Correlation effects for stochastic zeros of Sturm-Liouville equations

K. KREITH

1. Introduction

Stochastic Sturm-Liouville equations of the form

$$(1.1) \qquad (p(x, \omega)u')' + q(x, \omega)u = 0; \quad (x, \omega) \in [0, \infty) \times \Omega$$

arise naturally in mathematical models of vibrating systems whose physical properties (e.g. masses and spring constants) are known only in terms of probabilities. The most obvious deterministic approximation to such an equation is the classical Sturm-Liouville equation

$$(1.2) \qquad (P(x)v')' + Q(x)v = 0; \quad x \in [0, \infty)$$

where $P(x)$ and $Q(x)$ are the expected values of $p(x, \omega)$ and $q(x, \omega)$ relative to a given probability space Ω. The rather natural correspondence between (1.1) and (1.2) makes it important to understand the extent to which solutions of (1.2) do in fact approximate solutions of (1.1).

This paper is concerned with solutions of (1.1) and (1.2) which also satisfy initial conditions of the form

$$u(0, \omega) = 0 \quad \text{with probability 1;} \quad v(0) = 0.$$

Denoting the smallest positive zeros of such solutions of (1.1) and (1.2) by $\xi(\omega)$ and η, respectively, we shall focus on the relationship between X and η, where X denotes the expected value of $\xi(\omega)$ relative to Ω.

In [3] an analogous theory for eigenvalues serves as a basis for establishing criteria which assure that

$$(1.3) \qquad X \geqslant \eta.$$

While applying to very general classes of equations, the criteria of [3] do not take into account the nature of the correlation between $p(x, \omega)$ and $q(x, \omega)$ and, as a result, lead to rather restrictive hypotheses for assuring (1.3). The present paper focuses on the nature of the correlation between $p(x, \omega)$ and $q(x, \omega)$ in establishing criteria for (1.3).

It is assumed throughout that the coefficients $p(x, \omega)$ and $q(x, \omega)$ are

respectively of class C^2 and C on $[0, \infty)$ for each $\omega \in \Omega$ and that there exist positive constants a, b, c and d for which

$$0 < a \leqslant p(x, \omega) \leqslant b; \quad 0 < c \leqslant q(x, \omega) \leqslant d$$

for all $(x, \omega) \in [0, \infty) \times \Omega$. We also use "Pr" to denote probabilites of events in Ω.

2. Measures of correlation

In order to establish appropriate measures of correlation and to determine their effects on first zeros, we begin by considering the simpler equation

(2.1) $$p(\omega)y'' + q(\omega)y = 0; \quad y(0) = 0$$

with solution $\sin((q(\omega)/p(\omega))^{1/2}x)$ and first zero $X(\omega) = \pi(p(\omega)/q(\omega))^{1/2}$. Given a particuular pair of coefficients $p(\omega)$ and $q(\omega)$ we consider a closed rectangle in the (p, q)-plane,

$$R = \{(p, q): 0 < a \leqslant p \leqslant b; 0 < c \leqslant q \leqslant d\}$$

for which $\Pr((p, q) \in R) = 1$. Our measure of correlation between $p(\omega)$ and $q(\omega)$ will be formulated in terms of parameters μ and v satisfying

(2.2) $$c/b \leqslant \mu < \min(d/b, c/a) < v \leqslant d/a.$$

These parameters are used to partition R into disjoint sets

$$E_\mu = \{(p, q) \in R: q \leqslant \mu p\},$$
$$E_v = \{(p, q) \in R: q \geqslant vp\},$$

and

$$H = R \cap E_\mu^c \cap E_v^c.$$

Letting $f(p, q)$ denote the joint probability density function of p and q, the concentration of $f(p, q)$ in E_μ and E_v corresponds to a strong negative correlation between p and q, while the concentration of $f(p, q)$ in H corresponds to a strong positive correlation. Accordingly we introduce the notation

$$\alpha = \Pr((p, q) \in E_v)$$
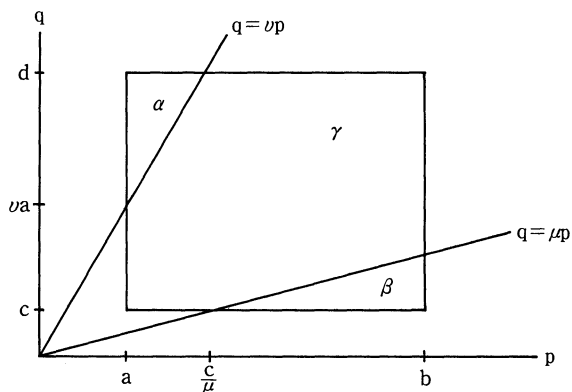$$\beta = \Pr((p, q) \in E_\mu)$$
$$\gamma = \Pr((p, q) \in H).$$

Figure 1.

2.1 THEOREM. *If*

(2.3)
$$\left(\frac{(\beta+\gamma)b+\alpha d/v}{(\beta+\gamma)c+\alpha va}\right)^{1/2} \leqslant \alpha(a/d)^{1/2} + \beta\mu^{-1/2} + \gamma v^{-1/2},$$

then $\eta \leqslant X$.

PROOF.   From Figure 1 we readily see that P is minimized if the probabilities α, β and γ are localized at $p=a$, $p=c/\mu$ and $p=a$, respectively. This shows that

$$(\alpha+\gamma)a + \beta(c/\mu) \leqslant P.$$

Proceeding in an analogous fashion, we can use the data represented in Figure 1 to establish that

$$(\alpha+\gamma)a + \beta(c/\mu) \leqslant P \leqslant (\beta+\gamma)b + \alpha(d/v)$$

$$(\beta+\gamma)c + \alpha va \leqslant Q \leqslant (\alpha+\gamma)d + \beta\mu b.$$

This shows that $\eta = \pi(P/Q)^{1/2}$

(2.4)
$$\pi\left(\frac{(\alpha+\gamma)a + \beta c/\mu}{(\alpha+\gamma)d + \beta\mu b}\right)^{1/2} \leqslant \eta \leqslant \pi\left(\frac{(\beta+\gamma)b + \alpha d/v}{(\beta+\gamma)c + \alpha va}\right)^{1/2}.$$

In order to obtain analogous bounds for X we make use of the transformation [2]

$$\begin{cases} t = (p/q)^{1/2} \\ s = (pq)^{1/2} \end{cases} \quad \text{or} \quad \begin{cases} p = st \\ q = s/t \end{cases}$$

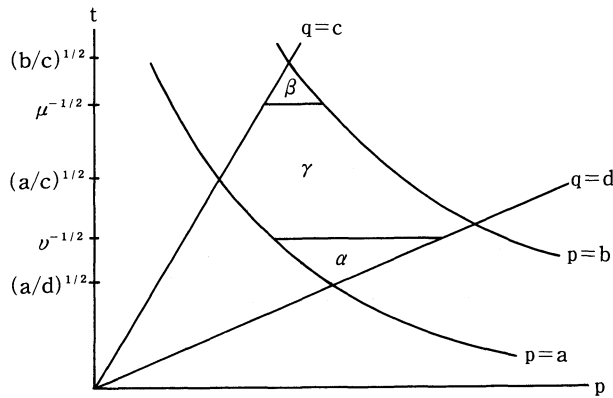with Jacobian $J(p, q)/(s, t) = -2q \neq 0$ which maps R onto a region R' in the (s, t)-

Figure 2.

plane as in Figure 2: Since the lines of $q=vp$ and $q=\mu p$ in Figure 1 map into horizontal lines $t=v^{-1/2}$ and $t=\mu^{-1/2}$, respectively, it is clear from Figure 2 that the expected value of $t(\omega)=(p(\omega)/q(\omega))^{1/2}$ is minimized if the probabilities α, β and γ are localized at $t=(a/d)^{1/2}$, $t=\mu^{-1/2}$, and $t=v^{-1/2}$, respectively. This shows that the expected value of t satifies

$$(2.5) \qquad\qquad T \geqslant \alpha(a/d)^{1/2} + \beta\mu^{-1/2} + \gamma v^{-1/2}.$$

Since $X=\pi T$, (2.3) follows directly from (2.5) and (2.4).

REMARKS

1.   From Figure 2 we could also show that

$$(2.5)' \qquad\qquad T \leqslant \alpha v^{-1/2} + \beta(b/c)^{1/2} + \gamma\mu^{-1/2}$$

and then seek to establish the complimentary inequality $X \leqslant \eta$ on the basis of

$$(2.6) \qquad \alpha v^{-1/2} + \beta(b/c)^{1/2} + \gamma\mu^{-1/2} \leqslant \left(\frac{(\alpha+\gamma)a+\beta c/\mu}{(\alpha+\gamma)d+\beta\mu b}\right)^{1/2}.$$

However, it is not clear that (2.6) has any nontrivial solutions—i.e. solutions in cases where (1.1) is truly stochastic. An example of a stochastic Sturm-Liouville equation of the form (1.1) for which $X < \eta$ is given in [3]; however, it is not known whether this is possible for equations of the special form (2.1).

2.   Whereas (2.6) may lack nontrivial solutions, this is *not* the case for (2.3). In particular, whenever $\beta>0$ we can exploit the convexity of the square root function to get solutions of (2.3) by choosing $\mu^{-1/2}=(c/b)^{1/2}$ sufficiently large. Since the concentration of the probability density function $f(p,q)$ in the neighborhood of (a,d)

and (b, c) corresponds to a strong negative correlation between $p(\omega)$ and $q(\omega)$, (2.3) can be interpreted as a correlation condition on these coefficients.

EXAMPLE. Suppose $\alpha = 1/4, \beta = 1/4, \gamma = 1/2, a = c = 1$ and $b = d = 2$. Choosing $\mu = 1/2$ and $\nu = 2$ implies that E_μ and E_ν consist of the single points $(2,1)$ and $(1, 2)$, respectively. Here (2.3) is *not* satisfied because

$$\left(\frac{(3/4) \cdot 2 + (1/4) \cdot 2/2}{(3/4) \cdot 1 + (1/4) \cdot 2 \cdot 1} \right)^{1/2} \nleqslant (3/4)\,(1/2)^{1/2} + (1/4)2^{1/2}.$$

We can, however, enchance the negative correlation between p and q by setting $\gamma = 0$ and $\alpha = \beta = 1/2$, in which case (2.3) is satisfied because

$$\left(\frac{(1/2) \cdot 2 + (1/2) \cdot (2/2)}{(1/2) \cdot 1 + (1/2) \cdot 2 \cdot 1} \right) = 1 < (1/2)2^{-1/2} + (1/2)2^{1/2} \approx 1.06.$$

The inequality (2.3) also remains valid with $\alpha = 1/4$ and $\beta = 3/4$ or with $\alpha = 3/4$ and $\beta = 1/4$.

3. Comparison theorem

Whereas the techniques of §2 cannot be applied to (1.1) directly, it is possible to use Sturmian comparison theorems to establish criteria for solutions of (1.1) to satisfy (1.3). Given

(3.1) $(p(x, \omega)u')' + q(x, \omega)u = 0; \quad \Pr(u(0, \omega) = 0) = 1$

with coefficients satisfying

$$\Pr(a \leqslant p(x, \omega) \leqslant b; \quad c \leqslant q(x, \omega) \leqslant d) = 1$$

for all $x \in [0, \infty)$, we enclose (3.1) between Sturmian majorants and minorants

(3.2) $p_1(\omega)y'' + q_1(\omega)y = 0; \quad y(0) = 0$

(3.3) $p_2(\omega)z'' + q_2(\omega)z = 0; \quad z(0) = 0$

whose coefficients satisfy

(3.4) $0 < a \leqslant p_2(\omega) \leqslant p(x, \omega) \leqslant p_1(\omega) \leqslant b$

$$0 < c \leqslant q_1(\omega) \leqslant q(x, \omega) \leqslant q_2(\omega) \leqslant d$$

with probability 1 for all $x \in [0, \infty)$. Denoting the first zeros of (3.2) and (3.3) by $\xi_1(\omega)$ and $\xi_2(\omega)$ respectively, let X_i denote the corresponding expected values for $i = 1, 2$. We also associate with (3.2) and (3.3) the deterministic equations

(3.5) $P_1 y'' + Q_1 y = 0; \quad y(0) = 0$

(3.6) $P_2 z'' + Q_2 z = 0;$ $z(0) = 0$

in which P_i and Q_i denote the expected values of $p_i(\omega)$ and $q_i(\omega)$, respectively, and η_i is the first zero of the corresponding solution. This notation leads to the following.

3.1 THEOREM.   *If* (3.4) *is valid for all* $x \in [0, \infty)$ *and*

(3.7) $X_2 \geqslant \eta_1,$

then $X \geqslant \eta.$

PROOF.   Applying the Sturm comparison theorem to (3.1) and (3.3), we see that (3.4) assures that $\xi_2(\omega) \leqslant \xi(\omega)$ for all $\omega \in \Omega$ and, as a result, that $X_2 \leqslant X$. The inequalities of (3.4) also assure that $P(x) \leqslant P_1, Q(x) \geqslant Q_1$, so that another application of the Sturm comparison theorem yields $\eta \leqslant \eta_1$. Combining these observations with (3.7) yields

$$\eta \leqslant \eta_1 \leqslant X_2 \leqslant X,$$

as was to be shown.

REMARK.   Hypothesis (3.7) runs counter to what one would expect from the ordering in (3.4). The usefulness of this theorem depends on showing that (3.7) can in fact be achieved. In [3], where correlation among the coefficients was not considered, Jensen's inequality provided a basis for establishing such inequalities. In the present context (3.7) can be attained by specifying appropriate forms of negative correlation among the coefficients of (3.2) and (3.3). Specific criteria for (3.7) can be established in terms of parameters $\alpha_i, \beta_i, \gamma_i, \mu_i$ and v_i which are defined relative to (3.2) and (3.3) in the same way as $\alpha, \beta, \gamma, \mu$ and v were defined relative to (2.1). The inequality (3.4) calls for a decrease in $p_i(\omega)$ and increase in $q_i(\omega)$, as i makes the transition from $i = 1$ to $i = 2$. In terms of Figure 1, this corresponds to a shift in the corresponding probability density functions "up and to the left" as i goes from 1 to 2–i.e. by $\alpha_2 \geqslant \alpha_1$ and $\beta_2 \leqslant \beta_1$ if μ and v are to remain fixed, or else by $\mu_2 \geqslant \mu_1$ and $v_2 \geqslant v_1$ if α, β and γ are to remain fixed. Referring to Figures 1 and 2, one readily sees that the hypotheses of Theorem 3.1 are satisfied whenever

(3.8) $\alpha_2(a/d)^{1/2} + \beta_2 \mu_2^{-1/2} + \gamma_2 v_2^{-1/2} \geqslant \left(\dfrac{(\beta_1 + \gamma_1)b + \alpha_1 d/v_1}{(\beta_1 + v_1)c + \alpha_1 v_1 a} \right)^{1/2}$

in the face of such shifts in parameters.

Because of the large number (ten) of indexed parameters which appear in (3.8), it is difficult to characterize all the relations between them. However, by holding some of the parameters fixed one can illustrate the importance of "negative correlation" by examining the constrained relations which result.

By way of example, consider again the case where $a = c = 1, b = d = 2$. Setting v_1

$= \nu_2 = 2$, $\alpha_1 = \alpha_2 = .5$, $\beta_1 = \beta_2 = .5$, $\gamma_1 = \gamma_2 = 0$ and $\mu_1 = 1/2$, we seek the values of μ_2 $> 1/2$ which are consistent with (3.8). In other words, we seek to satisfy

$$(1/2)^{3/2} + (1/2)\mu_2^{-1/2} \geqslant 1$$

by choosing $\mu_2 \leqslant .598$. In this way we define a region

$$E_\mu = \{(p, q): \quad q \leqslant .598p\}$$

contained in $R = \{(p, q): 1 \leqslant p, q \leqslant 2\}$ in which the trajectory corresponding to $(p(x, \omega), q(x, \omega))$ must be contained in order to assure that $\eta \leqslant X$.

If we vary this example to allow for a measure of positive correlation

$$\gamma_1 = \gamma_2 = .05; \quad \beta_1 = \beta_2 = .45,$$

This can be expected to reduce the latitude in choosing μ. Indeed, to satisfy

$$(1/2)^{3/2} + .45\mu_2^{-1/2} + (1/20)2^{-1/2} \geqslant 1$$

we must have $\mu_2 \leqslant .543$. However, the correlation implicit in

$$\gamma_1 = \gamma_2 = .1; \quad \beta_1 = \beta_2 = .4$$

does not allow us to satisfy (3.8) in the context of the values assigned to the other parameters.

References

[1]   A. T. Bharucha-Reid,   Probabilistic Methods in Applied Mathematics,   Academic Press, New York, 1968.
[2]   M. De Groot,   Probability and Statistics,   Addison-Wesley, Reading, Mass., 1975.
[3]   K. Kreith,   Lower estimates for zeros of stochastic Sturm-Liouville problems,   Proc. Amer. Math. Soc., **92** (1984), 515–518.

Department of Mathematics,
University of California
(Davis, CA 95616, U.S.A.)