ is a homogeneous polynomial of degree $m-n-|\alpha|$ and $h_\alpha(x)$ is a homogeneous function of degree $m-n-|\alpha|$, and for $|\alpha| \geqq m-n+1$, $H_\alpha(x) = 0$. Hence we have

$$(3.6) \qquad |D^\alpha L_m(x)| \leqq C|x|^{m-n-|\alpha|}(1+|\log|x||), \qquad \text{for} \quad |\alpha| \leqq m-n,$$

$$(3.7) \qquad |D^\alpha L_m(x)| \leqq C|x|^{m-n-|\alpha|}, \qquad \text{for} \quad |\alpha| \geqq m-n+1.$$

For an integer $k \leqq m-1$, we set

$$K_{m,k}(x, y) = K_m(x-y) - \sum_{|\gamma| \leqq k}(x^\gamma/\gamma!)D^\gamma K_m(-y),$$

where we regard the second term of the right-hand side as zero if $k \leqq -1$. For a multi-index α with $|\alpha| \leqq m-1$, it is easily seen that

$$(3.8) \quad D_x^\alpha K_{m,k}(x, y) = \begin{cases} D^\alpha K_m(x-y) - \sum_{|\gamma| \leqq k, \gamma \geqq \alpha} \dfrac{x^{\gamma-\alpha}}{(\gamma-\alpha)!} D^\gamma K_m(-y), & \text{for} \quad |\alpha| \leqq k, \\ D^\alpha K_m(x-y), \quad \text{for} \quad k+1 \leqq |\alpha| \leqq m-1, \end{cases}$$

where D_x^α denotes the differentiation with respect to x. Furthermore, for an integer k with $0 \leqq k \leqq m-1$, we set

$$L_{m,k}(x, y) = L_m(x-y) - \sum_{|\gamma| \leqq k}(x^\gamma/\gamma!)D^\gamma L_m(-y).$$

We put $l_x = \{tx; 0 \leqq t \leqq 1\}$ and denote by $d(y, l_x)$ the distance between y and l_x. For $y \notin l_x$ and $k \geqq 0$, it follows from (3.1) that

$$K_{m,k}(x, y) = (k+1)\sum_{|\gamma| = k+1} \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!}(x')^\gamma D^\gamma K_m(tx'-y)dt.$$

We shall give estimates of $K_{m,k}(x, y)$ and $L_{m,k}(x, y)$ for $d(y, l_x) \geqq |x|/2$.

LEMMA 3.1. (i) *Let k be a nonnegative integer with $k \leqq m-1$ and α be a multi-index with $|\alpha| \leqq k$. Then for $d(y, l_x) \geqq |x|/2$*

$$|D_x^\alpha K_{m,k}(x, y)| \leqq C \sum_{k-|\alpha|+1 \leqq l \leqq k+1} |x|^l |y|^{m-n-|\alpha|-l}.$$

(ii) *Let α be a multi-index. Then for $d(y, l_x) \geqq |x|/2$*

$$|D^\alpha K_m(x-y)| \leqq C|y|^{m-n-|\alpha|}.$$

(iii) *Let k be an integer with $m-n \leqq k \leqq m-1$ and α be a multi-index with $|\alpha| \leqq k$. Then for $d(y, l_x) \geqq |x|/2$*

$$|D_x^\alpha L_{m,k}(x, y)| \leqq \sum_{k-|\alpha|+1 \leqq l \leqq k+1} |x|^l |y|^{m-n-|\alpha|-l}.$$

PROOF. (i) It follows from (3.4) that

$$|D_x^\alpha K_{m,k}(x, y)| \leqq \sum_{k-|\alpha|+1 \leqq |\gamma| \leqq k} |(x^\gamma/\gamma!)D^{\gamma+\alpha} K_m(-y)|$$

$$+ (k+1)\sum_{|\gamma| = k+1} \left| \int_0^{|x|} \frac{(|x|-t)^k}{\gamma!}(x')^\gamma D^{\gamma+\alpha} K_m(tx'-y)dt \right|.$$

Using (3.5) we see that for $y \notin l_x$

$$|D_x^\alpha K_{m,k}(x, y)| \leqq \sum_{k-|\alpha|+1 \leqq |\gamma| \leqq k} C|x|^{|\gamma|}|y|^{m-n-|\gamma|-|\alpha|} + (k+1)\sum_{|\gamma|=k+1}\int_0^{|x|} C(|x|-t)^k|tx'-y|^{m-n-k-1-|\alpha|}dt.$$

We note that $d(y, l_x) \geqq |x|/2$ implies $|y|/3 \leqq |tx'-y| \leqq 3|y|$ for $0 \leqq t \leqq |x|$. Hence for $d(y, l_x) \geqq |x|/2$

$$\int_0^{|x|} (|x|-t)^k|tx'-y|^{m-n-k-1-|\alpha|}dt \leqq C|x|^{k+1}|y|^{m-n-k-1-|\alpha|}.$$

This concludes the proof of (i).

(ii) This follows from (3.5) and the fact that $d(y, l_x) \geqq |x|/2$ implies $|y|/3 \leqq |x-y| \leqq 3|y|$.

(iii) We note that $|\gamma+\alpha| \geqq k+1 \geqq m-n+1$ for $|\gamma| \geqq k-|\alpha|+1$. Therefore, using (3.7) we can prove (iii) in the same way as in the proof of (i).

We next study integrability properties of $K_{m,k}(x, y)$ and $L_{m,k}(x, y)$ as functions of y.

LEMMA 3.2. *Let* $m-(n/p)>0, \neq 1, 2, \cdots, m-1$ *and* $k=[m-(n/p)]$. *Then*:

(i) *For each multi-index* α *with* $|\alpha| \leqq k$,

$$\left(\int |D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'} \leqq C|x|^{m-(n/p)-|\alpha|}.$$

(ii) *For each multi-index* α *with* $|\alpha|=k$,

$$\left(\int |D_x^\alpha K_{m,k}(x, y)-D_x^\alpha K_{m,k}(z, y)|^{p'}dy\right)^{1/p'} \leqq C|x-z|^{m-(n/p)-|\alpha|}.$$

PROOF. (i) We have

$$\left(\int |D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'} \leqq \left(\int_{d(y,l_x)<|x|/2} |D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'} + \left(\int_{d(y,l_x)\geqq |x|/2} |D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'} = I_1+I_2.$$

First we estimate I_1. It follows from (3.5) and (3.8) that

$$|D_x^\alpha K_{m,k}(x, y)| \leqq C|x-y|^{m-n-|\alpha|} + C\sum_{|\gamma| \leq k, \gamma \geq \alpha} |x|^{|\gamma|-|\alpha|}|y|^{m-n-|\gamma|}.$$

Since $d(y, l_x) < |x|/2$ implies $|x-y| < (3/2)\,|x|$ and $|y| < (3/2)\,|x|$, we have

$$I_1 \leqq C\left(\int_{|x-y|<(3/2)|x|} |x-y|^{p'(m-n-|\alpha|)}dy\right)^{1/p'}$$
$$+ C\sum_{|\gamma| \leqq k, \gamma \geqq \alpha} |x|^{|\gamma|-|\alpha|}\left(\int_{|y|<(3/2)|x|} |y|^{p'(m-n-|\gamma|)}dy\right)^{1/p'}.$$

From $|\alpha| \leqq k$, $|\gamma| \leqq k$ and $m-(n/p) > k$ it follows that $p'(m-n-|\alpha|) > -n$ and $p'(m-n-|\gamma|) > -n$. Hence we see that

$$I_1 \leqq C|x|^{m-(n/p)-|\alpha|} + C\sum_{|\gamma| \leqq k, \gamma \geqq \alpha} |x|^{|\gamma|-|\alpha|}|x|^{m-(n/p)-|\gamma|}$$
$$= C|x|^{m-(n/p)-|\alpha|}.$$

Next, in order to estimate I_2 we apply Lemma 3.1 (i). Then using the fact that $d(y, l_x) \geqq |x|/2$ implies $|y| \geqq |x|/2$ we have

$$I_2 \leqq C\sum_{k-|\alpha|+1 \leqq l \leqq k+1} |x|^l\left(\int_{|y| \geqq |x|/2} |y|^{p'(m-n-l-|\alpha|)}dy\right)^{1/p'}$$

From $l \geqq k-|\alpha|+1$ and $m-(n/p) < k+1$, it follows that $p'(m-n-l-|\alpha|) < -n$. Hence we see that

$$I_2 \leqq C\sum_{k-|\alpha|+1 \leqq l \leqq k+1} |x|^l|x|^{m-(n/p)-l-|\alpha|} = C|x|^{m-(n/p)-|\alpha|}.$$

Thus we obtain (i)

(ii) For a multi-index α with $|\alpha| = k$, it follows from (3.8) that

$$D_x^\alpha K_{m,k}(x, y) - D_x^\alpha K_{m,k}(x, y) = D^\alpha K_m(x-y) - D^\alpha K_m(z-y).$$

Hence we have

$$\left(\int |D_k^\alpha K_{m,k}(x, y) - D_x^\alpha K_{m,k}(z, y)|^{p'}dy\right)^{1/p'}$$
$$= \left(\int |D^\alpha K_m(x-y) - D^\alpha K_m(z-y)|^{p'}dy\right)^{1/p'}$$
$$= \left(\int |D^\alpha K_m(x-z-y) - D^\alpha K_m(-y)|^{p'}dy\right)^{1/p'}$$
$$= \left(\int |D_x^\alpha K_{m,k}(x-z, y)|^{p'}dy\right)^{1/p'}.$$

Consequently, (i) gives (ii).

Let $m \geqq n$ and $k=(m-(n/p)]$. From $m-(n/p)=m-n+(n/p')$ it follows that $k \geqq m-n$. Hence using (3.6), (3.7) and Lemma 3.1 (iii) we can prove the following lemma in the same way as in the proof of Lemma 3.2 (i).

LEMMA 3.3. *If* $m \geqq n$, $m-(n/p) \neq 1, 2, \cdots, m-1$ *and* $k=[m-(n/p)]$, *then*
(i) *for* $|\alpha| \leqq m-n$

$$\left(\int |D_x^\alpha L_{m,k}(x, y)|^{p'} dy\right)^{1/p'} \leqq C|x|^{m-(n/p)-|\alpha|}(1+|\log|x||),$$

(ii) *for* $m-n+1 \leqq |\alpha| \leqq k$

$$\left(\int |D_x^\alpha L_{m,k}(x, y)|^{p'} dy\right)^{1/p'} \leqq C|x|^{m-(n/p)-|\alpha|}.$$

Next we consider the case $m-(n/p)=1, 2, \cdots, m-1$.

LEMMA 3.4. *If* $k=m-(n/p)$ *coincides with one of the numbers* $1, \cdots, m-1$, *then*
(i) *for each multi-index* α *with* $|\alpha| \leqq k-1$

$$\left(\int_{|y|<1} |D_x^\alpha K_{m,k-1}(x, y)|^{p'} dy\right)^{1/p'} \leqq c|x|^{k-|\alpha|}\left(1+\log^+\frac{1}{|x|}\right)^{1/p'},$$

where $\log^+ t=\log t$ *for* $t \geqq 1$ *and* $\log^+ t=0$ *for* $t<1$,

(ii) *for each multi-index* α *with* $|\alpha|=k-1$

$$\left(\int_{|y|<1} |D_x^\alpha K_{m,k-1}(x-y)-D_x^\alpha K_{m,k-1}(z, y)|^{p'} dy\right)^{1/p} \leqq C|x-z|\left(1+\log^+\frac{1}{|x-z|}\right)^{1/p'}.$$

PROOF. (i) We have

$$\begin{aligned}
&\left(\int_{|y|<1} |D_x^\alpha K_{m,k-1}(x, y)|^{p'} dy\right)^{1/p'} \\
&\leqq \left(\int_{|y|<1, d(y,l_x)<|x|/2} |D_x^\alpha K_{m,k-1}(x, y)|p' dy\right)^{1/p'} \\
&\quad +\left(\int_{|y|<1, d(y,\ell_x)\geqq |x|/2} |D_x^\alpha K_{m,k-1}(x, y)|^{p'} dy\right)^{1/p'} \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, by (3.5) and (3.8) we see that

$$I_1 \leqq C\left(\int_{|x-y|<(3/2)|x|} |x-y|^{p'(m-n-|\alpha|)} dy\right)^{1/p'}$$

$$+ C\sum_{|\gamma| \leqq k-1, \gamma \geqq \alpha} |x|^{|\gamma| - |\alpha|} \left(\int_{|y| < (3/2)|x|} |y|^{p'(m-n-|\gamma|)} dy \right)^{1/p'}.$$

From $|\alpha|, |\gamma| \leqq k-1$ it follows that $p'(m-n-|\alpha|), p'(m-n-|\gamma|) > -n$. Hence we have

$$I_1 \leqq C|x|^{m-(n/p)-|\alpha|} + C\sum_{|\gamma| \leqq k-1, \gamma \geqq \alpha} |x|^{|\gamma|-|\alpha|} |x|^{m-(n/p)-|\gamma|} = C|x|^{k-|\alpha|}.$$

For I_2, by Lemma 3.1 (i) we see that

$$I_2 \leqq C|x|^{k-|\alpha|} \left(\int_{|y|<1, |y| \geqq |x|/2} |y|^{p'(m-n-k)} dy \right)^{1/p'}$$

$$+ C\sum_{k-|\alpha|+1 \leqq l \leqq k} |x|^l \left(\int_{|y| \geqq |x|/2} |y|^{p'(m-n-l-|\alpha|)} dy \right)^{1/p'}$$

$$= I_{21} + I_{22}.$$

Since $p'(m-n-k) = -n$, we have

$$I_{21} = C|x|^{k-|\alpha|} \left(\int_{|x| \leqq |y| < 1} |y|^{-n} dy \right)^{1/p'} \leqq C|x|^{k-|\alpha|} \left(1 + \log^+ \frac{1}{|x|} \right)^{1/p'}.$$

Since $\ell \geqq k - |\alpha| + 1$ implies $p'(m-n-\ell-|\alpha|) < -n$, we have

$$I_{21} = C\sum_{k-|\alpha|+1 \leqq l \leqq k} |x|^l |x|^{m-(n/p)-l-|\alpha|} = C|x|^{k-|\alpha|}.$$

Thus we obtain (i).

(ii) For a multi-index α with $|\alpha| = k-1$, as in the proof of Lemma 3.2 (ii), We have

$$\left(\int_{|y|<1} |D_x^\alpha K_{m,k-1}(x, y) - D_x^\alpha K_{m,k-1}(z, y)|^{p'} dy \right)^{1/p'}$$

$$= \left(\int_{|z+y|<1} |D_x^\alpha K_{m,k-1}(x-z, y)|^{p'} dy \right)^{1/p'}$$

$$\leqq \left(\int_{|z+y|<1, d(y, l_{x-z}) < |x-z|/2} |D_x^\alpha K_{m,k-1}(x-z, y)|^{p'} dy \right)^{1/p'}$$

$$+ \left(\int_{|z+y|<1, d(y, l_{x-z}) \geqq |x-z|/2} |D_x^\alpha K_{m,k-1}(x-z, y)|^{p'} dy \right)^{1/p'}$$

$$= J_1 + J_2.$$

For J_1, in a way similar to the estimate for I_1 in (i) we have

$$J_1 \leqq C|x-z|.$$

For J_2, it follows from Lemma 3.1 (i) that

$$J_2 \leqq C|x-z|\left(\int_{|z+y|<1, |y| \geqq |x-z|/2} |y|^{p'(m-n-k)} dy\right)^{1/p'}$$

$$+ C\sum_{2\leqq l \leqq k} |x-z|^l \left(\int_{|y| \geqq |x-z|/2} |y|^{p'(m-n-l-k+1)} dy\right)^{1/p'}$$

$$= J_{21} + J_{22}.$$

For $|z| \geqq 2$, we see that

$$J_{21} \leqq C|x-z|\left(\int_{|z|-1 \leqq |y| \leqq |z|+1} |y|^{-n} dy\right)^{1/p'} \leqq C|x-z|.$$

For $|z|<2$, we have

$$J_{21} \leqq C|x-z|\left(\int_{|x-z|/2 \leqq |y| \leqq 3} |y|^{-n} dy\right)^{1/p'} \leqq C|x-z| \left(1+\log^+ \frac{1}{|x-z|}\right)^{1/p'}.$$

For J_{22}, noting that $p'(m-n-l-k+1) < -n$ for $2 \leqq l \leqq k$, we have

$$J_{22} = C\sum_{2\leqq l \leqq k} |x-x|^l |x-2|^{m-(n/p)-l-k+1} = C|x-z|.$$

We have completed the proof of (ii).

The proof of the next lemma is similar to that of Lemma 3.4 (i).

LEMMA 3.5. *If $m \geqq n$, $m-(n/p)=1, 2, \cdots, m-1$ and $k=m-(n/p)$, then*

(i) *for $|\alpha| \leqq m-n$*

$$\left(\int_{|y|<1} |D_x^\alpha L_{m,k-1}(x, y)|^{p'} dy\right)^{1/p'} \leqq C|x|^{k-|\alpha|}(1+|\log|x||),$$

(ii) *for $m-n+1 \leqq |\alpha| \leqq k$*

$$\left(\int_{|y|<1} |D_x^\alpha L_{m,k-1}(x, y)|^{p'} dy\right)^{1/p'} \leqq C|x|^{k-|\alpha|}\left(1+\log^+ \frac{1}{|x|}\right)^{1/p'}.$$

LEMMA 3.6. *If $k=m-(n/p)$ coincides with one of the numbers $1, \cdots, m-1$, then*

(i) *for each multi-index α with $|\alpha| \leqq k-1$*

$$\left(\int_{|y| \geqq 1} |D_x^\alpha K_{m,k}(x, y)|^{p'} dy\right)^{1/p'} \leqq C|x|^{k-|\alpha|}(1+\log^+|x|)^{1/p'},$$

(ii) *for each multi-index α with $|\alpha|=k-1$*

$$\left(\int_{|y| \geqq 1} |D_x^\alpha K_{m,k}(x, y) - D_x^\alpha K_{m,k}(z, y)|^{p'} dy\right)^{1/p'}$$

$$\leqq C|x-z|\left(1+\log^+|x-z|+\log^+\frac{|z|}{|x-z|}\right)^{1/p'}.$$

PROOF. (i) We have

$$\left(\int_{|y|\geqq 1}|D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'}$$
$$\leqq\left(\int_{|y|\geqq 1, d(y,l_x)<|x|/2}|D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'}$$
$$+\left(\int_{|y|\geqq 1, d(y,l_x)\geqq|x|/2}|D_x^\alpha K_{m,k}(x, y)|^{p'}dy\right)^{1/p'}$$
$$=I_1+I_2.$$

For I_1, by (3.5) and (3.8) we see that

$$I_1\leqq C\left(\int_{|x-y|<(3/2)|x|}|x-y|^{p'(m-n-|\alpha|)}dy\right)^{1/p'}$$
$$+\sum_{|\gamma|\leqq k-1,\gamma\geqq\alpha}C|x|^{|\gamma|-|\alpha|}\left(\int_{|y|<(3/2)|x|}|y|^{p'(m-n-|\gamma|)}dy\right)^{1/p'}$$
$$+\sum_{|\gamma|=k,\gamma\geqq\alpha}C|x|^{|\gamma|-|\alpha|}\left(\int_{1\leqq|y|<(3/2)|x|}|y|^{p'(m-n-k)}dy\right)^{1/p'}$$
$$=I_{11}+I_{12}+I_{13}.$$

Since $|\alpha|\leqq k-1$ and $|\gamma|\leqq k-1$ imply $p'(m-n-|\alpha|)>-n$ and $p'(m-n-|\gamma|)>-n$, respectively, we see that

$$\mathrm{I}_{11}\leqq C|x|^{k-|\alpha|}\quad\text{and}\quad I_{12}\leqq C|x|^{k-|\alpha|}.$$

For I_{13}, it follows from $p'(m-n-k)=-n$ that

$$I_{13}\leqq C|x|^{k-|\alpha|}(1+\log^+|x|)^{1/p'}.$$

Thus we have

$$I_1\leqq C|x|^{k-|\alpha|}(1+\log^+|x|)^{1/p'}.$$

By Lemma 3.1 (i) we see that

$$I_2\leqq C\sum_{k-|\alpha|+1\leqq l\leqq k+1}|x|^l\left(\int_{|y|\geqq|x|/2}|y|^{p'(m-n-l-|\alpha|)}dy\right)^{1/p'}\leqq C|x|^{k-|\alpha|},$$

since $p'(m-n-l-|\alpha|)<-n$ for $l\geqq k-|\alpha|+1$. Thus we obtain (i).

(ii) For a multi-index α with $|\alpha|=k-1$, it follows from (3.8) that

$$\begin{aligned} &D_x^\alpha K_{m,k}(x, y) - D_x^\alpha K_{m,k}(z, y) \\ &\quad = D^\alpha K_m(x-y) - D^\alpha K_m(z-y) - \textstyle\sum_{j=1}^n (x_j - z_j) D^{\alpha+e_j} K_m(-y). \end{aligned}$$

Hence we have

$$J = \left(\int_{|y| \geqq 1} |D_x^\alpha K_{m,k}(x, y) - D_x^\alpha K_{m,k}(z, y)|^{p'} dy \right)^{1/p'} \leqq J_1 + J_2,$$

where

$$J_i = \left(\int_{E_i} |D^\alpha K_m(x-z-y) - D^\alpha K_m(-y) - \textstyle\sum_{j=1}^n (x_j - z_j) D^{\alpha+e_j} K_m(-z-y)|^{p'} dy \right)^{1/p'}, \qquad i = 1, 2$$

with $E_1 = \{y;\ |z+y| \geqq 1,\ d(y, l_{x-z}) < |x-z|/2\}$ and $E_2 = \{y;\ |z+y| \geqq 1,\ d(y, l_{x-z}) \geqq |x-z|/2\}$. For J_1, it follows from (3.5) that

$$\begin{aligned} J_1 &\leqq C\left(\int_{|x-z-y| < (3/2)|x-z|} |x-z-y|^{p'(m-n-k+1)} dy \right)^{1/p'} \\ &\quad + C\left(\int_{|y| < (3/2)|x-z|} |y|^{p'(m-n-k+1)} dy \right)^{1/p'} \\ &\quad + C|x-z| \left(\int_{|z+y| \geqq 1,\ |y| < (3/2)|x-z|} |-z-y|^{p'(m-n-k)} dy \right)^{1/p'} \\ &= J_{11} + J_{12} + J_{13}. \end{aligned}$$

Since $p'(m-n-k+1) > -n$, we have

$$J_{11} + J_{12} \leqq C|x-z|.$$

We have

$$\begin{aligned} J_{13} &\leqq C|x-z| \left(\int_{|z+y| \geqq 1,\ |y| < (3/2)|x-z|,\ 3|x-z| < |z|} |z+y|^{-n} dy \right)^{1/p'} \\ &\quad + C|x-z| \left(\int_{|z+y| \geqq 1,\ |y| < (3/2)|x-z|,\ 3|x-z| \geqq |z|} |z+y|^{-n} dy \right)^{1/p'} \\ &= J_{131} + J_{132}. \end{aligned}$$

For J_{131}, since $|y| < (3/2)|x-z|$ and $3|x-z| < |z|$ imply $|z+y| \geqq |z|/2$, we see that

$$J_{131} \leqq C|x-z| \left(|z|^{-n} \int_{|y|<|z|/2} dy \right)^{1/p'} \leqq C|x-z|.$$

For J_{132}, since $|y|<(3/2)\ |x-z|$ and $|z| \leqq 3|x-z|$ imply $|z+y| \leqq (9/2)|x-z|$, we see that

$$J_{132} \leqq C|x-z| \int_{1 \leqq |z+y| \leqq (9/2)|x-z|} |z+y|^{-n} dy \leqq C|x-z|\ (1+\log^+|x-z|).$$

Thus we obtain

$$J_1 \leqq C|x-z|\ (1+\log^+|x-z|)^{1/p'}.$$

Next, we shall estimate J_2. We see that

$$\begin{aligned} J_2 \leqq & \left(\int_{d(y,l_{x-z}) \geqq |x-z|/2} |D^\alpha K_m(x-z-y) - D^\alpha K_m(-y) \right. \\ & \left. - \textstyle\sum_{j=1}^n (x_j - z_j) D^{\alpha+e_j} K_m(-y)|^{p'} dy \right)^{1/p'} \\ & + C|x-z| \textstyle\sum_{j=1}^n \left(\int_{E_2} D^{\alpha+e_j} K_m(-z-y) - D^{\alpha+e_j} K_m(-y)|^{p'} dy \right)^{1/p'} \\ = & J_{21} + J_{22}. \end{aligned}$$

We note that

$$D^\alpha K_m(x-z-y) - D^\alpha K_m(-y) - \textstyle\sum_{j=1}^n (x_j - z_j) D^{\alpha+e_j} K_m(-y) = D_x^\alpha K_{m,k}(x-z,\ y).$$

Hence it follows from Lemma 3.1 (i) that

$$\begin{aligned} J_{21} & \leqq C \textstyle\sum_{2 \leqq l \leqq k+1} |x-z|^l \left(\int_{|y| \geqq |x-z|/2} |y|^{p'(m-n-l-k+1)} dy \right)^{1/p'} \\ & \leqq C|x-z|, \end{aligned}$$

since $l \geqq 2$ implies $p'(m-n-l-k+1) < -n$. For J_{22}, we have

$$J_{22} \leqq J_{221} + J_{222},$$

where

$$J_{22i} = C|x-z| \textstyle\sum_{j=1}^n \left(\int_{E_{2i}} |D^{\alpha+e_j} K_m(-z-y) - D^{\alpha+e_j} K_m(-y)|^{p'} dy \right)^{1/p'}, \quad i=1,\ 2$$

with $E_{21} = \{y \in E_2;\ d(y,\ l_{-z}) < |z|/2\}$ and $E_{22} = \{y \in E_2;\ d(y,\ l_{-z}) \geqq |z|/2\}$. We note that

$$D^{\alpha+e_j} K_m(-z-y) - D^{\alpha+e_j} K_m(-y) = D_x^{\alpha+e_j} K_{m,k}(-z,\ y).$$

Therefore it follows from Lemma 3.1 (i) that

$$J_{222} \leqq C|x-z| \sum_{1 \leqq j \leqq n} \sum_{1 \leqq l \leqq k} |z|^l \left(\int_{|y| \geqq |z|/2} |y|^{p'(m-n-l-k)} dy \right)^{1/p'} \leqq C|x-z|,$$

since $l \geqq 1$ implies $p'(m-n-l-k) < -n$. For J_{221}, it follows from (3.5) that

$$\begin{aligned} J_{221} &\leqq C|x-z| \left(\int_{1 \leqq |z+y| \leqq (3/2)|z|} |z+y|^{p'(m-n-k)} dy \right)^{1/p'} \\ &\quad + C|x-z| \left(\int_{|x-z|/2 \leqq |y| \leqq (3/2)|z|} |y|^{p'(m-n-k)} dy \right)^{1/p'} \\ &= C|x-z| \, (1 + \log^+ (3/2)|z|)^{1/p'} + C|x-z| \left(\log^+ \frac{3|z|}{|x-z|} \right)^{1/p'} \\ &\leqq C|x-z| \left(1 + \log^+ |z| + \log^+ \frac{|z|}{|x-z|} \right)^{1/p'}. \end{aligned}$$

Consequently we have

$$J \leqq C|x-z| \left(1 + \log^+ |x-z| + \log^+ \frac{|z|}{|x-z|} \right)^{1/p'},$$

where we used the fact that $\log^+ a \leqq \log^+ b + \log^+ (a/b)$ for a, $b > 0$.

Corresponding to Lemma 3.6 (i), we can prove

LEMMA 3.7. *If* $m \geqq n$, $m-(n/p) = 1, 2, \cdots, m-1$ and $k = m-(n/p)$, *then*

(i) $$\left(\int_{|y| \geqq 1} |D_x^\alpha L_{m,k}(x, y)^{p'} dy \right)^{1/p'} \leqq C|x|^{k-|\alpha|}(1 + \log|x|) \quad \textit{for} \quad |\alpha| \leqq m-n,$$

(ii) $$\left(\int_{|y| \geqq 1} |D_x^\alpha L_{m,k}(x, y)| p' dy \right)^{1/p'} \leqq C|x|^{k-|\alpha|}(1 + \log^+ |x|)^{1/p'}$$

for $\quad m-n+1 \leqq |\alpha| \leqq k-1.$

REMARK 3.8. Let $m-(n/p) = 0, 1, \cdots, m-1$ *and* $k = m-(n/p)$. Then as an immediate consequence of Lemma 3.1 (i) we see that for each $|\alpha| \leqq k$

$$\left(\int_{d(y, l_x) \geqq |x|/2} |D_x^\alpha K_{m,k}(x, y)|^{p'} dy \right)^{1/p'} \leqq C|x|^{k-|\alpha|}.$$

For a locally integrable function f, K_m^f, $K_{m,k}^f$ and $L_{m,k}^f$ are defined by

$$K_m^f(x) = \int K_m(x-y) f(y) dy, \quad K_{m,k}^f(x) = \int K_{m,k}(x, y) f(y) dy$$

and

$$L^f_{m,k}(x)=\int L_{m,k}(x, y)f(y)dy$$

if they exist.

PROPOSITION 3.9. *Let* $k=[m-(n/p)]$ *and* $f\in L^p$. *Then*:

(i) *In case* $m-(n/p)>0$, $\neq 1,\cdots, m-1$, $K^f_{m,k}(x)$ *exists for any* x.

(ii) *In case* $m-(n/p)<0$, $K^f_{m,k}(x)=K^f_m(x)$ *exists for almost every* x.

(iii) *In case* $m-(n/p)=1, 2,\cdots, m-1$, $K^{f_1}_{m,k-1}(x)$ *and* $K^{f_2}_{m,k}(x)$ *exist for any* x *where* $f_1=f|_{B_1}$ *and* $f_2=f-f_1$.

(iv) *In case* $m-(n/p)=0$, $K^{f_1}_{m,-1}(x)=K^{f_1}_m(x)$ *and* $K^{f_2}_{m,0}(x)$ *exist for almost every* x.

PROOF. (i) By Lemma 3.2 (i), $K_{m,k}(x, y)$ is an $L^{p'}$-function as a function of y. Hence we obtain (i).

(ii) By (3.5), $|K_{m,k}(x, y)|=|K_m(x-y)|\leqq C|x-y|^{m-n}$ and

$$\int_{|y|\geqq 1}|y|^{m-n}|f(y)|dy\leqq\left(\int_{|y|\geqq 1}|y|^{p'(m-n)}dy\right)^{1/p'}\|f\|_p<\infty.$$

Therefore $K^f_{m,k}(x)=K^f_m(x)$ exists for almost every x (cf. N. S. Landkof [11; §3 in Chap. I]).

(iii) By Lemma 3.4 (i), $K_{m,k-1}(x, y)$ is an $L^{p'}$-function on B_1 as a function of y. Further by Lemma 3.6 (i), $K_{m,k}(x, y)$ is an $L^{p'}$-function on B^c_1 (the complement of B_1) as a function of y. Hence $K^{f_1}_{m,k-1}(x)$ and $K^{f_2}_{m,k}(x)$ exist for any x.

(iv) Since $K_{m,-1}(x, y)=K_m(x-y)$ and f_1 has compact support, $K^{f_1}_{m,-1}(x)=K^{f_1}_m(x)$ exists for almost every x. Moreover, we have

$$\begin{aligned}\int & K_{m,0}(x, y)f_2(y)dy\\ &=\int_{d(y,l_x)<|x|/2}K_m(x-y)f_2(y)dy-\int_{d(y,l_x)<|x|/2}K_m(-y)f_2(y)dy\\ &\quad+\int_{d(y,l_x)\geqq|x|/2}K_{m,0}(x, y)f_2(y)dy\\ &=I_1-I_2+I_3.\end{aligned}$$

Obviously I_1 exists for almost every x. For I_2, we see that

$$|I_2|\leqq\left(\int_{1\leqq|y|\leqq(3/2)|x|}|K_m(-y)|^{p'}dy\right)^{1/p'}\|f_2\|_p<\infty.$$

For I_3, it follows from Remark 3.8 that

$$|I_3| \leqq \left(\int_{d(y,l_x) \geqq |x|/2} K_{m,0}(x, y)|^{p'} dy \right)^{1/p'} \|f_2\|_p < \infty.$$

Thus $K_{m,0}^{f_2}(x)$ exists for almost every x.

By Lemmas 3.3, 3.5 and 3.7, we also have

PROPOSITION 3.10. *Let* $m \geqq n$, $k = [m - (n/p)]$ *and* $f \in L^p$. *Then*:

(i) *In case* $m - (n/p) \neq 1, 2, \cdots, m-1$, $L_{m,k}^f(x)$ *exists for any* x.

(ii) *In case* $m - (n/p) = 1, 2, \cdots, m-1$, $L_{m,k-1}^{f_1}(x)$ *and* $L_{m,k}^{f_2}(x)$ *exist for any* x *where* f_1 *and* f_2 *are as in Lemma* 3.9.

3.2. Integral inequalities

We shall investigate integral estimates for the operators $f \to K_m^f$, $f \to K_{m,k}^f$, and $f \to L_{m,k}^f$. In case $m - (n/p) < 0$, since

$$|K_m^f(x)| \leqq C \int |x-y|^{m-n} |f(y)| dy,$$

by the Hardy-Littlewood-Sobolev theorem on fractional integration ([8], [22]) we have

$$\|K_m^f\|_{p_m} \leqq C \|f\|_p$$

where $1/p_m = (1/p) - (m/n)$. This theorem was generalized by E. M. Stein and G. Weiss [25] and D. R. Adams [1] in the following form: If $m - (n/p) < 0$, then

$$\left(\int |x|^{-q(m-(n/p))-n} |K_m^f(x)|^q dx \right)^{1/q} \leqq C \|f\|_p \tag{3.9}$$

for $p \leqq q \leqq p_m$.

We shall give similar integral estimates for $K_{m,k}^f$ and $L_{m,k}^f$. The following inequality is due to G. O. Okikiolu [16; Theorem 2.1], which is useful for estimates of integral operators.

LEMMMA 3.11 *Let* (X, m_X) *and* (Y, m_Y) *be measure spaces, let* p, q, μ_1, μ_2 *be positive numbers such that*

$$1 < p \leqq q, \ (\mu_1/q) + (\mu_2/p') = 1$$

and let $K(x, y)$ *be a measurable function on* $X \times Y$. *Suppose that there are measurable functions* $\phi_1 > 0$ *on* X, $\phi_2 > 0$ *on* Y *and constants* $M_1 > 0$, $M_2 > 0$ *such that*

$$\int_Y \phi_2(y)^{p'}|K(x, y)|^{\mu_2}\mathrm{dm}_Y(y) \leqq M_1^{p'}\phi_1(x)^{p'}, \tag{3.10}$$

$$\int_X \phi_1(x)^q|K(x, y)|^{\mu_1}\mathrm{dm}_X(x) \leqq M_2^q\phi_2(y)^q \tag{3.11}$$

for all $x\in X$, $y\in Y$. If the operator K is defined by

$$Kf(x) = \int_Y K(x, y)f(y)\mathrm{dm}_Y(y),$$

then

$$\left(\int |Kf(x)|^q \mathrm{dm}_X(x)\right)^{1/q} \leqq M_1M_2\left(\int |f(y)|^p \mathrm{dm}_Y(y)\right)^{1/p}.$$

The following Lemmas 3.12, 3.13 and 3.15 are proved by applications of Lemma 3.11.

LEMMA 3.12. *Let $m>0$, $m-(n/p)\geqq 0$ and $p\leqq q<\infty$. Then*

(i) $$\left(\int |x|^{-q(m-(n/p))-n}\left|\int_{d(y,l_x)<|x|/2} |x-y|^{m-n}f(y)dy\right|^q dx\right)^{1/q} \leqq C\|f\|_p,$$

(ii) $$\left(\int (1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}\right.$$
$$\left.\times\left|\int_{d(y,l_x)<|x|/2} |x-y|^{m-n}|\log|x-y||\,|f(y)dy\right|^q dx\right)^{1/q}$$
$$\leqq C\|f\|_p.$$

PROOF. We take μ_1 and μ_2 such that $\mu_1>0$, $\mu_2>0$ and $(\mu_1/q)+(\mu_2/p')=1$. We choose μ_1 sufficiently small so that

$$(m-n)\mu_1+n>0. \tag{3.12}$$

We note that $m-(n/p)\geqq 0$ implies

$$(m-n)\mu_2+n>0. \tag{3.13}$$

We choose a number a such that

$$0<a<n/p'. \tag{3.14}$$

(i) We take $(X, m_X)=(R^n, |x|^{-q(m-(n/p))-n}dx)$ and $(Y, m_Y)=(R^n, dy)$. For $\phi_1(x) =|x|^{-a+((m-n)\mu_2/p')+(n/p')}$, $\phi_2(y)=|y|^{-a}$ and

$$K(x, y) = \begin{cases} |x-y|^{m-n}, & \text{for} \quad d(y, l_x) < |x|/2, \\ 0, & \text{for} \quad d(y, l_x) \geqq |x|/2, \end{cases}$$

we shall show (3.10) and (3.11). By (3.13) and (3.14) we have

$$\int \phi_2(y)^{p'}|K(x, y)|^{\mu_2} dm_Y(y) = \int_{d(y,l_x)<|x|/2} |y|^{-ap'}|x-y|^{(m-n)\mu_2} dy$$
$$\leqq |x|^{-ap'+(m-n)\mu_2+n} \int_{|z| \leqq 3/2} |z|^{-ap'}\left|\frac{x}{|x|}-z\right|^{(m-n)\mu_2} dz$$
$$= C|x|^{-ap'+(m-n)\mu_2+n} = C\phi_1(x)^{p'}.$$

Thus we obtain (3.10). Next, by (3.12) and (3.14) we have

$$\int \phi_1(x)^q |K(x, y)|^{\mu_1} dm_X(x)$$
$$= \int_{d(y,l_x)<|x|/2} |x|^{-aq-(m-n)\mu_1-n}|x-y|^{(m-n)\mu_1} dx$$
$$\leqq |y|^{-aq} \int_{|z|>2/3} |z|^{-aq-(m-n)\mu_1-n}\left|z-\frac{y}{|y|}\right|^{(m-n)\mu_1} dz$$
$$= C|y|^{-aq} = C\phi_2(y)^q.$$

Thus we obtain (3.11) and so (i) follows from Lemm 3.11.

(ii) We take $(X, m_X)=(R^n, (1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}dx)$ and $(Y, m_Y)=(R^n, dy)$. For $\phi_1(x)=|x|^{-a+((m-n)\mu_2/p')+(n/p')}(1+|\log|x||)^{\mu_2/p'}$, $\phi_2(y)=|y|^{-a}$ and

$$K(x, y) = \begin{cases} |x-y|^{m-n}|\log|x-y||, & \text{for} \quad d(y, l_x) < |x|/2, \\ 0, & \text{for} \quad d(y, l_x) \geqq |x|/2, \end{cases}$$

we verify (3.10) and (3.11). By (3.13) and (3.14) we have

$$\int \phi_2(y)^{p'}|K(x, y)|^{\mu_2} dm_Y(y)$$
$$= \int_{d(y,l_x)<|x|/2} |y|^{-ap'}|x-y|^{(m-n)\mu_2}|\log|x-y||^{\mu_2} dy$$
$$\leqq |x|^{-ap'+(m-n)\mu_2+n}(1+|\log|x||)^{\mu_2}$$
$$\times \int_{|z|<3/2} |z|^{-ap'}\left|\frac{x}{|x|}-z\right|^{(m-n)\mu_2}\left(1+\left|\log\left|\frac{x}{|x|}-z\right|\right|\right)^{\mu_2} dz$$

$$= C|x|^{-ap'+(m-n)\mu_2+n}(1+|\log|x||)^{\mu_2} = C\phi_1(x)^{p'}.$$

Thus we obtain (3.10). Next, by (3.12) and (3.14) we have

$$\int \phi_1(x)^q |K(x, y)|^{\mu_1} dm_X(x)$$

$$\leqq \int_{|x|>2|y|/3} |x|^{-aq-(m-n)\mu_1-n}(1+|\log|x||)^{(q\mu_2/p')-q}$$

$$\times |x-y|^{(m-n)\mu_1}|\log|x-y||^{\mu_1}dx$$

$$\leqq C|y|^{-aq}\int_{|z|>2/3} |z|^{-aq-(m-n)\mu_1-n}\left|\frac{y}{|y|}-z\right|^{(m-n)\mu_1}\left(1+\left|\log\left|\frac{y}{|y|}-z\right|\right|\right)^{\mu_1}dz$$

$$= C|y|^{-aq} = C\phi_2(y)^q.$$

Thus we obtain (3.11), and the proof of (ii) is completed by Lemma 3.11.

LEMMA 3.13. *If* $m>0$, $m-(n/p)>0$ *and* $p \leqq q < \infty$, *then*

(i) $$\left(\int |x|^{-q(m-(n/p))-n}\left|\int_{|y|<|x|} |y|^{m-n}f(y)dy\right|^q dx\right)^{1/q} \leqq C\|f\|_p,$$

(ii) $$\left(\int (1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}\left|\int_{|y|<|x|} |y|^{m-n}|\log|y||f(y)dy\right|^q dy\right)^{1/q}$$

$$\leqq C\|f\|_p.$$

PROOF. Let $l=n((1/p)-(1/q))$. Then $0 \leqq l < n$. We take $\mu_1=\mu_2=n/(n-l)$. Then $(\mu_1/q)+(\mu_2/p')=1$. Let $(X, m_X)=(Y, m_Y)=(R^n, dx)$ and $\phi_1(x)=\phi_2(x)=|x|^{-n^2/(p'q(n-l))}$.
(i) For

$$K(x, y) = \begin{cases} |x|^{-m+(n/p)-(n/q)}|y|^{m-n}, & \text{for } |y|<|x|, \\ 0, & \text{for } |y| \geqq |x|, \end{cases}$$

we shall show (3.10) and (3.11). We put

$$I = \int \phi_2(y)^{p'}K(x, y)^{n/(n-l)}dy$$

$$= |x|^{n(l-m)/(n-l)}\int_{|y|<|x|} |y|^{(m-n-(n/q))n/(n-l)}dy.$$

From $m-(n/p)>0$ it follows that $(m-n-(n/q))n/(n-l)>-n$. Hence we have

$$I = C|x|^{n(l-m)/(n-l)}|x|^{(m-(n/p))n/(n-l)} = C|x|^{-n^2/q(n-l)} = C\phi_1(x)^{p'}.$$

Thus we obtain (3.10). Since $m-(n/p)>0$ implies $-(n^2/p'(n-l))+n(l-m)/(n-l)<-n$, we see that

$$\int \phi_1(x)^q K(x, y)^{n/(n-l)}dx$$
$$= |y|^{(m-n)n/(n-l)} \int_{|x|>|y|} |x|^{-(n^2/p'(n-l))+n(l-m)/(n-l)}dx$$
$$= C|y|^{(m-n)n/(n-l)}|y|^{-(m-(n/p))/(n-l)} = C|y|^{-n^2/p'(n-l)} = C\phi_2(y)^q.$$

Hence we have (3.11), and (i) is proved by Lemma 3.11.

(ii) For

$$K(x, y) = \begin{cases} (1+|\log|x||)^{-1}|x|^{-m+(n/p)-(n/q)}|y|^{m-n}|\log|y||, & |y|<|x|, \\ 0, & |y| \geqq |x|, \end{cases}$$

we can prove (3.10) and (3.11) in the same way as in (i).

The next lemma is a consequence of Lemma 3.13 (i).

LEMMA 3.14. *If $m>0$, $m-(n/p)<0$ and $p \leqq q < \infty$, then*

$$\left(\int |x|^{-q(m-(n/p))-n}\left|\int_{|y|\geqq|x|} |y|^{m-n}f(y)dy\right|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

PROOF. Let $l=-m+(2n/p)$. Then $l-(n/p)>0$. If we apply Lemma 3.13 (i) to the function $f_1(z)=|z|^{-2n/p}f(z/|z|^2)$, then we obtain

$$\left(\int |w|^{-q(l-(n/p))-n}\left|\int_{|z|\leqq|w|} |z|^{l-n}f_1(z)dz\right|^q dw\right)^{1/q} \leqq C\|f_1\|_p.$$

Using the fact that the absolute value of the Jacobian of the transformation $z=y/|y|^2$ is $1/|y|^{2n}$, we see that the left hand side of the above inequality is equal to that in the lemma. Moreover, it is easy to see that $\|f_1\|_p=\|f\|_p$. Hence we have Lemma 3.14.

LEMMA 3.15. *If $p \leqq q < \infty$, then*

(i) *for $f \in L^p(B_1^c)$*

$$\left(\int_{|x|\geqq 1} (\log|x|)^{-(q/p')-1}|x|^{-n}\left|\int_{|y|<|x|} |y|^{-n/p'}f(y)dy\right|^q dx\right)^{1/q} \leqq C\|f\|_p,$$

(ii) *for $f \in L^p(B_1)$*

$$\left(\int_{|x|<1} (-\log|x|)^{-(q/p')-1}|x|^{-n}\left|\int_{|y|\geqq|x|} |y|^{-n/p'}f(y)dy\right|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

PROOF. We first note that (i) implies (ii). For $f \in L^p(B_1)$, it is clear that $f_1(z) = |z|^{-2n/p} f(z/|z|^2) \in L^p(B_1^c)$ and $\|f_1\|_p = \|f\|_p$. Applying (i) to the function f_1 and changing variables, we obtain (ii). In order to show (i), we apply Lemma 3.11. Let $(X, m_X) = (B_1^c, |x|^{-n}(\log|x|)^{-(q/p')-1}dx)$, $(Y, m_Y) = (B_1^c, dy)$, $\mu_1 > 0$, $\mu_2 > 0$ and $(\mu_1/q) + (\mu_2/p') = 1$. Taking $0 < a < 1$, we set $\phi_1(x) = (\log|x|)^{(1-a)/p'}$ and $\phi_2(y) = |y|^{-\mu_1 n/(p'q)}(\log|y|)^{-a/p'}$. For ϕ_1, ϕ_2 and

$$K(x, y) = \begin{cases} |y|^{-n/p'}, & 1 \leqq |y| < |x|, \\ 0, & 1 \leqq |x| \leqq |y|, \end{cases}$$

we shall show (3.10) and (3.11). First, from $a < 1$ we see that

$$\begin{aligned} &\int_{|y| \geqq 1} \phi_2(y)^{p'} K(x, y)^{\mu_2} dy \\ &= \int_{|x| > |y| \geqq 1} |y|^{-\mu_1 n/q} (\log|y|)^{-a} |y|^{-\mu_2 n/p'} dy \\ &= \int_{|x| > |y| \geqq 1} |y|^{-n} (\log|y|)^{-a} dy = C(\log|x|)^{1-a} = C\phi_1(x)^{p'}. \end{aligned}$$

Next, it follows from $a > 0$ that

$$\begin{aligned} &\int_{|x| \geqq 1} \phi_1(x)^q K(x, y)^{\mu_1} |x|^{-n} (\log|x|)^{-(q/p')-1} dx \\ &= |y|^{-\mu_1 n/p'} \int_{|x| > |y|} |x|^{-n} (\log|x|)^{-(aq/p')-1} dx \\ &= C|y|^{-\mu_1 n/p'} (\log|y|)^{-aq/p'} = C\phi_2(y)^q. \end{aligned}$$

Hence we obtain (3.10) and (3.11), and have the lemma.

REMARK 3.16. By Lemma 3.15 (i) the following inequality holds: For $f \in L^p(B_1^c)$

$$\left(\int_{|x| \geqq 1} (\log|x|)^{-(q/p')-q-1} |x|^{-n} \left| \int_{|y| < |x|} |y|^{-n/p'} (\log|y|) f(y) dy \right|^q dx \right)^{1/q} \leqq C\|f\|_p.$$

We are now in a position to prove the main theorems in this section.

THEOREM 3.17. *Let $m - (n/p) \neq 0, 1, \cdots, m-1$ and $k = [m - (n/p)]$. Then*

$$\left(\int |x|^{-q(m-(n/p))-n} |K_{m,k}^f(x)|^q dx \right)^{1/q} \leqq C\|f\|_p$$

for $p \leqq q < \infty$ *in case* $m-(n/p)>0$ *and for* $p \leqq q \leqq p_m$ *in case* $m-(n/p)<0$.

PROOF. If $m-(n/p)<0$, then $K^f_{m,k}=K^f_m$, and hence the assertion is nothing but (3.9). So, let $m-(n/p)>0$, $\neq 1, 2, \cdots, m-1$. From the definition of $K^f_{m,k}$ and (3.5), it follows that

$$\left(\int |x|^{-q(m-(n/p))-n}|K^f_{m,k}(x)|^q dx\right)^{1/q} \leqq CI_1 + CI_2 + I_3,$$

where

$$I_1 = \left(\int |x|^{-q(m-(n/p))-n}\left(\int_{d(y,l_x)<|x|/2} |x-y|^{m-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$I_2 = \sum_{|\gamma| \leqq k}\left(\int |x|^{-q(m-(n/p))-n}\left(\int_{d(y,l_x)<|x|/2} |x|^{|\gamma|}|y|^{m-|\gamma|-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$I_3 = \left(\int |x|^{-q(m-(n/p))-n}\left(\int_{d(y,l_x)\geqq |x|/2} |K_{m,k}(x, y)f(y)|dy\right)^q dx\right)^{1/q}.$$

By Lemma 3.12 (i), $I_1 \leqq C\|f\|_p$. For I_2, since $d(y, l_x)<|x|/2$ implies $|y|<(3/2)|x|$, we have

$$\begin{aligned} I_2 &\leqq \sum_{|\gamma| \leqq k}\left(\int |x|^{-q(m-|\gamma|-(n/p))-n}\left(\int_{|y|<3|x|/2} |y|^{m-|\gamma|-n}|f(y)|dy\right)^q dx\right)^{1/q} \\ &\leqq C\sum_{|\gamma| \leqq k}\|f\|_p = C\|f\|_p \end{aligned}$$

by Lemma 3.13 (i). Here, note that $m-(n/p)>0$, $\neq 1, 2, \cdots, m-1$ imply $m-|\gamma|-(n/p)>0$ for $|\gamma| \leqq k$. In order to estimate I_3, we apply Lemma 3.1 (i) and obtain

$$I_3 \leqq C\left(\int |x|^{-q(m-k-1-(n/p))-n}\left(\int_{|y|\geqq |x|/2} |y|^{m-k-1-n}|f(y)|dy\right)^q dx\right)^{1/q}.$$

Since $m-k-1-(n/p)<0$, it follows from Lemma 3.14 that $I_3 \leqq C\|f\|_p$. Thus the theorem is established.

Using (3.6), (3.7), Lemmas 3.1(iii), 3.12 (ii), 3.13 (ii) and 3.14, we can prove the following theorem in the same way as Theorem 3.17.

THEOREM 3.18. *Let* $m \geqq n$, $m-(n/p) \neq 1, 2, \cdots, m-1$, $k=[m-(n/p)]$ *and* $p \leqq q < \infty$. *Then*

$$\left(\int (1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}|L^f_{m,k}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

THEOREM 3.19. *If* $m-(n/p)$ *coincides with one of the numbers* $0, 1, \cdots, m-1$, k

$=m-(n/p)$ *and* $p \leqq q < \infty$, *then*

(i) *for* $f \in L^p(B_1)$

$$\left(\int(1+\log^+(1/|x|))^{-(q/p')-1}|x|^{-qk-n}|K^f_{m,k-1}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p,$$

(ii) *for* $f \in L^p(B_1^c)$

$$\left(\int(1+\log^+|x|)^{-(q/p')-1}|x|^{-qk-n}|K^f_{m,k}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

PROOF. (i) Let $f \in L^p(B_1)$. From the definition of $K^f_{m,k}$ and (3.5), it follows that

$$I=\left(\int(1+\log^+(1/|x|))^{-(q/p')-1}|x|^{-qk-n}|K^f_{m,k-1}(x)|^q dx\right)^{1/q} \leqq CI_1+CI_2+I_3,$$

where

$$I_1=\left(\int|x|^{-qk-n}\left(\int_{d(y,l_x)<|x|/2}|x-y|^{m-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$I_2=\sum_{|\gamma|\leqq k-1}\left(\int|x|^{-qk-n}\left(\int_{d(y,l_x)<|x|/2}|x|^{|\gamma|}|y|^{m-|\gamma|-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$I_3=\left(\int\left(1+\log^+\frac{1}{|x|}\right)^{-(q/p')-1}|x|^{-qk-n}\right.$$
$$\left.\times\left(\int_{d(y,l_x)\geqq|x|/2}|K_{m,k-1}(x,y)f(y)|dy\right)^q dx\right)^{1/q};$$

in case $k=0$ we set $I_2=0$. It follows from Lemma 3.12 (i) that $I_1 \leqq C\|f\|_p$. Since $m-(n/p)=k$, we have $m-|\gamma|-(n/p)>0$ for $|\gamma| \leqq k-1$ in case $k \geqq 1$. Hence by Lemma 3.13 (i) we see that $I_2 \leqq C\|f\|_p$. In case $k \geqq 1$, applying Lemma 3.1 (i) to I_3, we have

$$I_3 \leqq C\left(\int\left(1+\log^+\frac{1}{|x|}\right)^{-(q/p')-1}|x|^{-qk-n}\right.$$
$$\left.\times\left(\int_{d(y,l_x)\geqq|x|/2}|x|^k|y|^{m-k-n}|f(y)|dy\right)^q dx\right)^{1/q}$$
$$\leqq C\left(\int_{|x|\leqq 2}\left(1+\log^+\frac{1}{|x|}\right)^{-(q/p')-1}|x|^{-n}\right.$$
$$\left.\times\left(\int_{|y|\geqq|x|/2,\,|y|<1}|y|^{-n/p'}|f(y)|dy\right)^q dx\right)^{1/q}.$$

In case $k=0$, applying Lemma 3.1 (ii) to I_3 and noting that $m-n=-n/p'$, we have

$$I_3 \leqq C\left(\int_{|x|\leqq 2}\left(1+\log^+\frac{1}{|x|}\right)^{-(q/p')-1}|x|^{-n} \times\left(\int_{|y|\geqq |x|/2,\,|y|<1}|y|^{-n/p'}|f(y)|dy\right)^q dx\right)^{1/q}.$$

Hence by Lemma 3.15 (ii) we have $I_3 \leqq C\|f\|_p$, and thus $I\leqq C\|f\|_p$. We have completed the proof of (i).

(ii) Let $f\in L^p(B_1^c)$. From the definition of $K^f_{m,k}$ and (3.5), it follows that

$$\left(\int(1+\log^+|x|)^{-(q/p')-1}|x|^{-qk-n}|K^f_{m,k}(x)|^q dx\right)^{1/q} \leqq CJ_1+CJ_2+J_3,$$

where

$$J_1=\left(\int|x|^{-qk-n}\left(\int_{d(y,l_x)<|x|/2}|x-y|^{m-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$J_2=\sum_{|\gamma|\leqq k}\left(\int(1+\log^+|x|)^{-(q/p')-1}|x|^{-qk-n} \times\left(\int_{d(y,l_x)<|x|/2}|x|^{|\gamma|}|y|^{m-|\gamma|-n}|f(y)|dy\right)^q dx\right)^{1/q},$$

$$J_3=\left(\int|x|^{-qk-n}\left(\int_{d(y,l_x)\geqq |x|/2}|K_{m,k}(x,y)f(y)|dy\right)^q dx\right)^{1/q}.$$

It follows from Lemma 3.12 (i) that $J_1\leqq C\|f\|_p$. Since $m-k-n=-n/p'$.

$$J_2\leqq \sum_{|\gamma|\leqq k-1}\left(\int|x|^{-q(k-|\gamma|)-n}\left(\int_{|y|<3|x|/2}|y|^{m-|\gamma|-n}|f(y)|dy\right)^q dx\right)^{1/q}$$
$$+\left(\int_{|x|>2/3}(1+\log^+|x|)^{-(q/p')-1}|x|^{-n} \times\left(\int_{|y|<3|x|/2,\,|y|>1}|y|^{-n/p'}|f(y)|dy\right)^q dx\right)^{1/q}$$
$$=J_{21}+J_{22}.$$

Since $m-|\gamma|-(n/p)>0$ for $|\gamma|\leqq k-1$, it follows from Lemma 3.13 (i) that

$$J_{21}\leqq C\sum_{|\gamma|\leqq k-1}\|f\|_p=C\|f\|_p.$$

By Lemma 3.15 (i) we have $J_{22}\leqq C\|f\|_p$. By Lemma 3.1 (i) and 3.14, we have

$$J_3 \leqq C\left(\int |x|^{-qk-n}\left(\int_{d(y,l_x)\geqq |x|/2} |x|^{k+1}|y|^{m-k-1-n}|f(y)|dy\right)^q dx\right.$$

$$\leqq C\left(\int |x|^{q-n}\left(\int_{|y|\geqq |x|/2} |y|^{m-k-1-n}|f(y)|dy\right)^q dx\right)^{1/q} \leqq C\|f\|_p.$$

Thus we obtain the desired result.

REMARK 3.20. Let $m-(n/p)=0, 1, \cdots, m-1, k=m-(n/p)$ and $p\leqq q<\infty$. Since $p \leqq q$ implies $(q/p')+1\leqq q$, by Theorem 3.19 we have the following inequalities:

(i) For $f\in L^p(B_1)$

$$\left(\int(1+\log^+(1/|x|))^{-q}|x|^{-qk-n}|K^f_{m,k-1}(x)^q dx\right)^{1/q} \leqq C\|f\|_p.$$

(ii) For $f\in L^p(B_1^c)$

$$\left(\int(1+\log^+|x|)^{-q}|x|^{-qk-n}|K^f_{m,k}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

Using (3.6), (3.7), Lemma 3.1 (iii), 3.12 (i) (ii), 3.14, 3.15 and Remark 3.16, we can prove the following theorem in the same way as Theorem 3.19.

THEOREM 3.21. *Let $m\geqq n$, $m-(n/p)=1, 2,\cdots, m-1$, $k=m-(n/p)$ and $p\leqq q<\infty$. Then*

(i) *for $f\in L^p(B_1)$*

$$\left(\int(1+|\log|x||)^{-q}|x|^{-qk-n}|L^f_{m,k-1}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p,$$

(ii) *for $f\in L^p(B_1^c)$*

$$\left(\int(1+|\log|x||)^{-(q/p')-q-1}|x|^{-qk-n}|L^f_{m,k}(x)|^q dx\right)^{1/q} \leqq C\|f\|_p.$$

3.3 Riesz potentials of order (m, k)

As in the introduction, for a positive integer m, the Riesz kernel of order m is defined by

$$\kappa_m(x)=\begin{cases} |x|^{m-n}, & m<n \quad\text{or}\quad m\geqq n,\ m-n \quad\text{odd},\\ (\delta_{m,n}-\log|x|)|x|^{m-n}, & m\geqq n,\ m-n \quad\text{even},\end{cases}$$

where

$$\delta_{m,n}=\begin{cases}\dfrac{\Gamma'(m/2)}{2\Gamma(m/2)}+\dfrac{1}{2}\Big(1+\dfrac{1}{2}+\cdots+\dfrac{1}{(m-n)/2}-\mathscr{C}\Big)-\log\pi, & m>n,\ m-n \text{ even},\\ \dfrac{\Gamma'(m/2)}{2\Gamma(m/2)}-\dfrac{\mathscr{C}}{2}-\log\pi, & m=n\end{cases}$$

and $\mathscr{C}$ is Euler's constant. We note that $\Delta^m(\rho_{m,n}\kappa_{2m})=\delta$, where

$$\rho_{m,n}=\begin{cases}(-1)^m\Gamma((n/2)-m)2^{-2m}\pi^{-n/2}((m-1)!)^{-1}, & 2m<n \quad\text{or}\quad 2m\geqq n,\ n \quad \text{odd},\\ (-1)^{n/2}2^{-2m+1}\pi^{-n/2}((m-1)!)^{-1}((m-(n/2))!)^{-1}, & 2m\geqq n,\ n \quad \text{even}\end{cases}$$

and δ is the Dirac measure at the origin (cf. [21; §10 in Chap VII]). For an integer $k \leqq m-1$, we set

$$\kappa_{m,k}(x,y)=\begin{cases}\kappa_m(x-y)-\sum_{|\gamma|\leqq k}(x^\gamma/\gamma!)D^\gamma\kappa_m(-y), & 0\leqq k\leqq m-1,\\ \kappa_m(x-y), & k\leqq -1.\end{cases}$$

For a locally integrable function f, U_m^f and $U_{m,k}^f$ are defined by

$$U_m^f(x)=\int\kappa_m(x-y)f(y)dy \quad\text{and}\quad U_{m,k}^f(x)=\int\kappa_{m,k}(x,y)f(y)dy$$

if they exist. Applying Theorems 3.17 and 3.18 to the Riesz potentials of order (m,k), we have

COROLLARY 3.22. *If $m-(n/p)\neq 0,1,\cdots,m-1$ and $k=[m-(n/p)]$, then for any $f\in L^p$, $U_{m,k}^f$ exists and satisfies the following estimates:*

(i) *When $m<n$ or $m\geqq n$, $m-n$ is odd,*

$$\left(\int|x|^{-q(m-(n/p))-n}|U_{m,k}^f(x)|^q dx\right)^{1/q}\leqq \|f\|_p$$

for $p\leqq q<\infty$ in case $m-(n/p)>0$ and for $p\leqq q\leqq p_m$ in case $m-(n/p)<0$.

(ii) *When $m\geqq n$ and $m-n$ is even,*

$$\left(\int(1+|\log|x||)^{-q}|x|^{-q(m/p))-n}|U_{m,k}^f(x)|^q dx\right)^{1/q}\leqq C\|f\|_p$$

for $p\leqq q<\infty$.

By Theorems 3.17, 3.18, 3.19 and 3.21 we also obtain

COROLLARY 3.23. *If $m-(n/p)=0,1,\cdots,m-1$, $k=m-(n/p)$ and $p\leqq q<\infty$, then for any $f\in L^p(B_1)$ and any $g\in L^p(B_1^c)$, $U_{m,k-1}^f$ and $U_{m,k}^g$ exist, and satisfy the following* *estimates:*

(i) *When* $m<n$ *or* $m\geqq n$, $m-n$ *is odd*,

$$\left(\int(1+\log^+(1/|x|))^{-(q/p')-1}|x|^{-q(m-(n/p))-n}|U^f_{m,k-1}(x)|^q dx\right)^{1/q}\leqq C\|f\|_p,$$

$$\left(\int(1+\log^+|x|)^{-(q/p')-1}|x|^{-q(m-(n/p))-n}|U^g_{m,k}(x)|^q dx\right)^{1/q}\leqq C\|g\|_p.$$

(ii) *When* $m\geqq n$ *and* $m-n$ *is even*,

$$\left(\int(1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}|U^f_{m,k-1}(x)^q dx\right)^{1/q}\leqq C\|f\|_p,$$

$$\left(\int(1+|\log|x||)^{-(q/p')-q-1}|x|^{-q(m-(n/p))-n}|U^g_{m,k}(x)|^q dx\right)^{1/q}\leqq C\|g\|_p.$$

Concerning smoothness of $U^f_{m,k}$, we have

PROPOSITION 3.24. (i) *If* $m-(n/p)>0$, $\neq 1, 2,\cdots, m-1$, $k=[m-(n/p)]$ *and* $f\in L^p$, *then* $U^f_{m,k}\in C^k$ *and* $D^\beta U^f_{m,k}(0)=0$ *for* $|\beta|\leqq k$.

(ii) *If* $m-(n/p)=1, 2,\cdots, m-1$, $k=m-(n/p)$ *and* $f\in L^p$, *then* $U^{f_1}_{m,k-1}$, $U^{f_2}_{m,k}\in C^{k-1}$ *and* $D^\beta U^{f_1}_{m,k-1}(0)=D^\beta U^{f_2}_{m,k}(0)=0$ *for* $|\beta|\leqq k-1$, *where* f_1 *and* f_2 *are as in Proposition* 3.9.

PROOF. (i) is a consequence of Lemmas 3.2 (i) and 3.3. (ii) follows from Lemmas 3.4 (i), 3.5, 3.6 (i) and 3.7.

Finally, we discuss partial derivatives of $U^f_{m,k}$ in the sense of distribution.

PROPOSITION 3.25. (I) *Let* $m-(n/p)\neq 0, 1,\cdots, m-1$, $k=[m-(n/p)]$ *and* $f\in L^p$. *Then for* $|\alpha|\leqq m-1$

$$D^\alpha U^f_{m,k}(x)=\int D^\alpha_x\kappa_{m,k}(x, y)f(y)dy \qquad in \quad \mathscr{D}'.$$

(II) *Let* $m-(n/p)=0, 1,\cdots, m-1$ *and* $k=m-(n/p)$. *Then*:

(i) *For* $f\in L^p(B_1)$ *and* $|\alpha|\leqq m-1$

$$D^\alpha U^f_{m,k-1}(x)=\int D^\alpha_x\kappa_{m,k-1}(x, y)f(y)dy \qquad in \quad \mathscr{D}'.$$

(ii) *For* $f\in L^p(B_1^c)$ *and* $|\alpha|\leqq m-1$

$$D^\alpha U^f_{m,k}(x)=\int D^\alpha_x\kappa_{m,k}(x, y)f(y)dy \qquad in \quad \mathscr{D}'.$$

PROOF. We shall give only the proof of (I), because the proof of (II) is similar. Let $|\alpha| \leqq m-1$. For $f \in \mathscr{D}$, we easily see that

$$D^{\alpha}U^{f}_{m,k}(x) = \int D^{\alpha}_{x}\kappa_{m,k}(x, y)f(y)dy \tag{3.15}$$

for all $x \in R^n$. For $f \in L^p$, we take a sequence $\{f_N\}_{N=1,2,\cdots} \subset \mathscr{D}$ such that f_N converges to f as $N \to \infty$ in L^p-norm. From Corollary 3.22 it follows that $U^{f_N}_{m,k}$ converges to $U^{f}_{m,k}$ in $\mathscr{D}'$ as $N \to \infty$. Hence $D^{\alpha}U^{f_N}_{m,k}$ converges to $D^{\alpha}U^{f}_{m,k}$ in $\mathscr{D}'$ as $N \to \infty$. On the other hand, by (3.8) and (3.15) we have

$$D^{\alpha}U^{f_N}_{m,k}(x) = \int (D^{\alpha}\kappa_m(x-y) - \textstyle\sum_{|\beta| \leqq k-|\alpha|} (x^{\beta}/\beta!)\,(D^{\beta}D^{\alpha}\kappa_m)\,(-y))f_N(y)dy.$$

Since $m-|\alpha|-(n/p) \neq 0, 1, \cdots, m-1$ and $[m-|\alpha|-(n/p)] = k-|\alpha|$, it follows from Theorems 3.17 and 3.18 that $D^{\alpha}U^{f_N}_{m,k}(x)$ converges to

$$\int D^{\alpha}_{x}\kappa_{m,k}(x, y)f(y)dy$$

in $\mathscr{D}'$ as $N \to \infty$. Hence we obtain the desired result.

§4. Higher Riesz transforms

The Riesz transforms R_j $(j=1, 2, \cdots, n)$ are defined as follows ([24]): For $f \in L^p$

$$R_j f(x) = \lim_{\varepsilon \to 0} c_n \int_{|x-y| \geqq \varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)dy$$

with $c_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$. The Riesz transforms are bounded operators on L^p. For $f \in L^p$ the Fourier transform of $R_j f$ is given by

$$\mathscr{F}(R_j f)\,(x) = -\frac{ix_j}{|x|}\hat{f}(x), \quad j=1, 2, \cdots, n, \quad i=\sqrt{-1}. \tag{4.1}$$

Moreover, relationship between partial derivatives of the Riesz potential of order 1 and the Riesz transforms is given as follows ([5; Theorem 2 in Chap. III] and [15; Theorem 9.6]): For $f \in L^p$ $(p<n)$,

$$D_j U^{f}_{1} = (1-n)c_n^{-1} R_j f, \qquad j=1, 2, \cdots, n.$$

In this section we are concerned with relationship between partial derivatives of the Riesz potentials of order (m, k) and higher Riesz transforms. Following S. G. Samko [19; §4], for a multi-index α with $|\alpha| = m$ we set

$$R^\alpha = R_1^{\alpha_1} \cdots R_n^{\alpha_n}.$$

We call R^α ($|\alpha| = m$) the higher Riesz transforms of degree m (cf. [24; §3 in Chap. III]). Since the Riesz transforms are bounded operators on L^p, we see that $R^\alpha f \in L^p$ for $f \in L^p$. For $f \in L^2$, it follows from (4.1) that

$$\mathscr{F}(R^\alpha f)(x) = \left(\frac{-ix_1}{|x|}\right)^{\alpha_1} \cdots \left(\frac{-ix_n}{|x|}\right)^{\alpha_n} \hat{f}(x) = \frac{(-i)^m x^\alpha}{|x|^m} \hat{f}(x) \tag{4.2}$$

The following lemma is easily seen by taking Fourier transform.

LEMMA 4.1. *For* $f \in L^p$,

$$\sum_{|\alpha| = m} (m!/\alpha!) R^{2\alpha} f = (-1)^m f.$$

For functions f such that $D^\alpha f \in L^p$ for any $|\alpha| = m$, we set

$$D^m f = \sum_{|\alpha| = m} (m!/\alpha!) R^\alpha D^\alpha f$$

(cf. [19; Corollary in §4]).

LEMMA 4.2. *For* $f \in \mathscr{D}$, *the following equalities hold:*

(i) $\mathscr{F}(D^m f)(x) = (2\pi)^m |x|^m \hat{f}(x)$.

(ii) $D^{2l} f = (-1)^l \Delta^l f$, $l = 1, 2, \cdots$.

(iii) (cf. [19; Theorem 7]) *For a multi-index* α *with* $|\alpha| = m$,

$$R^\alpha D^m f = (-1)^m D^\alpha f.$$

PROOF. (i) It follows from (4.2) that

$$\begin{aligned}
\mathscr{F}(D^m f)(x) &= \sum_{|\alpha| = m} (m!/\alpha!) \mathscr{F}(R^\alpha D^\alpha f)(x) \\
&= \sum_{|\alpha| = m} (m!/\alpha!) (-i)^m \frac{x^\alpha}{|x|^m} (2\pi i x)^\alpha \hat{f}(x) \\
&= (2\pi)^m \sum_{|\alpha| = m} \frac{m! x^{2\alpha}}{\alpha! |x|^m} \hat{f}(x) = (2\pi)^m |x|^m \hat{f}(x)
\end{aligned}$$

(ii) This follows from (i).

(iii) It follows from (4.2) and (i) that

$$\mathscr{F}(R^\alpha D^m f)(x) = (-i)^m \frac{x^\alpha}{|x|^m} (2\pi)^m |x|^m \hat{f}(x) = (-1)^m (2\pi i x)^\alpha \hat{f}(x) = (-1)^m \mathscr{F}(D^\alpha f)(x).$$

In order to given an expression of the Fourier transform of κ_m, we introduce the pseudo function Pf. $|x|^{-m}$, which is defined as follows ([21; §3 in Chap. III]): If $m < n$, Pf. $|x|^{-m} = |x|^{-m}$; if $m - n$ is a positive odd number,

$$<\mathrm{Pf.}|x|^{-m}, \phi> = \lim_{\varepsilon \to 0}\left\{\int_{|x|\geqq\varepsilon} |x|^{-m}\phi(x)dx + \sum_{0\leqq k\leqq(m-n-1)/2} H_k\Delta^k\phi(0)\frac{\varepsilon^{-m+n+2k}}{-m+n+2k}\right\}$$

for $\phi\in\mathscr{D}$; and if $m-n$ is a nonnegative even number,

$$<\mathrm{Pf.}\ |x|^{-m}, \phi> = \lim_{\varepsilon \to 0}\left\{\int_{|x|\geqq\varepsilon} |x|^{-m}\phi(x)dx + \sum_{0\leqq k\leqq(m-n-2)/2} H_k\Delta^k\phi(0)\frac{\varepsilon^{-m+n+2k}}{-m+n+2k} + H_{(m-n)/2}\Delta^{(m-n)/2}\phi(0)\log\varepsilon\right\}$$

for $\phi\in\mathscr{D}$, where $H_k=\pi^{-n/2}2^{2k-1}k!\Gamma((n/2)+k)$. We note that for a homogeneous polynomial $P(x)$ of degree m,

$$P(x)\ (Pf.\ |x|^{-m}) = \frac{P(x)}{|x|^m}. \tag{4.3}$$

The Fourier transform of the Riesz kernel κ_m is given by

$$\hat{\kappa}_m(x) = \gamma_{m,n}\mathrm{Pf.}\ |x|^{-m} \qquad \text{in} \quad \mathscr{S}', \tag{4.4}$$

where

$$\gamma_{m,n} = \begin{cases} \pi^{-m+(n/2)}\Gamma(m/2)/\Gamma((n-m)/2, & m<n \quad \text{or} \quad m\geqq n,\ m-n \quad \text{odd},\\ \pi^{-m+(n/2)}\Gamma(m/2)\ (-1)^{(m-n)/2}2^{-1}\left(\frac{m-n}{2}\right)!, & m\geqq n,\ m-n \quad \text{even} \end{cases}$$

(see [21; §7 in Chap. VII]).

Now we state a relation between the m-th partial derivative of $U^f_{m,k}$ and $R^\alpha f$.

LEMMA 4.3. *Let $f\in\mathscr{D}$. For a multi-index α with $|\alpha|=m$,*

$$D^\alpha U^f_m = (-2\pi)^m\gamma_{m,n}R^\alpha f.$$

PROOF. It follows from (4.2), (4.3) and (4.4) that

$$\mathscr{F}(D^\alpha U^f_m) = \mathscr{F}(D^\alpha(\kappa_m * f)) = (2\pi ix)^\alpha\hat{\kappa}_m\hat{f} = (2\pi ix)\gamma_{m,n}(\mathrm{Pf.}\ |x|^{-m})\hat{f}$$

$$= (-2\pi)^m\gamma_{m,n}\frac{(-i)^m x^\alpha}{|x|^m}\hat{f} = (-2\pi)^m\gamma_{m,n}\mathscr{F}(R^\alpha f).$$

Hence we obtain the lemma.

THEOREM 4.4. *Let* $k=[m-(n/p)]$ *and* $|\alpha|=m$.

(i) *If* $m-(n/p)\neq 0, 1, \cdots, m-1$, *then* $D^{\alpha}U^{f}_{m,k}=(-2\pi)^{m}\gamma_{m,n}R^{\alpha}f$ *for* $f\in L^{p}$.

(ii) *If* $m-(n/p)=0, 1, \cdots, m-1$, *then* $D^{\alpha}U^{f}_{m,k-1}=(-2\pi)^{m}\gamma_{m,n}R^{\alpha}f$ *for* $f\in L^{p}(B_1)$ *and* $D^{\alpha}U^{g}_{m,k}=(-2\pi)^{m}\gamma_{m,n}R^{\alpha}g$ *for* $g\in L^{p}(B_1^c)$.

PROOF. (i) For $f\in L^p$, we take a sequence $\{f_N\}_{N=1,2,\dots}\subset\mathscr{D}$ such that $\|f-f_N\|_p\to 0$ as $N\to\infty$. Since

$$U^{f_N}_{m,k}(x)=\begin{cases} U^{f_N}_{m}(x)-\sum_{|\gamma|\leqq k}(x^{\gamma}/\gamma!)\int D^{\gamma}\kappa_m(-y)f_N(y)dy, & k\geqq 0,\\ U^{f_N}_{m}(x), & k<0,\end{cases}$$

it follows from Lemma 4.3 that

$$D^{\alpha}U^{f_N}_{m,k}=D^{\alpha}U^{f_N}_{m}=(-2\pi)^{m}\gamma_{m,n}R^{\alpha}f_N.$$

By Corollary 3.22 we see that $U^{f_N}_{m,k}$ tends to $U^{f}_{m,k}$ in $\mathscr{D}'$ as $N\to\infty$. Hence $D^{\alpha}U^{f_N}_{m,k}$ tends to $D^{\alpha}U^{f}_{m,k}$ in $\mathscr{D}'$ as $N\to\infty$. On the other hand, $R^{\alpha}f_N$ tends to $R^{\alpha}f$ as $N\to\infty$ in L^p-norm. Therefore we obtain the required equality.

(ii) Using Lemma 4.3 and Corollary 3.23 we can prove (ii) in the same way as above.

By Lemma 4.1 and Theorem 4.4 we obtain

COROLLARY 4.5. *Let* $k=[m-(n/p)]$.

(i) *If* $m-(n/p)\neq 0, 1, \cdots, m-1$, *then* $D^{m}U^{f}_{m,k}=(2\pi)^{m}\gamma_{m,n}f$ *for* $f\in L^{p}$.

(ii) *If* $m-(n/p)=0, 1, \cdots, m-1$, *then* $D^{m}U^{f}_{m,k-1}=(2\pi)^{m}\gamma_{m,n}f$ *for* $f\in L^{p}(B_1)$ *and* $D^{m}U^{g}_{m,k}=(2\pi)^{m}\gamma_{m,n}g$ *for* $g\in L^{p}(B_1^c)$.

§5. Primitives of higher order

In this section we discuss primitives of higher order of L^p-functions. We begin with some observations on primitives of smooth functions. Let m be a positive integer and $\{f_\alpha\}_{|\alpha|=m}\subset C^{\infty}$ be a family of functions such that $D_j f_\alpha=D_k f_\beta$ whenever $\alpha+e_j=\beta+e_k$ and $|\alpha|=|\beta|=\mathrm{m}$. Set

$$\phi(x)=\sum_{|\alpha|=m}(m/\alpha!)\int_0^{|x|}(x')^{\alpha}(|x|-t)^{m-1}f_{\alpha}(tx')dt.$$

Then $\phi\in C^\infty$ and for each multi-index α with $|\alpha|=m$, $D^\alpha\phi=f_\alpha$. Next, let $\{g_j\}_{j=1,\cdots,n}\subset\mathscr{D}$ be a family of functions such that $D_jg_k=D_kg_j$ for all $j,k=1,\cdots,n$. Taking $\xi\in R^n$ with $|\xi|=1$, we set

$$\psi(x)=\sum_{1\leq j\leq n}\int_0^\infty \xi_j g_j(x-t\xi)dt.$$

By Stokes' theorem we see that $\psi(x)$ is independent of ξ and $D_j\psi=g_j$ for $j=1,\cdots,n$, which shows that $\psi\in\mathscr{D}$. From this fact, we see the following: Let $\{g_\alpha\}_{|\alpha|=m}\subset\mathscr{D}$ be family of functions such that $D_jg_\alpha=D_kg_\beta$ for $\alpha+e_j=\beta+e_k$ and $|\alpha|=|\beta|=$ m. Defining

$$\chi(x)=\sum_{|\alpha|=m}(m/\alpha!)\int_0^\infty \xi^\alpha t^{m-1}g_\alpha(x-t\xi)dt,$$

we see that $\chi\in\mathscr{D}$ and $D^\alpha\chi=g_\alpha$ for any multi-index α with $|\alpha|=m$. We note that $\chi(x)$ is independent of ξ. Using this fact, we see that

$$\begin{aligned}\chi(x)&=\sigma_n^{-1}\int_{|\xi|=1}\chi(x)dS(\xi)\\&=\sum_{|\alpha|=m}m/(\sigma_n\alpha!)\int_{|\xi|=1}\int_0^\infty \xi^\alpha t^{m-1}g_\alpha(x-t\xi)dtdS(\xi)\\&=\sum_{|\alpha|=m}m/(\sigma_n\alpha!)\int\frac{y^\alpha}{|y|^n}g_\alpha(x-y)dy=\sum_{|\alpha|=m}m/(\sigma_n\alpha!)\int\frac{(x-y)^\alpha}{|x-y|^n}g_\alpha(y)dy.\end{aligned}$$

Thus we obtain the following result (cf. Yu. G. Reshetnyak [18; Lemma 6.2]).

PROPOSITION 5.1. *Let $\{f_\alpha\}_{|\alpha|=m}\subset\mathscr{D}$ be a family of functions such that $D_jf_\alpha=D_kf_\beta$ for $\alpha+e_j=\beta+e_k$ and $|\alpha|=|\beta|=m$. If we set*

$$\phi(x)=\sum_{|\alpha|=m}m/(\sigma_n\alpha!)\int\frac{(x-y)^\alpha}{|x-y|^n}f_\alpha(y)dy,$$

then $\phi\in\mathscr{D}$ and $D^\alpha\phi=f_\alpha$ for any $|\alpha|=m$.

Concerning a primitive of distributions, from L. Schwartz [21; §6 in Chap. II] we have

PROPOSITION 5.2. *Let $\{u_\alpha\}_{|\alpha|=m}$ be a family of distributions such that $D_ju_\alpha=D_ku_\beta$ for $\alpha+e_j=\beta+e_k$ and $|\alpha|=|\beta|=m$. Then there exists a distribution u such that $D^\alpha u=u_\alpha$ for any $|\alpha|=m$.*

The following lemma is due to S. L. Sobolev [23].

LEMMA 5.2. *Let u be a distribution such that $D^\alpha u\in L^p$ for all $|\alpha|=m$. Then there exists a sequence $\{\phi_N\}\subset\mathscr{D}$ such that $D^\alpha\phi_N$ converges to $D^\alpha u$ in L^p-norm as $N\to\infty$ for* *any* $|\alpha|=m$.

COROLLARY 5.4. *Let* $\{f_\alpha\}_{|\alpha|=m}$ *be a family of* L^p*-functions such that* $D_j f_\alpha = D_k f_\beta$ *for* $\alpha+e_j=\beta+e_k$ *and* $|\alpha|=|\beta|=m$. *Then there exists a family of functions* $\{\phi_{\alpha,N}\}_{|\alpha|=m, N=1,2\cdots}\subset\mathscr{D}$ *such that*

$$D_j\phi_{\alpha,N}=D_k\phi_{\beta,N} \quad \text{for} \quad \alpha+e_j=\beta+e_k \quad \text{and} \quad |\alpha|=|\beta|=m, \tag{5.1}$$

$$\phi_{\alpha,N}\longrightarrow f_\alpha \quad \text{as} \quad N\to\infty \quad \text{in } L^p\text{-norm for each} \quad |\alpha|=m. \tag{5.2}$$

PROOF. By Proposition 5.2 there exists a distribution u such that $D^\alpha u=f_\alpha\in L^p$ for all $|\alpha|=m$. By Lemma 5.3 there exists a sequence $\{\phi_N\}_{N=1,2,\cdots}\subset\mathscr{D}$ such that $D^\alpha\phi_N$ converges to $D^\alpha u=f_\alpha$ as $N\to\infty$ in L^p-norm. If we set $\phi_{\alpha,N}=D^\alpha\phi_N$ for each $|\alpha|=m$, then the family $\{\phi_{\alpha,N}\}_{|\alpha|=m,N=1,2,\cdots}$ satisfies conditions (5.1) and (5.2).

Now we study primitives of L^p-functions. For a multi-index α with $|\alpha|=m$ we put

$$\kappa_\alpha(x)=m/(\sigma_n\alpha!)\frac{x^\alpha}{|x|^n}.$$

The function $\kappa_\alpha(x)$ is a homogeneous function of degree $m-n$. For an integer $k\leqq m-1$ we set

$$\kappa_{\alpha,k}(x,y)=\begin{cases}\kappa_\alpha(x-y)-\sum_{|\gamma|\leqq k}(x^\gamma/\gamma!)D^\gamma\kappa_\alpha(-y), & 0\leqq k\leqq m-1\\ \kappa_\alpha(x-y), & k\leqq -1.\end{cases}$$

First, we consider the case $m-(n/p)\neq 0, 1,\cdots, m-1$.

THEOREM 5.5. *Let* $m-(n/p)\neq 0,1,\cdots,m-1$ *and* $k=[m-(n/p)]$. *We assume that* $F=\{f_\alpha\}_{|\alpha|=m}$ *is a family of* L^p*-functions such that* $D_jf_\alpha=D_lf_\beta$ *for* $\alpha+e_j=\beta+e_l$ *and* $|\alpha|=|\beta|=m$. *If we set*

$$V^F_{m,k}(x)=\sum_{|\alpha|=m}\int\kappa_{\alpha,k}(x,y)f_\alpha(y)dy,$$

then $V^F_{m,k}$ *satisfies the following conditions*:

$$D^\alpha V^F_{m,k}=f_\alpha \qquad \text{for any} \quad |\alpha|=m. \tag{5.3}$$

$$\left(\int|x|^{-q(m-(n/p))-n}|V^F_{m,k}(x)|^q dx\right)^{1/q}\leqq C\sum_{|\alpha|=m}\|f_\alpha\|_p \tag{5.4}$$

for $p\leqq q<\infty$ *in case* $m-(n/p)>0$ *and for* $p\leqq q\leqq p_m$ *in case* $m-(n/p)<0$.

$$V^F_{m,k}\in C^k \quad \text{and} \quad D^\beta V^F_{m,k}(0)=0 \quad \text{for any} \quad |\beta|\leqq k \quad \text{if} \quad k\geqq 0. \tag{5.5}$$

PROOF. By Corollary 5.4 there exists a family of functions $\{\phi_{\alpha,N}\}_{|\alpha|=m, N=1,2,\dots} \subset \mathscr{D}$ which satisfies (5.1) and (5.2). We write $\Phi_N = \{\phi_{\alpha,N}\}_{|\alpha|=m}$. We see that

$$V^{\Phi_N}_{m,k}(x) = \sum_{|\alpha|=m} \int \kappa_{\alpha,\kappa}(x, y)\phi_{\alpha,N}(y)dy$$

$$= \begin{cases} \sum_{|\alpha|=m} \int \kappa_\alpha(x-y)\phi_{\alpha,N}(y)dy - \sum_{|\gamma| \leqq k} \\ \qquad \left(\sum_{|\alpha|=m} \int D^\gamma \kappa_\alpha(-y)\phi_{\alpha,N}(y)dy\right)(x^\gamma/\gamma!), & k \geqq 0, \\ \sum_{|\alpha|=m} \int \kappa_\alpha(x-y)\phi_{\alpha,N}(y)dy, & k \leqq -1. \end{cases}$$

Hence it follows from Proposition 5.1 that for any $|\alpha| = m$.

$$D^\alpha V^{\Phi_N}_{m,k} = \phi_{\alpha,N}.$$

On account of Theorem 3.17 we see that $V^{\Phi_N}_{m,k}$ converges to $V^F_{m,k}$ in $\mathscr{D}'$ as $N \to \infty$, so that $D^\alpha V^{\Phi_N}_{m,k}$ converges to $D^\alpha V^F_{m,k}$ in $\mathscr{D}'$ as $N \to \infty$. On the other hand, $\phi_{\alpha,N}$ converges to f_α as $N \to \infty$ in L^p-norm. Consequently we obtain (5.3). Assertions (5.4) and (5.5) follow from Theorem 3.17 and Lemma 3.2 (i), respectively.

REMARK 5.6. The function which satisfies (5.3) and (5.4) is unique.

LEMMA 5.7. *Let $\{f_\alpha\}_{|\alpha|=m}$ be a family of L^p-functions. A necessary and sufficient condition that $D_j f_\alpha = D_k f_\beta$ for $\alpha + e_j = \beta + e_k$ and $|\alpha| = |\beta| = m$ is that there exists an L^p-function f such that $f_\alpha = R^\alpha f$ for any $|\alpha| = m$. In this case, the function f is given by*

$$f = (-1)^m \sum_{|\alpha|=m} (m!/\alpha!) R^\alpha f_\alpha.$$

PROOF. First we assume that there exists an L^p-function f such that $f_\alpha = R^\alpha f$ for any $|\alpha| = m$. We take a sequence $\{f_N\}_{N=1,2,\dots} \subset \mathscr{D}$ such that f_N tends to f as $N \to \infty$ in L^p-norm. For each multi-index α with $|\alpha| = m$ we put $f_{\alpha,N} = R^\alpha f_N$. Then we see that

$$\mathscr{F}(D_j f_{\alpha,N})\ (x) = 2\pi i x_j \frac{(-i)^m x^\alpha}{|x|^m} \hat{f}_N(x) = 2\pi i(-i)^m \frac{x^{\alpha+e_j}}{|x|^m} \hat{f}_N(x).$$

Hence for $\alpha + e_j = \beta + e_k$ and $|\alpha| = |\beta| = m$ we have $D_j f_{\alpha,N} = D_k f_{\beta,N}$. Letting $N \to \infty$, we get $D_j f_\alpha = D_k f_\beta$. Next we assume that $D_j f_\alpha = D_k f_\beta$ for $\alpha + e_j = \beta + e_k$ and $|\alpha| = |\beta| = m$. By Corollary 5.4 there exists a family of functions $\{\phi_{\alpha,N}\}_{|\alpha|=m, N=1,2,\dots} \subset \mathscr{D}$ which satisfies (5.1) and (5.2). It follows from (5.1) and Proposition 5.1 that there is a function $\phi_N \in \mathscr{D}$ such that $D^\alpha \phi_N = \phi_{\alpha,N}$ for any $|\alpha| = m$. If we put

$$f = (-1)^m \sum_{|\beta|=m} (m!/\beta!) R^\beta f_\beta,$$

then f belongs to L^p. Since

$$(-1)^m D^m \phi_N = (-1)^m \sum_{|\beta|=m} (m!/\beta!) R^\beta D^\beta \phi_N = (-1)^m \sum_{|\beta|=m} (m!/\beta!) R^\beta \phi_{\beta,N},$$

it follows from (5.2) that $(-1)^m D^m \phi_N$ tends to f as $N \to \infty$ in L^p-norm. By Lemma 4.2 (iii), for each multi-index α with $|\alpha| = m$ we have

$$(-1)^m R^\alpha D^m \phi_N = (-1)^{2m} D^\alpha \phi_N = \phi_{\alpha,N}.$$

Since $(-1)^m R^\alpha D^m \phi_N$ tends to $R^\alpha f$ and $\phi_{\alpha,N}$ tends to f_α as $N \to \infty$ in L^p-norm, we have $R^\alpha f = f_\alpha$.

THEOREM 5.8. *Let $m-(n/p) \neq 0, 1, \cdots, m-1$ and $k = [m-(n/p)]$.*

(i) *Let $f \in L^p$ and $F = \{R^\alpha f\}_{|\alpha|=m}$. Then $U^f_{m,k} = (-2\pi)^m \gamma_{m,n} V^F_{m,k}$.*

(ii) *Let $F = \{f_\alpha\}_{|\alpha|=m}$ be a family of L^p-functions such that $D_j f_\alpha = D_l f_\beta$ for $\alpha + e_j = \beta + e_l$ and $|\alpha| = |\beta| = m$. If we set $f = (-1)^m \sum_{|\alpha|=m} (m!/\alpha!) R^\alpha f_\alpha$, then $U^f_{m,k} = (-2\pi)^m \gamma_{m,n} V^F_{m,k}$.*

PROOF. (i) It follows from Theorems 4.4 (i) and 5.5 that for any multi-index α with $|\alpha| = m$.

$$D^\alpha U^f_{m,k} = (-2\pi)^m \gamma_{m,n} R^\alpha f = (-2\pi)^m \gamma_{m,n} D^\alpha V^F_{m,k}.$$

Hence $U^f_{m,k} - (-2\pi)^m \gamma_{m,n} V^F_{m,k}$ is a polynomial P of degree $m-1$. Moreover by Corollary 3.22 and Theorem 5.5 we see that

$$\int (1 + |\log|x||)^{-p} |x|^{-mp} |P(x)|^p dx < \infty.$$

This implies $P(x) = 0$.

(ii) Using Lemma 5.7 we obtain (ii) from (i).

COROLLARY 5.9 (cf. Corollary 3.22). *If $m-(n/p) \neq 0, 1, \cdots, m-1$, $k = [m-(n/p)]$ and $|\beta| \leqq m-1$, then*

$$\left(\int |x|^{-q(m-|\beta|-(n/p))-n} |D^\beta U^f_{m,k}(x)|^q dx \right)^{1/q} \leqq \|f\|_p$$

for $p \leqq q < \infty$ in case $m-(n/p) > 0$ and for $p \leqq q \leqq p_m$ in case $m-(n/p) < 0$.

PROOF. If we put $F = \{R^\alpha f\}_{|\alpha|=m}$, then by Theorems 5.8 and 3.17 we see that

$$\left(\int |x|^{-q(m-|\beta|-(n/p))-n} |D^\beta U^f_{m,k}(x)|^q dx \right)^{1/q}$$

$$= C \left(\int |x|^{-q(m-|\beta|-(n/p))-n} |D^\beta V^F_{m,k}(x)|^q dx \right)^{1/q}$$

$$\leqq C\sum_{|\alpha|=m}\|R^{\alpha}f\|_p \leqq C\|f\|_p.$$

REMARK 5.10. Let $m-n<0$ and $f\in\mathscr{D}$. Then, from Theorem 4.4, Proposition 5.1 and Theorem 5.8 it follows that $U_m^f\in\mathscr{D}$ if and only if $R^{\alpha}f\in\mathscr{D}$ for all $|\alpha|=m$.

Next we shall discuss the case $m-(n/p)=0, 1,\cdots, m-1$.

THEOREM 5.11. *Let* $m-(n/p)=0, 1,\cdots, m-1$ *and* $k=m-(n/p)$.

(I) *Let* $F=\{f_\alpha\}_{|\alpha|=m}\subset L^p(B_1)$ *be a family of functions such that* $D_jf_\alpha=D_lf_\beta$ *for* $\alpha+e_j=\beta+e_l$ *and* $|\alpha|=|\beta|=m$. *Then* $V^F_{m,k-1}$ *satisfies the following conditions*:

(i) $$D^{\alpha}V^F_{m,k-1}=f_\alpha \qquad \text{for any} \quad |\alpha|=m.$$

(ii) $$\left(\int(1+\log^+(1/|x|))^{-(q/p)-1}|x|^{-qk-n}|V^F_{m,k-1}(x)|^q dx\right)^{1/q}\leqq C\sum_{|\alpha|=m}\|f_\alpha\|_p$$

for $p\leqq q<\infty$.

(iii) $V^F_{m,k-1}\in C^{k-1}$ *and* $D^{\alpha}V^F_{m,k-1}(0)=0$ *for any* $|\beta|\leqq k-1$ *if* $k\geqq 1$.

(II) *Let* $F=\{f_\alpha\}_{|\alpha|=m}\subset L^p(B_1^c)$ *be a family of functions such that* $D_jf_\alpha=D_lf_\beta$ *for* $\alpha+e_j=\beta+e_l$ *and* $|\alpha|=|\beta|=m$. *Then* $V^F_{m,k}$ *satisfies the following conditions*:

(i) $$D^{\alpha}V^F_{m,k}=f_\alpha \qquad \text{for any} \quad |\alpha|=m.$$

(ii) $$\left(\int(1+\log^+|x|)^{-(q/p')-1}|x|^{-qk-n}|V^F_{m,k}(x)|^q dx\right)^{1/q}\leqq C\sum_{|\alpha|=m}\|f_\alpha\|_p \quad \text{for} \quad p$$

$\leqq q<\infty$.

(iii) $V^F_{m,k}\in C^{k-1}$ *and* $D^{\beta}V^F_{m,k}(0)=0$ *for any* $|\beta|\leqq k-1$ $(k\geqq 1)$.

PROOF. We will only give the proof of (I). By Corollary 5.4 there exists a family of functions $\{\phi_{\alpha,N}\}_{|\alpha|=m,N=1,2,\cdots}\subset\mathscr{D}$ which satisfies (5.1) and (5.2). We put $\Phi_N=\{\phi_{\alpha,N}\}_{|\alpha|=m}$. Moreover we denote $\phi^1_{\alpha,N}=\phi_{\alpha,N}|_{B_1}$ and $\phi^2_{\alpha,N}=\phi_{\alpha,N}-\phi^1_{\alpha,N}$. We set

$$V^{\Phi_N}_{m,k-1}(x)=\sum_{|\alpha|=m}\int\kappa_{\alpha,k-1}(x,y)\phi_{\alpha,N}(y)dy=V^1_N+V^2_N+V^3_N,$$

where

$$V^i_N=\sum_{|\alpha|=m}\left(\int\kappa_\alpha(x-y)\phi^i_{\alpha,N}(y)dy-\sum_{|\gamma|\leq k-1}(x^\gamma/\gamma!)\int D^\gamma\kappa_\alpha(-y)\phi^i_{\alpha,N}(y)dy\right),$$

$i=1, 2$, and

$$V^3_N=\sum_{|\alpha|=m}\sum_{|\gamma|=k}(x^\gamma/\gamma!)\int D^\gamma\kappa_\alpha(-y)\,\phi^2_{\alpha,N}(y)dy.$$

Since $\phi^1_{\alpha,N}, f_\alpha \in L^p(B_1)$ and $\|\phi^1_{\alpha,N}-f_\alpha\|_p \to 0$ as $N\to\infty$, by Theorem 3.19 (i) we see that V^1_N tends to $V^F_{m,k-1}$ in $\mathscr{D}'$ as $N\to\infty$. Since $\phi^2_{\alpha,N}\in L^p(B^c_1)$ and $\|\phi^2_{\alpha,N}\|_p\to 0$ as $N\to\infty$, it follows from Theorem 3.19 (ii) that $V^2_N\to 0$ in $\mathscr{D}'$ as $N\to\infty$. Hence for any $|\alpha|=m$ we have

$$D^\alpha V^{\Phi_N}_{m,k-1} = D^\alpha V^1_N + D^\alpha V^2_N \longrightarrow D^\alpha V^F_{m,k-1} \qquad \text{in } \mathscr{D}'$$

as $N\to\infty$. On the other hand, $D^\alpha V^{\Phi_N}_{m,k-1} = \phi_{\alpha,N}$ tends to f_α as $N\to\infty$ in L^p-norm. Hence $D^\alpha V^F_{m,k-1} = f_\alpha$, which shows (i). Assertions (ii) and (iii) follow from Theorem 3.19 (i) and Lemma 3.4 (i), respectively.

In case $m-(n/p)=0, 1, \cdots, m-1$, it still remains to discuss primitives of a family $\{f_\alpha\}_{|\alpha|=m}$ of L^p-functions on the whole space R^n such that $D_j f_\alpha = D_l f_\beta$ for $\alpha+e_j=\beta+e_l$ and $|\alpha|=|\beta|=m$. We need the following lemma.

LEMMA 5.12 ([2; Theorem 4.14]). *Let m be a positive integer. Suppose that $D^\alpha u\in L^p$ for any $|\alpha|=m$. Then for a multi-index β with $|\beta|\leqq m-1$ and $r>0$, we have*

$$\left(\int_{Br} |D^\beta u(x)|^p dx\right)^{1/p} \leqq C\left(\left(\int_{Br} |u(x)|^p dx\right)^{1/p} + \sum_{|\alpha|=m}\left(\int_{Br} |D^\alpha u(x)|^p dx\right)^{1/p}\right).$$

THEOREM 5.13. *Let $m-(n/p)=0, 1, \cdots, m-1$ and $k=m-(n/p)$. If $F=\{f_\alpha\}_{|\alpha|=m}$ is a family of L^p-functions such that $D_j f_\alpha = D_l f_\beta$ for $\alpha+e_j=\beta+e_l$ and $|\alpha|=|\beta|=m$, then there exists a function v which satisfies the following conditions*:

(5.6) $$D^\alpha v = f_\alpha \qquad \text{for any} \quad |\alpha|=m.$$

(5.7) $$\left(\int (1+|\log|x||)^{-(q/p')-1} |x|^{-qk-n} |v(x)|^q dx\right)^{1/q} \leqq C\sum_{|\alpha|=m} \|f_\alpha\|_p.$$

(5.8) $$v\in C^{k-1} \quad \textit{and} \quad D^\beta v(0)=0 \quad \textit{for any} \quad |\beta|\leqq k-1 \quad \textit{if} \quad k\geqq 1.$$

PROOF. We put $f=(-1)^m\sum_{|\alpha|=m}(m!/\alpha!)R^\alpha f_\alpha$, $f_1=f|_{B_1}$ and $f_2=f-f_1$. If we set $u=U^{f_1}_{m,k-1}+U^{f_2}_{m,k}$, then it follows from Theorem 4.4 (ii) and Lemma 5.7 that for any $|\alpha|=m$

$$D^\alpha u = (-2\pi)^m \gamma_{m,n}(R^\alpha f_1 + R^\alpha f_2) = (-2\pi)^m\gamma_{m,n}R^\alpha f = (2\pi)^m\gamma_{m,n} f_\alpha.$$

We take functions ϕ_1 and ϕ_2 such that $\phi_1,\phi_2\in C^\infty$, $\phi_1\geqq 0$, $\phi_2\geqq 0$, $\phi_1+\phi_2=1$, $\phi_1(x)=0$ for $|x|\geqq 2$ and $\phi_2(x)=0$ for $|x|<1/2$. We put $u_i=\phi_i u$ and $f_{i,\alpha}=(2\pi)^{-m}\gamma^{-1}_{m,n}D^\alpha u_i$ for each multi-index α with $|\alpha|=m$ and $i=1, 2$. It is easily seen that $f_{1,\alpha}\in L^p(B_2)$, $f_{2,\alpha}\in L^p(B^c_{1/2})$, $f_{1,\alpha}+f_{2,\alpha}=f_\alpha$ and $D_j f_{1,\alpha}=D_l f_{1,\beta}$, $D_j f_{2,\alpha}=D_l f_{2,\beta}$ for $\alpha+e_j=\beta+e_l$ and $|\alpha|=|\beta|=m$. We shall show

(5.9) $$\|f_{i,\alpha}\|_p \leqq C\sum_{|\gamma|=m}\|f_\gamma\|_p, \qquad i=1, 2.$$

By Leibniz's formula we have

$$f_{i,\alpha}=D^{\alpha}(\phi_i u)=\sum_{\delta\leqq\alpha}\binom{\alpha}{\delta}D^{\alpha-\delta}\phi_i D^{\delta}u,\qquad i=1,2.$$

Hence using Lemma 5.12

$$\|f_{i,\alpha}\|_p\leqq C\sum_{\delta<\alpha}\left(\int_{B_2}|D^{\delta}u(x)|^p dx\right)^{1/p}+\left(\int|D^{\alpha}u(x)|^p dx\right)^{1/p}$$
$$\leqq C\left(\left(\int_{B_2}|u(x)|^p dx\right)^{1/p}+\sum_{|\gamma|=m}\|f_\gamma\|_p\right).$$

Since $u=U^{f_1}_{m,k-1}+U^{f_2}_{m,k}$, it follows from Corollary 3.23 that

$$\left(\int_{B_2}|u(x)|^p dx\right)^{1/p}\leqq\left(\int_{B_2}|U^{f_1}_{m,k-1}(x)|^p dx\right)^{1/p}+\left(\int_{B_2}|U^{f_2}_{m,k}(x)|^p dx\right)^{1/p}$$
$$\leqq C(\|f_1\|_p+\|f_2\|_p)\leqq C\|f\|_p\leqq C\sum_{|\gamma|=m}\|R^{\gamma}f_\gamma\|_p\leqq C\sum_{|\gamma|=m}\|f_\gamma\|_p.$$

Thus we obtain (5.9). If we set $F_1=\{f_{1,\alpha}\}_{|\alpha|=m}$, $F_2=\{f_{2,\alpha}\}_{|\alpha|=m}$ and $v=V^{F_1}_{m,k-1}+V^{F_2}_{m,k}$, then by Theorem 5.11 and (5.9), v satisfies the conditions (5.6), (5.7) and (5.8).

REMARK 5.14. The function which satisfies (5.6) and (5.7) is unique up to a homogeneous polynomial of degree k. In fact, if v_1 and v_2 satisfy (5.6) and (5.7), then v_1-v_2 is polynomial P satisfying

$$\int(1+|\log|x||)^{-p}|x|^{-kp-n}|P(x)|^p dx<\infty.$$

Hence v_1-v_2 must be of the form $P(x)=\sum_{|\gamma|=k}a_\gamma x^\gamma$. Moreover we note that for the uniqueness of v condition (5.7) can be replaced by

$$\int(1+|\log|x||)^{-(q/p')-q-1}|x|^{-qk-n}|v(x)|^q dx<\infty.$$

PROPOSITION 5.15 (cf. Corollary 3.23). *If* $m-(n/p)=0,1,\cdots,m-1$, $k=m-(n/p)$ *and* $p\leqq q<\infty$, *then*

(i) *for* $f\in L^p(B_1)$ *and* $|\beta|\leqq m-1$

$$\left(\int(1+|\log|x||)^{-(q/p')-1}|x|^{-q(m-|\beta|-(n/p))-n}|D^{\beta}U^{f}_{m,k-1}(x)|^q dx\right)^{1/q}\leqq C\|f\|_p,$$

(ii) *for* $g\in L^p(B_1^c)$ *and* $|\beta|\leqq m-1$

$$\left(\int(1+|\log|x||)^{-(q/p')-1}|x|^{-q(m-|\beta|-(n/p))-n}|D^{\beta}U^{g}_{m,k}(x)|^q dx\right)^{1/q}\leqq C\|g\|_p.$$

PROOF. We give only the proof of (i), because the proof of (ii) is similar. Let $f\in L^p(B_1)$ and $F=\{R^\alpha f\}_{|\alpha|=m}$ By Theorem 5.13 there exists a function v which satisfies

$$D^\alpha v = R^\alpha f \qquad \text{for any} \quad |\alpha| = m, \tag{5.10}$$

$$\left(\int (1+|\log|x||)^{-q}|x|^{-q(m-(n/p))-n}|v(x)^q dx\right)^{1/q} \leqq C\|f\|_p. \tag{5.11}$$

On the other hand, by Corollary 3.23 and Theorem 4.4 the function $u=(-2\pi)^{-m}\gamma_{m,n}^{-1}U^f_{m,k-1}$ also satisfies (5.10) and (5.11). Hence by Remark 5.14 we have $u(x)-v(x)=\sum_{|\gamma|=k}(a_\gamma/\gamma!)x^\gamma$. Since $D^\gamma u - D^\gamma v = a_\gamma$ for any multi-index γ with $|\gamma|=k$, it follows from Lemma 5.12 and (5.10) that

$$\begin{aligned} |a_\gamma| &= \omega_n^{-1/p}\left(\int_{B_1} |a_\gamma|^p dx\right)^{1/p} \\ &\leqq C\left(\int_{B_1} |D^\gamma u(x)|^p dx\right)^{1/p} + C\left(\int_{B_1} |D^\gamma v(x)|^p dx\right)^{1/p} \leqq C\|f\|_p. \end{aligned}$$

Therefore in view of the proof of Theorem 5.13 and Theorem 3.19 we have

$$\begin{aligned} &\left(\int (1+|\log|x||)^{-(q/p')-1}|x|^{-q(m-|\beta|-(n/p))-n}|D^\beta U^f_{m,k-1}(x)|^q dx\right)^{1/q} \\ &\leqq C\left(\int (1+|\log|x||)^{-(q/p')-1}|x|^{-q(m-|\beta|-(n/p))-n}|D^\beta v(x)|^q dx\right)^{1/q} \\ &\quad + C\sum_{|\gamma|=k}|a_\gamma|\left(\int (1+|\log|x||)^{-(q/p')-1}|x|^{-n} dx\right)^{1/q} \\ &\leqq C\|f\|_p + C\sum_{|\gamma|=k}\|f\|_p = C\|f\|_p. \end{aligned}$$

Thus we obtain (i).

§6. Beppo Levi spaces

For a positive integer m and $p>1$, the space $\mathscr{L}^p_m$ is defined by

$$\mathscr{L}^p_m = \{u\in\mathscr{D}';\ D^\alpha u\in L^p \quad \text{for any} \quad |\alpha|=m\}.$$

We call $\mathscr{L}^p_m$ a Beppo Levi space and elements of $\mathscr{L}^p_m$ Beppo Levi functions. Beppo Levi functions are locally integrable. First, we give potential representations of Beppo Levi functions.

THEOREM 6.1. *Let $m-(n/p)\neq 0, 1, \cdots, m-1$ and $k=[m-(n/p)]$. Then $u\in\mathscr{L}_m^p$ can be represented as*

$$(6.1)\qquad u=\sum_{|\gamma|\leqq m-1}a_\gamma x^\gamma+U_{m,k}^f \qquad \text{with} \quad f=(2\pi)^{-m}\gamma_{m,n}^{-1}D^m u\in L^p.$$

Moreover, for $p\leqq q<\infty$ in case $m-(n/p)>0$ and for $p\leqq q\leqq p_m$ in case $m-(n/p)<0$

$$(6.2)\qquad \left(\int |x|^{-q(m-(n/p))-n}|U_{m,k}^f(x)|^q dx\right)^{1/q}\leqq\sum_{|\alpha|=m}\|D^\alpha u\|_p,$$

and for $|\gamma|\leqq m-1$

$$(6.3)\qquad |a_\gamma|\leqq C\left(\left(\int_{B1}|u(y)|^p dy\right)^{1/p}+\sum_{|\alpha|=m}\|D^\alpha u\|_p\right).$$

Furthermore, if $k\geqq 0$, then $u\in C^k$ and for $|\gamma|\leqq k$

$$(6.4)\quad |a_\gamma|=\left|\frac{D^\gamma u(0)}{\gamma!}\right|\leqq C\left(\left(\int_{B1}|u(y)|^p dy\right)^{1/p}+\sum_{|\alpha|=m}\left(\int_{B1}|D^\alpha u(y)|^p dy\right)^{1/p}\right).$$

PROOF. By Lemma 5.3 there is a sequence $\{\phi_N\}_{N=1,2,\cdots}\subset\mathscr{D}$ such that $D^\alpha\phi_N$ converges to $D^\alpha u$ as $N\to\infty$ in L^p-norm for all $|\alpha|=m$. By Lemma 4.2 (iii) we have $R^\alpha D^m\phi_N=(-1)^m D^\alpha\phi_N$. Letting $N\to\infty$, we get $(2\pi)^m\gamma_{m,n}R^\alpha f=(-1)^m D^\alpha u$. On the other hand, by Theorem 4.4 (i) we have $D^\alpha U_{m,k}^f=(-2\pi)^m\gamma_{m,n}R^\alpha f$. Therefore $D^\alpha u=D^\alpha U_{m,k}^f$ for all $|\alpha|=m$. Hence $u-U_{m,k}^f=\sum_{|\gamma|\leqq m-1}a_\gamma x^\gamma$. If $k\geqq 0$, then from Proposition 3.24 it follows that $U_{m,k}^f\in C^k$ and $D^\gamma U_{m,k}^f(0)=0$ for any $|\gamma|\leqq k$. Hence we see that $u\in C^k$ and $a_\gamma=D^\gamma u(0)/\gamma!$ for $|\gamma|\leqq k$. Assertion (6.2) follows from Corollary 3.22 (i) and Corollary 5.9. We shall show (6.4). We take a function $\eta\in\mathscr{D}$ such that $\eta(x)=1$ for $|x|<1/2$ and $\eta(x)=0$ for $|x|\geqq 1$. Since $\Delta^m(\rho_{m,n}\kappa_{2m})=\delta$ (see §3.3), we have

$$(6.5)\qquad \delta=\Delta^m((1-\eta)\rho_{m,n}\kappa_{2m})+\Delta^m(\eta\rho_{m,n}\kappa_{2m}).$$

Obviously, $\zeta=\Delta^m((1-\eta)\rho_{m,n}\kappa_{2m})\in C^\infty$ and supp $\zeta\subset\{|x|\leqq 1\}$. Moreover, taking a function $\phi\in\mathscr{D}$ such that $0\leqq\phi\leqq 1$ and $\phi(x)=1$ for $|x|<1$, we put $v=\phi u$. By (6.5) we see that for $|\gamma|\leqq k$

$$D^\gamma v=D^\gamma\zeta*v+\sum_{|\alpha|=m}(m!/\alpha!)D^{\gamma+\alpha}(\eta\rho_{m,n}\kappa_{2m})*D^\alpha v.$$

Since $D^\beta u(y)=D^\beta v(y)$ for $|y|<1$ and any multi-index β,

$$|D^\gamma u(0)|\leqq\int_{B_1}|D^\gamma\zeta(-y)v(y)|dy+C\sum_{|\alpha|=m}\int_{B_1}|D^{\gamma+\alpha}(\eta\rho_{m,n}\kappa_{2m})(-y)D^\alpha v(y)|dy$$
$$\leqq C\left(\int_{B_1}|u(y)|^p dy\right)^{1/p}$$

$$+C\sum_{|\alpha|=m}\left(\int_{B_1}|D^{\gamma+\alpha}(\eta\rho_{m,n}\kappa_{2m})\,(-y)|^{p'}dy\right)^{1/p'}\left(\int_{B_1}|D^{\alpha}u(y)|^{p}dx\right)^{1/p}.$$

Since

$$\int_{B_1}|D^{\gamma+\alpha}(\eta\rho_{m,n}\kappa_{2m})\,(-y)|^{p'}dy<\infty$$

for $|\gamma|\leqq k$, we obtain (6.4). Finally we show (6.3). By (6.4) it suffices to show (6.3) for $k+1\leqq|\gamma|\leqq m-1$. First, let $k+1\leqq|\gamma|=m-1$. By (6.1) and Proposition 3.25 (I) we have

$$D^{\gamma}u(x)=a_{\gamma}+D^{\gamma}U^{f}_{m,k}(x)=a_{\gamma}+\int D^{\gamma}_{x}\kappa_{m,k}(x,\,y)f(y)dy.$$

Hence we obtain

$$|a_{\gamma}|\leqq C\Bigg(\left(\int_{B_1}|D^{\gamma}u(x)|^{p}dx\right)^{1/p}+\left(\int_{B_1}\left|\int D^{\gamma}_{x}\kappa_{m,k}(x,\,y)f(y)dy\right|^{p}dx\right)^{1/p}\Bigg)=I_1+I_2.$$

It follows from Lemma 5.12 that

$$I_1\leqq C\Bigg(\left(\int_{B_1}|u(x)|^{p}dx\right)^{1/p}+\sum_{|\alpha|=m}\left(\int_{B_1}|D^{\alpha}u(x)|^{p}dx\right)^{1/p}\Bigg).$$

Since $|\gamma|\geqq k+1$, $D^{\gamma}_{x}\kappa_{m,k}(x,\,y)=(D^{\gamma}\kappa_{m})\,(x-y)$ and $D^{\gamma}\kappa_{m}(x)$ is homogeneous of degree $m-n-|\gamma|=1-n$. Since $k\neq m-1$ implies $p<n$,

$$I_2\leqq C\left(\int|x|^{-p}\left|\int D^{\gamma}\kappa_{m}(x-y)f(y)dy\right|^{p}dx\right)^{1/p}\leqq C\|f\|_{p}\leqq C\sum_{|\alpha|=m}\|D^{\alpha}u\|_{p}.$$

Therefore we obtain (6.3) for $|\gamma|=m-1$. Repeating the above procedure for $|\gamma|=m-2,\,m-3,\cdots,\,k+1$, we obtain (6.3), which completes the proof of Theorem 6.1.

COROLLARY 6.2. *Let $m-(n/p)\neq 0,\,1,\cdots,\,m-1$ and $k=[m-(n/p)]$. If $u\in\mathscr{L}^{p}_{m}$, then u can be represented as*

$$u=\sum_{|\gamma|\leqq m-1}a_{\gamma}x^{\gamma}+V^{F}_{m,k},$$

where $F=\{D^{\alpha}u\}_{|\alpha|=m}$.

PROOF. This corollary is an immediate consequence of Theorem 5.8 (ii) and the above theorem.

THEOREM 6.3. *Let $m-(n/p)=0,1,\cdots,m-1$ and $k=m-(n/p)$. Then $u\in\mathscr{L}^p_m$ can be represented as*

$$u=\sum_{|\alpha|\leqq m-1}a_\gamma x^\gamma+U^{f_1}_{m,k-1}+U^{f_2}_{m,k} \tag{6.6}$$

with $f=(2\pi)^{-m}\gamma_{m,n}^{-1}D^m u\in L^p$ where $f_1=f|_{B_1}$ and $f_2=f-f_1$. Moreover, for $p\leqq q<\infty$

$$\left(\int(1+|\log|x||)^{-(q/p')-1}|x|^{-qk-n}|U^{f_1}_{m,k-1}(x)|^q dx\right)^{1/q}\leqq C\sum_{|\alpha|=m}\|D^\alpha u\|_p$$

(6.7)

$$\left(\int(1+|\log|x||)^{-(q/p')-1}|x|^{-qk-n}|U^{f_2}_{m,k}(x)|^q dx\right)^{1/q}\leqq C\sum_{|\alpha|=m}\|D^\alpha u\|_p$$

and for $|\gamma|\leqq m-1$

$$|a_\gamma|\leqq C\left(\left(\int_{B_1}|u(y)|^p dy\right)^{1/p}+\sum_{|\alpha|=m}\|D^\alpha u\|_p\right). \tag{6.8}$$

Furthermore, if $k\geqq 1$, then $u\in C^{k-1}$ and for $|\gamma|\leqq k-1$

$$|a_\gamma|=\left|\frac{D^\gamma u(0)}{\gamma!}\right|\leqq C\left(\left(\int_{B_1}|u(y)|^p dy\right)^{1/p}+\sum_{|\alpha|=m}\left(\int_{B_1}|D^\alpha u(y)|^p dy\right)^{1/p}\right). \tag{6.9}$$

PROOF. Assertions (6.6), (6.8) and (6.9) can be proved in a way similar to the proofs of (6.1), (6.3) and (6.4). Assertion (6.7) follows from Proposition 5.15.

For $u\in\mathscr{L}^p_m$, write

$$|u|_{m,p}=\sum_{|\alpha|=m}\|D^\alpha u\|_p \quad\text{and}\quad \|u;\mathscr{L}^p_m\|=\left(\int_{B_1}|u(x)|^p dx\right)^{1/p}+|u|_{m,p}.$$

We shall say that $u_N\to 0$ in $\mathscr{L}^p_m$ as $N\to\infty$ if $u_N\to 0$ in $\mathscr{D}'$ and $|u_N|_{m,p}\to 0$ as $N\to\infty$.

PROPOSITION 6.4. *For $\{u_N\}\subset\mathscr{L}^p_m$, $u_N\to 0$ in $\mathscr{L}^p_m$ as $N\to\infty$ if and only if $\|u_N;\mathscr{L}^p_m\|\to 0$ as $N\to\infty$.*

PROOF. The "if" part is easily seen by Theorems 6.1 and 6.3. We shall show the "only if" part. Let $0<r_1<r_2$. We take a function $\eta\in\mathscr{D}$ such that $\eta(x)=1$ for $|x|<r_1$ and $\eta(x)=0$ for $|x|\geqq r_2$. Moreover, taking a function $\phi\in\mathscr{D}$ such that $\phi(x)=1$ for $|x|<1+r_2$, we put $v_N=\phi u_N$. As in the proof of Theorem 6.1,

$$v_N=\zeta*v_N+\sum_{|\alpha|=m}(m!/\alpha!)D^\alpha(\eta\rho_{m,n}\kappa_{2m})*D^\alpha v_N$$

with $\zeta\in C^\infty$. Since $u_N\to 0$ in $\mathscr{D}'$ as $N\to\infty$, we see that $v_N\to 0$ in $\mathscr{E}'$ as $N\to\infty$, and hence $\zeta*v_N\to 0$ in $\mathscr{E}$ as $N\to\infty$ (L. Schwartz [21; §4 in Chap. VI]). In particular, we have

$$\int_{B_1} |\zeta * v_N(x)|^p dx \to 0 \quad (N \to \infty).$$

Since $D^\alpha(\eta\rho_{m,n}\kappa_{2m}) * D^\alpha v_N(x) = D^\alpha(\eta\rho_{m,n}\kappa_{2m}) * D^\alpha u_N(x)$ for $|x|<1$, by Young's inequality we have

$$\left(\int_{B_1} D^\alpha(\eta\rho_{m,n}\kappa_{2m}) * D^\alpha v_N(x)|^p dx\right)^{1/p}$$

$$= C\left(\int_{B_1} |D^\alpha(\eta\rho_{m,n}\kappa_{2m}) * D^\alpha u_N(x)|^p dx\right)^{1/p} \leqq C\|D^\alpha u_N\|_p \to 0 \quad (N \to \infty).$$

Hence we see that

$$\left(\int_{B_1} |u_N(x)|^p dx\right)^{1/p} = \left(\int_{B_1} |v_N(x)|^p dx\right)^{1/p} \to 0 \quad (N \to \infty).$$

The proof of the proposition is completed.

Henceforth, we consider $\mathscr{L}^p_m$ as a normed space equipped with the norm $\|\cdot; \mathscr{L}^p_m\|$. The space $\mathscr{L}^p_m$ is a Banach space.

The following lemma is an improvement of P. I. Lizorkin [12; Theorem 3].

LEMMA 6.5. *If* $u \in \mathscr{L}^p_m$ *and*

$$\int (\log(e+|x|))^{-p}(1+|x|)^{-mp}|u(x)|^p dx < \infty, \tag{6.10}$$

then for $\max(0, k+1) \leqq |\beta| \leqq m-1$

$$\left(\int |x|^{-(m-|\beta|)p}|D^\beta u(x)|^p dx\right)^{1/p} \leqq C|u|_{m,p}$$

and for $0 \leqq |\beta| \leqq k$

$$\left(\int (1+|x|)^{-(m-|\beta|)p}|D^\beta u(x)|^p dx\right)^{1/p} \leqq C\|u; \mathscr{L}^p_m\|$$

if $m-(n/p) \neq 0, 1, \cdots, m-1$,

$$\left(\int (\log(e+|x|))^{-p}(1+|x|)^{-(m-|\beta|)p}|D^\beta u(x)|^p dx\right)^{1/p} \leqq \|u; \mathscr{L}^p_m\|$$

if $m-(n/p) = 0, 1, \cdots, m-1$.

PROOF. Let $m-(n/p) \neq 0, 1, \cdots, m-1$. By Theorem 6.1 and (6.10) we see that

$u \in C^k$ and

$$u(x) = \sum_{|\gamma| \leqq k} a_\gamma x^\gamma + U^f_{m,k}(x),$$

where $a_\gamma = D^\gamma u(0)/\gamma!$ and $f = (2\pi)^{-m} \gamma^{-1}_{m,n} D^m u$. For $\max(0, k+1) \leqq |\beta| \leqq m-1$ we obtain $D^\beta u(x) = D^\beta U^f_{m,k}(x)$. Hence by Proposition 3.25 (I) and Theorem 3.17 we have

$$\left(\int |x|^{-(m-|\beta|)p} |D^\beta u(x)|^p dx \right)^{1/p} = \left(\int |x|^{-(m-|\beta|)p} |D^\beta U^f_{m,k}(x)|^p dx \right)^{1/p}$$
$$\leqq C\|f\|_p \leqq C|u|_{m,p}.$$

For $0 \leqq |\beta| \leqq k$ we obtain

$$D^\beta u(x) = \sum_{|\gamma| \leqq k, \gamma \geqq \beta} (D^\gamma u(0)/(\gamma-\beta)!) x^{\gamma-\beta} + D^\beta U^f_{m,k}(x).$$

By Proposition 3.25 (I), Theorem 3.17 and Corollary 5.9 we have

$$\left(\int |x|^{-(m-|\beta|)p} |D^\beta U^f_{m,k}(x)|^p dx \right)^{1/p} \leqq C\|f\|_p \leqq C|u|_{m,p}.$$

Moreover, we see that

$$\left(\int (1+|x|)^{-(m-|\beta|)p} \left| \sum_{|\gamma| \leqq k, \gamma \geqq \beta} (D^\gamma u(0)/(\gamma-\beta)!) x^{\gamma-\beta} \right|^p dx \right)^{1/p}$$
$$\leqq \sum_{|\gamma| \leqq k, \gamma \geqq \beta} |D^\gamma u(0)| \left(\int (1+|x|)^{-(m-|\gamma|)p} dx \right)^{1/p} \leqq C\|u; \mathscr{L}^p_m\|$$

by (6.4). In case $m-(n/p) = 0, 1, \cdots, m-1$, using Theorem 3.17, Proposition 3.25 (II), Proposition 5.15 and Theorem 6.3 we obtain the required inequalities. The proof of the lemma is completed.

The following lemma is proved in [10]. See also V. G. Maz'ya [13; Lemma 2.4 in Kapitel 6].

LEMMA 6.6. *There exists a sequence* $\{f_N\}_{N=2,3,\cdots}$ *which satisfies the following conditions*:

(i) $f_N \in \mathscr{D}$, $0 \leqq f_N \leqq 1$, $f_N(x) = 1$ *for* $|x| \leqq N$ *and* $f_N(x) = 0$ *for* $|x| \geqq N^2$.

(ii) *If*

$$\int (\log(e+|x|))^{-p} (1+|x|)^{-mp} |u(x)|^p dx < \infty,$$

then $\|u D^\alpha f_N\|_p \to 0$ *as* $N \to \infty$ *for* $|\alpha| = m$.

We denote by L_m^p the closure of $\mathscr{D}$ in $\mathscr{L}_m^p$. The following theorem gives characterizations of this space.

THEOREM 6.7. *Assume that* $u \in \mathscr{L}_m^p$.

(I) *Let* $m-(n/p) \neq 0, 1, \cdots, m-1$ *and* $k=[m-(n/p)]$. *Then the following four conditions are equuivalent*:

(i) $u \in L_m^p$.

(ii) *There exist an* L^p*-function* f *and real numbers* $a_\gamma (|\gamma| \leqq k)$ *such that*

$$u=\sum_{|\gamma| \leqq k} a_\gamma x^\gamma + U_{m,k}^f$$

(iii)

$$\int (1+|x|)^{-mp} |u(x)|^p dx < \infty.$$

(iv)

$$\int (\log(e+|x|))^{-p} (1+|x|)^{-mp} |u(x)|^p dx < \infty.$$

(II) *Let* $m-(n/p)=0, 1, \cdots, m-1$ *and* $k=m-(n/p)$. *Then the following three conditions are equivalent*:

(i) $u \in L_m^p$.

(ii) *There exist an* L^p*-function* f *and real numbers* $a_\gamma (|\gamma| \leqq k)$ *such that*

$$u=\sum_{|\gamma| \leqq k} a_\gamma x^\gamma + U_{m,k-1}^{f_1} + U_{m,k}^{f_1}$$

where $f_1=f|_{B_1}$ *and* $f_2=f-f_1$.

(iii) $$\int (\log(e+|x|))^{-p} (1+|x|)^{-mp} |u(x)|^p dx < \infty.$$

PROOF. We only give the proof of (I). First we prove (i)⇒(ii). By the assumption, there exists a sequence $\{u_N\}_{N=1,2,\dots} \subset \mathscr{D}$ such that u_N converges to u as $N \to \infty$ in $\mathscr{L}_m^p$. Let $f_N=(2\pi)^{-m} \gamma_{m,n}^{-1} D^m u_N$. Then f_N converges to $f=(2\pi)^{-m} \gamma_{m,n}^{-1} D^m u$ as $N \to \infty$ in L^p-norm. Since

$$\int (1+|x|)^{-mp} |U_{m,k}^{f_N}(x)|^p dx < \infty$$

by Corollary 5.9, we see from the proof of Lemma 6.5 that.

$$u_N(x)=\sum_{|\gamma| \leqq k} (D^\gamma u_N(0)/\gamma!) x^\gamma + U_{m,k}^{f_N}(x).$$

By Theorem 6.1, we see that $u \in C^k$ and $D^\gamma u_N(0) \to D^\gamma u(0)$ and $U^{f_N}_{m,k} \to U^f_{m,k}$ in $\mathscr{L}^p_m$ as $N \to \infty$. Consequently by Proposition 6.4, $u_N \to u$ in $\mathscr{L}^p_m$ and

$$u(x) = \sum_{|\gamma| \leqq k} (D^\gamma u(0)/\gamma!) x^\gamma + U^f_{m,k}(x).$$

The implication (ii)⇒(iii) is clear by Corollary 3.22 (i) and Corollary 5.9. The implication (iii)⇒(iv) is trivial. Finaly we shall show (iv)⇒(i). It is enough to show that u can be approximated by a sequence of functions in $\mathscr{L}^p_m$ which have compact support. Taking a sequence $\{f_N\}_{N=1,2,\dots}$ which satisfies the conditions in Lemma 6.6, we put $u_N = f_N u$. It is clear that

$$\int_{B_1} |u_N(x) - u(x)|^p dx = 0$$

For $|\alpha| = m$, by Leibniz's formula we have

$$\begin{aligned} I_N &= \left(\int |D^\alpha (u(x) - u_N(x))|^p dx \right)^{1/p} \\ &\leqq \left(\int |(1 - f_N(x)) D^\alpha u(x)|^p dx \right)^{1/p} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left(\int |D^{\alpha-\beta}(1 - f_N(x)) D^\beta u(x)|^p dx \right)^{1/p} \\ &\leqq \left(\int_{|x| \geqq N} |D^\alpha u(x)|^p dx \right)^{1/p} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left(\int |D^{\alpha-\beta} f_N(x) D^\beta u(x)|^p dx \right)^{1/p}. \end{aligned}$$

By the assumption and Lemma 6.5 we have

$$\int (1 + |x|)^{-(m-|\beta|)p} |D^\beta u(x)|^p dx < \infty.$$

Hence it follows from Lemma 6.6 (ii) that $I_N \to 0$ as $N \to \infty$. Therefore u_N converges to u as $N \to \infty$ in $\mathscr{L}^p_m$. Thus we obtain (I).

REMARK 6.8. P. I. Lizorkin [12] proved the equivalence of (i) and (iii) in Theorem 6.7 (I).

REMARK 6.9. (i) Let $m - (n/p) < 0$. By Theorem 6.7 (I) we have $\mathrm{L}^p_m = \{U^f_m;\ f \in \mathrm{L}^p\}$. If $u = U^f_m \in \mathrm{L}^p_m$, then $f = (2\pi)^{-m} \gamma^{-1}_{m,n} D^m u$. Hence by Corollary 3.22 (i) we have

$$\left(\int_{B_1} |u(x)|^p dx \right)^{1/p} \leqq \left(\int |x|^{-mp} |U^f_m(x)|^p dx \right)^{1/p} \leqq C \|f\|_p \leqq C |u|_{m,p}.$$

Consequently for $u \in \mathrm{L}^p_m$ we see that

$$|u|_{m,p} \leqq \|u;\ \mathscr{L}^p_m\| \leqq C|u|_{m,p}.$$

Also we see that for $u \in \mathrm{L}^p_m$

$$C^{-1}\|u;\ \mathscr{L}^p_m\| \leqq \left(\int (\log(e+|x|))^{-p}(1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} + |u|_{m,p}$$
$$\leqq \left(\int (1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} + |u|_{m,p} \leqq C\|u;\ \mathscr{L}^p_m\|.$$

(ii) Let $m-(n/p)>0, \neq 1, \cdots, m-1$ and $k=[m-(n/p)]$. From Theorems 6.1 and 6.7 (I) it follows that for $u \in \mathrm{L}^p_m$

$$C^{-1}\|u;\ \mathscr{L}^p_m\| \leqq \left(\int (1+\log(e+|x|)^{-p}(1+|x|)^{-mp}|u(x)^p dx\right)^{1/p} + |u|_{m,p}$$
$$\leqq \left(\int (1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} + |u|_{m,p} \leqq C\|u;\ \mathscr{L}^p_m\|.$$

Moreover, for $u \in \mathrm{L}^p_m$ we shall show

$$C^{-1}\|u;\ \mathscr{L}^p_m\| \leqq \sum_{|\gamma| \leqq k}|D^\gamma u(0)| + |u|_{m,p} \leqq C\|u;\ \mathscr{L}^p_m\|. \tag{6.11}$$

By the proof of Theorem 6.7 (I), $u \in \mathrm{L}^p_m$ can be represented as

$$u = \sum_{|\gamma| \leqq k}(D^\gamma u(0)/\gamma!)x^\gamma + U^f_{m,k}, \qquad f = (2\pi)^{-m}\gamma^{-1}_{m,n}D^m u.$$

Hence by Corollary 3.22 (i) and Corollary 5.9 we have

$$\left(\int_{B1} |u(x)|^p dx\right)^{1/p} \leqq \sum_{|\gamma| \leqq k}|D^\gamma u(0)/\gamma!|\left(\int_{B_1} |x|^{|\gamma|p} dx\right)^{1/p} + \left(\int_{B_1} U^f_{m,k}(x)|^p dx\right)^{1/p}$$
$$\leqq C\sum_{|\gamma| \leqq k}|D^\gamma u(0)| + \left(\int_{B_1} |x|^{-mp}|U^f_{m,k}(x)|^p dx\right)^{1/p}$$
$$\leqq C\sum_{|\gamma| \leqq k}|D^\gamma u(0)| + C\|f\|_p \leqq C(\sum_{|\gamma| \leqq k}|D^\gamma u(0)| + |u|_{m,p}).$$

Hence, together with (6.4) we obtain (6.11).

(iii) Let $m-(n/p)=0, 1, \cdots, m-1$. Then it follows from Theorems 6.3 and 6.7 (II) that for $u \in \mathrm{L}^p_m$

$$C^{-1}\|u;\ \mathscr{L}^p_m\| \leqq \left(\int (\log(e+|x|))^{-p}(1+|x|)^{-mp}|u(x)|^p dx\right)^{1/p} + |u|_{m,p} \leqq C\|u;\ \mathscr{L}^p_m\|.$$

REMARK 6.10. Let $u \in \mathrm{L}^p_m$ and $|\beta|=j$ with $0 \leqq j \leqq m$. By Lemma 6.5 and Theorem 6.7 we have $D^\beta u \in \mathrm{L}^p_{m-j}$ and

$$\|D^\beta u;\ \mathscr{L}^p_{m-j}\| \leqq C\|u;\ \mathscr{L}^p_m\|$$

where $\mathscr{L}_0^p$ and L_0^p mean L^p. Namely, the differential operator D^β is a bounded operator from L_m^p to L_{m-j}^p.

Now, as a consequence of Theorem 6.7 (I), we obtain a potential representation of a solution of the equation (1.1) for arbitrary positive integer l.

THEOREM 6.11 (i) *If* $2l-(n/p)\neq 0,1,\cdots,2l-1, k=[2l-(n/p)]$ *and* $f\in L^p$, *then we have* $\Delta^l U_{2l,k}^f=(-1)^l(2\pi)^{2l}\gamma_{2l,n}f$.

(ii) *If* $2l-(n/p)=0,\ 1,\cdots,\ 2l-1,\ k=2l-(n/p)$ *and* $f\in \mathrm{L}^p$, *then we have* $\Delta^l(U_{2l,k-1}^{f_1}+U_{2l,k}^{f_2})=(-1)^l(2\pi)^{2l}\gamma_{2l,n}f$ *where* $f_1=f|_{B_1}$ *and* $f_2=f-f_1$.

PROOF. We only give the proof of (i). Since $U_{2l,k}^f\in \mathrm{L}_{2l}^p$ by Theorem 6.7 (I), from Lemma 4.2 (ii) it is easily seen that $D^{2l}U_{2l,k}^f=(-1)^l\Delta^l U_{2l,k}^f$. Hence the conclusion follows from Corollary 4.5 (i).

§7. Embedding and interpolation theorems

In this section we are concerned with embedding and interpolation theorems for the spaces L_m^p. First we establish an embedding theorem for L_m^p.

THEOREM 7.1. *Let* $k=[m-(n/p)]$. *If* $\max(k+1,0)\leqq l\leqq m$, *then* $\mathrm{L}_m^p\subset \mathrm{L}_l^{p_{m-l}}$ *and whenever* $u\in \mathrm{L}_m^p$,

$$\|u;\ \mathscr{L}_l^{p_{m-l}}\|\leqq C\|u;\ \mathscr{L}_m^p\|.$$

PROOF. The case $l=m$ is trivial. Let $\max(k+1,0)\leqq l\leqq m-1$. First we assume that $m-(n/p)\neq 0,1,\cdots,m-1$. Let $u\in \mathrm{L}_m^p$. Then by the proof of Theorem 6.7 (I) u can be represented as

$$u=\sum_{|\gamma|\leqq k}a_\gamma x^\gamma+U_{m,k}^f$$

where $f=(2\pi)^{-m}\gamma_{m,n}^{-1}D^m u$. For $|\beta|=l$, it follows from Proposition 3.25 (I) that

$$D^\beta u(x)=D^\beta U_{m,k}^f(x)=\int D^\beta\kappa_m(x-y)f(y)dy.$$

Since $|\beta|\geqq\max(k+1,0)$, $D^\beta\kappa_m(x)$ is homogeneous of degree $m-l-n$. Since $m-l-(n/p)\leqq m-k-1-(n/p)<0$, it follows from Theorem 3.17 that

$$\|D^\beta u\|_{p_{m-l}}\leqq C\|f\|_p\leqq C|u|_{m,p}.$$

Therefore we have $u\in\mathscr{L}_l^{p_{m-l}}$ and

$$|u|_{l,p_{m-l}}\leqq C|u|_{m,p}.$$

Moreover, we have

$$\left(\int(1+|x|)^{-lp_{m-l}}|u(x)|^{p_{m-l}}dx\right)^{1/p_{m-l}}$$

$$\leqq\sum_{|\gamma|\leqq k}|a_\gamma|\left(\int(1+|x|)^{(|\gamma|-l)p_{m-l}}dx\right)^{1/p_{m-l}}$$

$$+\left(\int|x|^{-lp_{m-l}}|U^f_{m,k}(x)|^{p_{m-l}}dx\right)^{1/p_{m-l}}=I_1+I_2.$$

Since

$$\int(1+|x|)^{(|\gamma|-l)p_{m-l}}dx<\infty$$

for $0\leqq|\gamma|\leqq k$, it follows from (6.3) that $I_1\leqq C\|u;\ \mathscr{L}^p_m\|$. Since $-lp_{m-l}=-p_{m-l}(m-(n/p))-n$ and $p_{m-l}\geqq p$, by Theorem 3.17 and Corollary 5.9, we obtain $I_2\leqq C\|f\|_p\leqq C|u|_{m,p}$. Noting that $l-(n/p_{m-l})=m-(n/p)\neq 0, 1,\cdots, m-1$, by Theorem 6.7 (I) and Remark 6.9 (i), (ii) we have $u\in \mathrm{L}^{p_{m-l}}_l$ and

$$\|u;\ \mathscr{L}^{p_{m-l}}_l\|\leqq C\|u;\ \mathscr{L}^p_m\|.$$

Next let $m-(n/p)=0, 1,\cdots, m-1$ and $u\in\mathrm{L}^p_m$. By Theorem 6.7 (II) we have

$$u(x)=\sum_{|\gamma|\leqq k}a_\gamma x^\gamma+U^{f_1}_{m,k-1}(x)+U^{f_2}_{m,k}(x),$$

where $f=(2\pi)^{-m}\gamma^{-1}_{m,n}D^m u$, $f_1=f|_{B_1}$ and $f_2=f-f_1$. For $|\beta|=l$, it follows from Proposition 3.25 (II) that

$$D^\beta u(x)=D^\beta U^{f_1}_{m,k-1}(x)+D^\beta U^{f_2}_{m,k}(x)=\int D^\beta\kappa_m(x-y)f(y)dy.$$

Since $m-l-(n/p)<0$, by Theorem 3.17 we have

$$\|D^\beta u\|_{p_{m-l}}\leqq C\|f\|_p\leqq C|u|_{m,p}.$$

We note that $l-(n/p_{m-l})=0, 1,\cdots, m-1$. In order to show $u\in\mathrm{L}^{p_{m-l}}_l$, in view of Theorem 6.7 (II) it is enough to prove

(7.1) $$J=\left(\int(\log(e+|x|))^{-p_{m-l}}(1+|x|)^{-lp_{m-l}}|u(x)|^{p_{m-l}}dx\right)^{1/p_{m-l}}<\infty.$$

We have

$$J\leqq\sum_{|\gamma|\leqq k}|a_\gamma|\left(\int(\log(e+|x|))^{-p_{m-l}}(1+|x|)^{(|\gamma|-l)p_{m-l}}dx\right)^{1/p_{m-l}}$$

$$+\left(\int(1+|\log|x||)^{-p_{m-l}}|x|^{-lp_{m-l}}|U^{f_1}_{m,k-1}(x)|^{p_{m-l}}dx\right)^{1/p_{m-l}}$$

$$+\left(\int(1+|\log|x||)^{-p_{m-l}}|x|^{-lp_{m-l}}|U^{f_2}_{m,k}(x)|^{p_{m-l}}dx\right)^{1/p_{m-l}}$$

$$=J_1+J_2+J_3.$$

Since $(|\gamma|-l)p_{m-1}\leqq -n$ for $|\gamma|\leqq k$ and $p_{m-l}>1$, it follows from (6.8) that $J_1\leqq C\|u; \mathscr{L}^p_m\|$. Moreover, since $lp_{m-l}=p_{m-l}(m-(n/p))+n$ and $p_{m-l}\geqq p$, by Proposition 5.15 we see that $J_2\leqq C\|f_1\|_p\leqq C|u|_{m,p}$. Similarly, we have $J_3\leqq C|u|_{m,p}$. Thus we obtain (7.1). Hence $u\in \mathrm{L}^{p_{m-l}}_l$ and Remark 6.9 gives

$$\|u; \mathscr{L}^{p_{m-l}}_l\|\leqq C\|u; \mathscr{L}^p_m\|.$$

We have completed the proof of Theorem 7.1.

Next we establish an interpolation theorem for L^p_m. We cite here two known results as lemmas.

LEMMA 7.2 ([7; Theorem 9.3]). *If* $0<l<m$, $p_1, p_2>1$ *and* $1/p=(1/p_1)(1-(l/m))+(1/p_2)(l/m)$, *then for* $\phi\in\mathscr{D}$ *and* $|\alpha|=l$

$$\|D^\alpha\phi\|_p\leqq C(\|\phi\|_{p_1}+|\phi|_{m,p_2}).$$

For a nonnegative measurable function w on R^n, we put

$$\mathrm{L}^p(w)=\{f;\ \|f\|_{p,w}=\left(\int|f(x)|^p w(x)dx\right)^{1/p}<\infty\}.$$

LEMMA 7.3 ([4; Theorem 5.5.1]). *Assume that* $p_1, p_2>1, 0\leqq\theta\leqq 1$ *and* $1/p=(1-\theta)/p_1+\theta/p_2$. *If for nonnegative measurable functions* w_1, w_2 *we set* $w(x)=w_1(x)^{p(1-\theta)/p_1}w_2(x)^{p\theta/p_2}$, *then we have*

$$\mathrm{L}^{p_1}(w_1)\cap\mathrm{L}^{p_2}(w_2)\subset\mathrm{L}^p(w)$$

and whenever $f\in\mathrm{L}^{p_1}(w_1)\cap\mathrm{L}^{p_2}(w_2)$,

$$\|f\|_{p,w}\leqq C(\|f\|_{p_1,w_1}+\|f\|_{p_2,w_2}).$$

THEOREM 7.4. (i) *Let* m_1, m_2 *and* m *be nonnegative integers such that* $m_1\leqq m\leqq m_2$, $m_1\neq m_2$, *and let* p_1, $p_2>1$. *If we set* $\theta=(m-m_1)/(m_2-m_1)$ *and* $1/p=(1-\theta)/p_1+\theta/p_2$ *then we have* $\mathrm{L}^{p_1}_{m_1}\cap\mathrm{L}^{p_2}_{m_2}\subset\mathrm{L}^p_m$ *and whenever* $u\in\mathrm{L}^{p_1}_{m_1}\cap\mathrm{L}^{p_2}_{m_2}$,

$$\|u; \mathscr{L}^p_m\|\leqq C(\|u; \mathscr{L}^{p_1}_{m_1}\|+\|u; \mathscr{L}^{p_2}_{m_2}\|).$$

(ii) *Let* m *be a nonnegative integer and* $1<p_1<p_2$. *Then for* $p_1\leqq p\leqq p_2$ *we have*

$L_m^{p_1} \cap L_m^{p_2} \subset L_m^p$ *and whenever* $u \in L_m^{p_1} \cap L_m^{p_2}$,

$$\|u;\ \mathscr{L}_m^p\| \leqq C(\|u;\ \mathscr{L}_m^{p_1}\| + \|u;\ \mathscr{L}_m^{p_2}\|).$$

PROOF. First let $m_1 = 0 < m \leqq m_2$. For $u \in L^{p_1} \cap L_{m_2}^{p_2}$, as in the proof of Theorem 6.7 we can find a sequence $\{\phi_N\}_{N=1,2\dots} \subset \mathscr{D}$ such that ϕ_N tends to u as $N \to \infty$ in L^{p_1} and $D^\beta \phi_N$ tends to $D^\beta u$ as $N \to \infty$ in L^{p_2} for $|\beta| = m_2$. Hence by Lemma 7.2 we see that $u \in L_m^p$ and for $|\alpha| = m$

$$\|D^\alpha u\|_p \leqq C(\|u\|_{p_1} + |u|_{m_2, p_2}).$$

Since

$$\left(\int_{B_1} |u(x)|^p dx\right)^{1/p} \leqq C\left(\left(\int_{B_1} |u(x)|^{p_1} dx\right)^{1/p_1} + \left(\int_{B_1} |u(x)|^{p_2} dx\right)^{1/p_2}\right),$$

(cf. Lemma 7.3), we have $\|u;\ \mathscr{L}_m^p\| \leqq C(\|u\|_{p_1} + \|u;\ \mathscr{L}_{m_2}^{p_2}\|)$. Next, let $1 \leqq m_1 \leqq m \leqq m_2$, $m_1 \neq m_2$ and $u \in L_{m_1}^{p_1} \cap L_{m_2}^{p_2}$. For $|\beta| = m_1$ and $m_3 = m_2 - m_1$, we easily see (cf. Remark 6.10) that $D^\beta u \in L_{m_3}^{p_2}$ and

$$\|D^\beta u;\ \mathscr{L}_{m_3}^{p_2}\| \leqq C\|u;\ \mathscr{L}_{m_2}^{p_2}\|.$$

Consequently $D^\beta u \in L^{p_1} \cap L_{m_3}^{p_2}$. Let $m_4 = m - m_1$. Then by the argument in the first case we see that $L^{p_1} \cap L_{m_3}^{p_2} \subset L_{m_4}^p$ since $1/p = (1/p_1)\,(1 - (m_4/m_3)) + (1/p_2)\,(m_4/m_3)$, and whenever $v \in L^{p_1} \cap L_{m_3}^{p_2}$,

$$\|v;\ \mathscr{L}_{m_4}^p\| \leqq C(\|v\|_{p_1} + \|v;\ \mathscr{L}_{m_2}^{p_2}\|).$$

Thus $D^\beta u \in L_{m_4}^p$ and

$$|D^\beta u|_{m_4, p} \leqq C(\|D^\beta u\|_{p_1} + \|u;\ \mathscr{L}_{m_2}^{p_2}\|),$$

so that $u \in \mathscr{L}_m^p$ and

$$|u|_{m,p} \leqq C(\|u;\ \mathscr{L}_{m_1}^{p_1}\| + \|u;\ \mathscr{L}_{m_2}^{p_2}\|.$$

Let $w_1 = (\log(e + |x|))^{-p_1}(1 + |x|)^{-m_1 p_1}$, $w_2 = (\log(e + |x|))^{-p_2}(1 + |x|)^{-m_2 p_2}$ and $w = (\log(e + |x|)^{-p}(1 + |x|)^{-mp}$. Then, since $w_1^{p(1-\theta)/p_1} w_2^{p\theta/p_2} = w$, Lemma 7.3 implies that $u \in L^{p_1}(w_1) \cap L^{p_2}(w_2) \subset L^p(w)$ and

$$\|u;\ \mathscr{L}_m^p\| \leqq C\|u\|_{p,w} + |u|_{m,p} \leqq C(\|u;\ \mathscr{L}_{m_1}^{p_1}\| + \|u;\ \mathscr{L}_{m_2}^{p_2}\|)$$

on account of Remark 6.9. Hence by Theorem 6.7 we see that $u \in L_m^p$.

(ii) This follows from Lemma 7.3, Theorem 6.7 and Remark 6.9.

REMARK 7.5. Let m_1, m_2, m be nonnegative integers such that $m_1 \leqq m \leqq m_2$, $m_1 \neq m_2$, and let $p_1, p_2 > 1$, $\theta = (m - m_1)/(m_2 - m_1)$, $1/p = (1 - \theta)/p_1 + \theta/p_2$. Assume that $m_1 - (n/p_1) < 0$ and $m_2 - (n/p_2) < 0$. Then $m - (n/p) < 0$. Hence by Remark 6.9

(i) and Theorem 7.4 (i) we see that for $u \in \mathrm{L}_{m_1}^{p_1} \cap \mathrm{L}_{m_2}^{p_2}$

$$|u|_{m,p} \leqq C(|u|_{m_1,p_1} + |u|_{m_2,p_2}).$$

REMARK 7.6. Let m_1, m_2, m be nonnegative integers such that $m_1 \leqq m \leqq m_2$, $m_1 \neq m_2$, and let $p_1, p_2 > 1$, $\theta = (m - m_1)/(m_2 - m_1)$, $1/p = (1 - \theta)/p_1 + \theta/p_2$. Assume that $m_2 - (n/p_2) < m$. If we put $1/q = 1/p_2 - (m_2 - m)/n$, then for $p \leqq r \leqq q$ we have $\mathrm{L}_{m_1}^{p_1} \cap \mathrm{L}_{m_2}^{p_2} \subset \mathrm{L}_m^r$. Indeed, by Theorem 7.4 (i) we have $\mathrm{L}_{m_1}^{p_1} \cap \mathrm{L}_{m_2}^{p_2} \subset \mathrm{L}_m^p$. Moreover, it follows from Theorem 7.1 that $\mathrm{L}_{m_2}^{p_2} \subset \mathrm{L}_m^q$. Hence by Theorem 7.4 (ii) we see that $\mathrm{L}_{m_1}^{p_1} \cap \mathrm{L}_{m_2}^{p_2} \subset \mathrm{L}_m^p \cap \mathrm{L}_m^q \subset \mathrm{L}_m^r$.

For any nonnegative integer m and any $p > 1$, the Sobolev space W_m^p is defined by

$$\mathrm{W}_m^p = \{u \in \mathscr{D}';\ \|u\|_{m,p} = \textstyle\sum_{|\alpha| \leqq m} \|D^\alpha u\|_p < \infty\}.$$

The following corollary is an improvement of Sobolev's embedding theorem [2; Theorem 5.4].

COROLLARY 7.7. *Let* $k = [m - (n/p)]$. *If* $\max(k+1, 0) \leqq l \leqq m$, *then* $\mathrm{L}_m^p \cap \mathrm{L}_{m-l}^p \subset \mathrm{W}_l^{p_{m-l}}$ *and whenever* $u \in \mathrm{L}_m^p \cap \mathrm{L}_{m-l}^p$,

$$\|u\|_{l,p_{m-l}} \leqq C(|u|_{m,p} + |u|_{m-l,p}).$$

PROOF. By Theorem 7.4 (i) we have

$$\mathrm{L}_m^p \cap \mathrm{L}_{m-l}^p = \textstyle\bigcap_{j=m-l}^m \mathrm{L}_j^p.$$

Since $\max(k+1, 0) \leqq l \leqq m$, the condition $m - l \leqq j \leqq m$ implies that $\max([j - (n/p)] + 1, 0) \leqq j - (m - l) \leqq j$. Hence by Theorem 7.1 we see that

$$\textstyle\bigcap_{j=m-l}^m \mathrm{L}_j^p \subset \bigcap_{j=m-l}^m \mathrm{L}_{j-(j-(m-l))}^{p_{j-(m-l)}} = \bigcap_{j=m-l}^m \mathrm{L}_{j-(m-l)}^{p_{m-l}} = \bigcap_{j=0}^l \mathrm{L}_j^{p_{m-l}} = \mathrm{W}_l^{p_{m-l}}.$$

Consequently we have $\mathrm{L}_m^p \cap \mathrm{L}_{m-l}^p \subset \mathrm{W}_l^{p_{m-l}}$. Furthermore, it follows from Theorems 7.1 and 7.4 (i) that

$$\|u\|_{l,p_{m-l}} \leqq \textstyle\sum_{0 \leqq j \leqq l} \|u;\ \mathscr{L}_j^{p_{m-l}}\| \leqq C \sum_{m-l \leqq j \leqq m} \|u;\ \mathscr{L}_j^p\|$$
$$\leqq C(\|u;\ \mathscr{L}_m^p\| + \|u;\ \mathscr{L}_{m-l}^p\|).$$

Since $l \geqq \max(k+1, 0)$ implies $m - l - (n/p) < 0$, by Remark 6.9 (i) we have

$$\|u;\ \mathscr{L}_m^p\| + \|u;\ \mathscr{L}_{m-l}^p\| \leqq C(|u|_{m,p} + |u|_{m-l,p}).$$

The proof of Corollary 7.7 is completed.

Finally, we establish smooth function space embeddings. Let m be a nonnegtive integer. We define spaces B^m and E^m by

$$\mathrm{B}^m=\{u\in C^m;\ |u|_{m,\infty}=\sum_{|\alpha|=m}\sup_{x\in R^n}|D^\alpha u(x)|<\infty\},$$
$$\mathrm{E}^m=\{u\in C^m;\ \|u\|_{m,\infty}=\sum_{|\alpha|\leqq m}\sup_{x\in R^n}|D^\alpha u(x)|<\infty\}$$

and write

$$\|u;\ \mathrm{B}^m\|=\sum_{|\gamma|\leqq m-1}|D^\gamma u(0)|+|u|_{m,\infty}.$$

Moreover, for a positive and nonintegral number r, we define spaces B^r and E^r by

$$\mathrm{B}^r=\left\{u\in C^{[r]};\ |u|_{r,\infty}=\sum_{|\alpha|=[r]}\sup_{x\neq y}\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^{r-[r]}}<\infty\right\},$$
$$\mathrm{E}^r=\{u\in C^{[r]};\ \|u\|_{r,\infty}=\sum_{|\alpha|\leqq[r]}\sup_{x\in R^n}|D^\alpha u(x)|+|u|_{r,\infty}<\infty\}$$

and write

$$\|u;\ \mathrm{B}^r\|=\sum_{|\gamma|\leqq[r]}|D^\gamma u(0)|+|u|_{r,\infty}.$$

THEOREM 7.8. *Let* $m-(n/p)>0,\ \neq 1,\cdots,\ m-1$. *Then* $\mathrm{L}^p_m\subset\mathrm{B}^{m-(n/p)}$ *and whenever* $u\in\mathrm{L}^p_m$,

$$\|u;\ \mathrm{B}^{m-(n/p)}\|\leqq C\|u;\ \mathscr{L}^p_m\|.$$

PROOF. Let $k=[m-(n/p)]$ and $u\in\mathrm{L}^p_m$. By Theorem 6.1, $u\in C^k$ and by Theorem 6.7 (I) we see that

$$u(x)=\sum_{|\gamma|\leqq k}(D^\gamma u(0)/\gamma!)x^\gamma+U^f_{m,k}(x)$$

where $f=(2\pi)^{-m}(\gamma_{m,n})^{-1}D^m u$. Hence it follows from Lemma 3.2 (ii) and Theorem 5.8 that for $|\beta|=k$

$$|D^\beta u(x)-D^\beta u(y)|=|D^\beta U^f_{m,k}(x)-D^\beta U^f_{m,k}(y)|\leqq C|x-y|^{m-(n/p)-k}\|f\|_p.$$

Therefore we obtain

$$|u|_{m-(n/p),\infty}=\sum_{|\beta|=k}\sup_{x\neq y}\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^{m-(n/p)-k}}\leqq C\|f\|_p\leqq C|u|_{m,p}.$$

Furthermore, by Theorem 6.1 we see that

$$\|u;\ \mathrm{B}^{m-(n/p)}\|=\sum_{|\gamma|\leqq k}|D^\gamma u(0)|+|u|_{m-(n/p),\infty}$$
$$\leqq C(\|u;\ \mathscr{L}^p_m\|+|u|_{m,p})\leqq C\|u;\ \mathscr{L}^p_m\|.$$

Thus we obtain the theorem.

The proof of the following lemma is found in [6; Proposition 5.23].

LEMMA 7.9. *Let* $0<h<1$. *Then* $\mathrm{B}^h\cap\mathrm{L}^p\subset\mathrm{B}^0$ *and whenever* $u\in\mathrm{B}^h\cap\mathrm{L}^p$,

$$|u|_{0,\infty} \leqq C(|u|_{h,\infty} + \|u\|_p).$$

The following corollary is an improvement of the smooth function space embedding theorem for the Sobolev spaces ([2; Theorem 5.4]).

COROLLARY 7.10. *Let* $m-(n/p)>0\neq 1,\cdots, m-1$ *and* $k=[m-(n/p)]$. *Then* $\mathrm{L}_m^p \cap \mathrm{L}_{m-k-1}^p \subset \mathrm{E}^{m-(n/p)}$ *and whenever* $u \in \mathrm{L}_m^p \cap \mathrm{L}_{m-k-1}^p$,

$$\|u\|_{m-(n/p),\infty} \leqq C(|u|_{m,p} + |u|_{m-k-1,p}).$$

PROOF. First we show that if $u \in \mathrm{L}_{j-1}^p \cap \mathrm{L}_j^p$ and $m-k \leqq j \leqq m$, then $u \in \mathrm{B}^{j-(n/p)}$ and

$$|D^\beta u|_{0,\infty} + |D^\beta u|_{m-(n/p)-k,\infty} \leqq C(\|u;\ \mathscr{L}_{j-1}^p\| + \|u;\ \mathscr{L}_j^p\|) \tag{7.2}$$

for $|\beta| = [j-(n/p)]$. We see from Theorem 7.8 that

$$D^\beta u \in \mathrm{B}^{j-(n/p)-[j-(n/p)]} = \mathrm{B}^{m-(n/p)-k} \quad \text{and} \quad |D^\beta u|_{m-(n/p)-k} \leqq C\|u;\ \mathscr{L}_{j-1}^p\|.$$

On the other hand, Theorem 7.1 implies that

$$\mathrm{L}_{j-1}^p \subset \mathrm{L}_{[j-(n/p)]}^{p_{j-1-[j-(n/p)]}} \subset \mathrm{L}_{[j-(n/p)]}^{p_{m-k-1}} \quad \text{and} \quad \|u;\ \mathscr{L}_{[j-(n/p)]}^{p_{m-k-1}}\| \leqq C\|u;\ \mathscr{L}_{j-1}^p\|.$$

Hence, by Lemma 7.9 we see that $D^\beta u \in \mathrm{B}^{m-(n/p)-k} \cap \mathrm{L}^{p_{m-k-1}} \subset \mathrm{B}^0$ and

$$\begin{aligned} |D^\beta u|_{0,\infty} &\leqq C(|D^\beta u|_{m-(n/p)-k,\infty} + \|D^\beta u\|_{p_{m-k-1}}) \\ &\leqq C\|u;\ \mathscr{L}_j^p\| + \|u;\ \mathscr{L}_{j-1}^p\|), \end{aligned}$$

which yields (7.2).

By (7.2) and Theorem 7.4 we see that

$$\mathrm{L}_m^p \cap \mathrm{L}_{m-k-1}^p = \textstyle\bigcap_{m-k \leqq j \leqq m} (\mathrm{L}_{j-1}^p \cap \mathrm{L}_j^p) \subset \mathrm{E}^{m-(n/p)}$$

and

$$\begin{aligned} \|u\|_{m-(n/p),\infty} &= \textstyle\sum_{j=0}^k |u|_{j,\infty} + |u|_{m-(n/p),\infty} \\ &\leqq C \textstyle\sum_{j=m-k}^m (\|u;\ \mathscr{L}_j^p\| + \|u;\ \mathscr{L}_{j-1}^p\|) \leqq C(|u|_{m,p} + |u|_{m-k-1,p}). \end{aligned}$$

The proof of the corollary is completed.

To consider the case $m-(n/p) = 1, 2, \cdots, m-1$, we introduce the space $\mathrm{B}^{m,r}$ for a nonnegative integer m and $r>0$, which is defined by

$$\mathrm{B}^{m,r} = \{u \in C^m;\ |u|_{m,r,\infty} = \textstyle\sum_{|\alpha|=m} \sup_{R \geqq e} (\log R)^{-r} \sup_{x,y \in B_R, x \neq y} \dfrac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|(1+|\log|x-y||)^r} < \infty\}$$

with the norm $\|u;\ \mathrm{B}^{m,r}\| = \sum_{|\alpha| \leqq m} |D^\alpha u(0)| + |u|_{m,r,\infty}$.

THEOREM 7.11. *Let $m-(n/p)=1, 2, \cdots, m-1$ and $k=m-(n/p)$. Then* $\mathrm{L}_m^p \subset \mathrm{B}^{k-1,1/p'}$ *and whenever* $u \in \mathrm{L}_m^p$, $\|u;\ \mathrm{B}^{k-1,1/p'}\| \leqq C\|u;\ \mathscr{L}_m^p\|$.

PROOF. Let $u \in \mathrm{L}_m^p$. By Theorem 6.7 (II) we have

$$u(x) = \sum_{|\gamma| \leqq k} a_\gamma x^\gamma + U_{m,k-1}^{f_1}(x) + U_{m,k}^{f_2}(x),$$

where $f = (2\pi)^{-m}(\gamma_{m,n})^{-1} D^m u$, $f_1 = f|_{B_1}$ and $f_2 = f - f_1$. For any multi-index β with $|\beta| = k-1$, we see that

$$|D^\beta u(x) - D^\beta u(y)| \leqq \sum_{|\gamma| = k} |a_\gamma||x-y| + |D^\beta U_{m,k-1}^{f_1}(x) - D^\beta U_{m,k-1}^{f_1}(y)|$$
$$+ |D^\beta U_{m,k}^{f_2}(x) - D^\beta U_{m,k}^{f_2}(x)|.$$

Hence in view of Lemmas 3.4, 3.6, Proposition 3.25 (II) and Remark 5.14 we have

$$|D^\beta u(x) - D^\beta u(y)| \leqq \sum_{|\gamma| = k} |a_\gamma||x-y| + C\|f\|_p |x-y| \left(1 + \log^+ \frac{1}{|x-y|}\right)^{1/p'}$$
$$+ C\|f\|_p |x-y| \left(1 + \log^+|x-y| + \log^+ \frac{|x|}{|x-y|}\right)^{1/p'}.$$

Therefore using (6.9), we have

$$\sup_{x,y \in B_R, x \neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|(1 + |\log|x-y||)^{1/p'}}$$
$$\leqq C\|u;\ \mathscr{L}_m^p\| \left\{1 + \sup_{x,y \in B_R, x \neq y} \left(\frac{1 + \log^+|x-y| + \log^+(|y|/|x-y|)}{1 + |\log|x-y||}\right)^{1/p'}\right\}$$
$$\leqq C\|u;\ \mathscr{L}_m^p\|(1 + \log R)^{1/p'}.$$

Consequently we have

$$|u|_{k-1,1/p'} \leqq C\|u;\ \mathscr{L}_m^p\| \sup_{R \geqq e} \left(\frac{1 + \log R}{\log R}\right)^{1/p'} = C\|u;\ \mathscr{L}_m^p\| 2^{1/p'}.$$

Since $\sum_{|\gamma| \leqq k-1} |D^\gamma u(0)| \leqq C\|u;\ \mathscr{L}_m^p\|$, we obtain the theorem.

Acknowlodgement. The author is deeply indebted to Probessor Fumi-Yuki Maeda and Dr. Yoshihiro Mizuta who read the manuscript carefully and suggested many improvements.

References

[1] D. R. Adams, Traces of potentials arising from translation invariant operators, Ann. Scuola Norm. Sup. Pisa **25** (1971), 203–217.
[2] R. A. Adams, Sobolev spaces, Academic Press, New York, 1975.
[3] N. Aronszajn, T. M. Creese and L. J. Lipkin, Polyharmonic functions, Claredon Press, Oxford, 1983.
[4] J. Bergh and J. Löfström, Interpolation spaces, Springer-Verlag, Berlin, 1976.
[5] A. P. Calderon and A. Zygmund, On the existence of certain singular integrals, Acta Math. **88** (1952), 85–139.
[6] G. B. Folland, Lectures on partial differential equations, Tata Inst. Bombay, 1983.
[7] A. Friedman, Partial differential equations, R. E. Krieger Publishing Company, New York, 1976.
[8] G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals I, Math. Zeit. **27** (1927), 565–606; II, ibid. **34** (1934), 403–439.
[9] W. K. Hayman and P. B. Kennedy, Subharmonic functions, Academic Press, London, 1976.
[10] T. Kurokawa, A note on Beppo Levi functions and $L_{m,p}$-spaces, Sci. Rep. Kagoshima Univ. **31** (1982), 25–31.
[11] N. S. Landkof, Foundations of modern potential theory, Springer-Verlag, Berlin, 1972.
[12] P. I. Lizorkin, The behavior at infinity of functions in Liouville classes. On Riesz potentials of arbitrary order, Proc. Steklov Inst. Math. 1981, Issue **4**, 185–209.
[13] V. G. Maz'ya, Zur Theorie Sobolewscher Räume, Teubner-texte zur Mathematik, Leipzig, 1981.
[14] Y. Mizuta, Integral representations of Beppo Levi functions of higher order, Hiroshima Math. J. **4** (1974), 375–396.
[15] M. Ohtsuka, Extremal length and precise functions in 3-space, Lecture notes, Hiroshima Univ., 1973.
[16] G. O. Okikiolu, On inequalities for integral operators, Glasgow Math. J. **11** (1970), 126–133.
[17] Premalatha and V. Anandam, On the integral representation of superharmonic functions in R^n, Acad. Roy. Belg. Bull. Cl. Sci. (5) **66** (1980), 116–129.
[18] Yu. G. Reshetnyak, The concept of capacity in the theory of functions with generalized derivatives, Siberian Math. J. **10** (1969), 818–942.
[19] S. G. Samko, On spaces of Riesz potentials, Math. USSR Izv. **10** (1976), 1089–1117.
[20] E. Sawyer, Multipliers of Besov and power-weighted L^2 spaces, Indiana Univ. Math. J. **33** (1984), 353–366.
[21] L. Schwartz, Théorie des distributions, Herman, Paris, 1966.
[22] S. L. Sobolev, On a theorem in functional analysis, Mat. Sob. **46** (1938), 471–497.
[23] S. L. Sobolev, The density of finite functions in the $L_p^{(m)}$ space, Soviet Math. Dokl. **4** (1963), 313–316.
[24] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, 1970.
[25] E. M. Stein and G. Weiss, Fractional integrals on n-dimensional Euclidean space, J. Math. Mech. **7** (1958), 503–514.
[26] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, Princeton, 1971.

Department of Mathematics,
College of Liberal Arts,
Kagoshima University