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# Oscillation theory of higher order linear functional differential equations of neutral type

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## 1. Introduction

This paper is concerned with the oscillatory (and nonoscillatory) behavior of proper solutions of neutral linear functional differential equations of the type

$$\text{(A)} \qquad \frac{d^n}{dt^n}[x(t)-h(t)x(\tau(t))] + \sigma \sum_{i=1}^{N} p_i(t)x(g_i(t)) = 0,$$

where $n \geqq 2$, $\sigma = 1$ or $-1$, and the following conditions are assumed to hold without further mention:

(a) $h$: $[a, \infty) \to \boldsymbol{R}$ is continuous and satisfies $|h(t)| \leqq \lambda$ on $[a, \infty)$ for some constant $\lambda < 1$;

(b) $\tau$: $[a, \infty) \to \boldsymbol{R}$ is continuous and incresing, $\tau(t) < t$ for $t \geqq a$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(c) each $p_i$: $[a, \infty) \to (0, \infty)$ is continuous, $1 \leqq i \leqq N$;

(d) each $g_i$: $[a, \infty) \to \boldsymbol{R}$ is continuous and satisfies $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leqq i \leqq N$.

By a proper solution of (A) we mean a function $x$: $[T_x, \infty) \to \boldsymbol{R}$ which satisfies (A) (so that$x(t)-h(t)x(\tau(t))$ is $n$-times continuously differentiable) for all sufficiently large $t$ and $\sup\{|x(t)|: t \geqq T\} > 0$ for any $T \geqq T_x$. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory.

In recent years there has been a growing interest in oscillation theory of functional differential equations of neutral type; see, for example, the papers [1–6, 9–15] and the references cited therein. Most of the literature, however, is focused on first order linear equations with constant coefficients and deviations, and very few results are available for higher order equations with variable coefficients and deviations. To the best of the authors' knowledge, the first step toward a systematic investigation of the second kind was taken by Ruan [12] who studied the existence of nonoscillatory solutions of second order equations of the form

$$\frac{d^2}{dt^2}[x(t)-\lambda x(t-\tau)] + p(t)x(g(t)) = 0,$$

where $\lambda$ and $\tau$ are positive constants.

The purpose of this paper is twofold. First, we want to generalize Ruan's results [12] to $n$-th order equations of the form (A) involving much more general deviating arguments. This is done in Sections 2 and 3; we classify the possible nonoscillatory solutions of (A) according to their asymptotic behavior as $t\to\infty$, and construct, with the aid of fixed point techniques, nonoscillatory solutions $x(t)$ having the following types of asymptotic behavior

$$(\mathrm{I})_k \qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^k}=\text{const}\neq 0 \qquad \text{for some } k\in\{0,1,\dots,n-1\};$$

$$(\mathrm{II})_l \qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^l}=0, \quad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^{l-1}}=\infty \text{ or } -\infty$$

$$\text{for some } l\in\{1,2,\dots,n-1\}.$$

Secondly, we intend to establish criteria for oscillation of all proper solutions of equation(A). This is done in Section 4; we first derive "ordinary" functional differential inequalities of the type

$$\{\sigma u^{(n)}(t)+p(t)u(g(t))\}\operatorname{sgn} u(t)\leqq 0,$$

which must possess nonoscillatory solutions if equation (A) is assumed to have a nonoscillatory solution, and then show that such a situation is impossible by applying nonexistence criteria for the above inequalities recently obtained by Kitamura [8]

## 2. Classification of nonoscillatory solutions

A) *Kiguradze's lemma.* We begin by classifying all possible nonoscillatory solutions of equation (A) according to their asymptotic behavior as $t\to\infty$, on the basis of a well known lemma of Kiguradze [7] stated below.

LEMMA 2.1. *Let $u\in C^n[t_0,\infty)$ be such that*

$$(2.1) \qquad u(t)\neq 0 \quad \text{and} \quad \sigma u(t)u^{(n)}(t)<0 \quad \text{for} \quad t\geqq t_0.$$

*Then, there exist an integer $l\in\{0,1,\dots,n\}$ and a $t_1\geqq t_0$ such that $(-1)^{n-l-1}\sigma=1$ and*

$$(2.2)_l \qquad \begin{cases} u(t)u^{(i)}(t)>0, & t\geqq t_1, \quad 0\leqq i\leqq l, \\ (-1)^{i-l}u(t)u^{(i)}(t)>0, & t\geqq t_1, \quad l\leqq i\leqq n. \end{cases}$$

A function $u(t)$ satisfying $(2.2)_l$ is termed a function of (Kiguradze) degree $l$. The asymptotic behavior of a function of degree $l$ is as follows:

i) If $l=0$ (which is possible only when $\sigma=1$ and $n$ is odd, or $\sigma=-1$ and $n$ is even), then either

$$\lim_{t\to\infty} u(t)=\text{const}\neq 0 \quad \text{or} \quad \lim_{t\to\infty} u(t)=0;$$

ii) If $1\leqq l\leqq n-1$, then one of the following three cases holds:

$$\lim_{t\to\infty}\frac{u(t)}{t^l}=\text{const}\neq 0$$

$$\lim_{t\to\infty}\frac{u(t)}{t^{l-1}}=\text{const}\neq 0$$

$$\lim_{t\to\infty}\frac{u(t)}{t^l}=0 \quad \text{and} \quad \lim_{t\to\infty}\frac{u(t)}{t^{l-1}}=\infty \quad \text{or} \quad -\infty;$$

iii) If $l=n$ (which is possible only when $\sigma=-1$), then either

$$\lim_{t\to\infty}\frac{u(t)}{t^{n-1}}=\text{const}\neq 0 \quad \text{or} \quad \lim_{t\to\infty}\frac{u(t)}{t^{n-1}}=\infty \quad \text{or} \quad -\infty.$$

In what follows we use the notation

$$\tau^0(t)=t,\ \tau^i(t)=\tau(\tau^{i-1}(t)),\ \tau^{-i}(t)=\tau^{-1}(\tau^{-(i-1)}(t)),\quad i=1,2,\dots, \tag{2.3}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

B) *Classification of nonoscillatory solutions.* Let $x(t)$ be a nonoscillatory solution of equation (A). From (A) it is clear that $x(t)-h(t)x(\tau(t))$ is eventually one-signed, so that either

$$x(t)[x(t)-h(t)x(\tau(t))]>0 \tag{2.4}$$

or

$$x(t)[x(t)-h(t)x(\tau(t))]<0 \tag{2.5}$$

for all sufficiently large $t$. If (2.4) holds, then the function $u(t)=x(t)-h(t)x(\tau(t))$ satisfies (2.1) for all large $t$, and so, by Lemma 2.1 $u(t)$ is a function of Kiguradze degree $l$ for some $l\in\{0,1,\dots,n\}$ with $(-1)^{n-l-1}\sigma=1$. Let us denote by $\mathcal{N}_l^+$ the set of solutions $x(t)$ of (A) which satisfy (2.4) and for which $x(t)-h(t)x(\tau(t))$ are of degree $l$. On the other hand, if (2.5) holds, then $u(t)=h(t)x(\tau(t))-x(t)$ satisfies (2.1) (with $\sigma$ replaced by $-\sigma$) for all large $t$. However, the Kiguradze degree of $u(t)$ must be zero. In fact, from (2.5) we have $|x(t)|\leqq|h(t)x(\tau(t))|\leqq\lambda|x(\tau(t))|$, and hence $|x(\tau^{-m}(t))|$

$\leqq \lambda^m|x(t)|$, $m=1, 2,\ldots$, which implies $\lim_{t\to\infty}x(t)=0$. Thus the set of all solutions $x(t)$ of (A) satisfying (2.5) is denoted by $\mathcal{N}_0^-$. It is clear that the class $\mathcal{N}_0^-$ for (A) is empty if $(-1)^{n-1}\sigma=1$, that is, if $\sigma=1$ and $n$ is odd, or if $\sigma=-1$ and $n$ is even. Summarizing the above observations, we have the following general classification relations for the set $\mathcal{N}$ of all nonoscillatory solutions of (A):

$$
\begin{aligned}
(2.6)\qquad & \mathcal{N}=\mathcal{N}_1^+\cup\mathcal{N}_3^+\cup\cdots\cup\mathcal{N}_{n-1}^+\cup\mathcal{N}_0^- && \text{for}\quad \sigma=1 \text{ and } n \text{ even},\\
& \mathcal{N}=\mathcal{N}_0^+\cup\mathcal{N}_2^+\cup\cdots\cup\mathcal{N}_{n-1}^+ && \text{for}\quad \sigma=1 \text{ and } n \text{ odd},\\
& \mathcal{N}=\mathcal{N}_0^+\cup\mathcal{N}_2^+\cup\cdots\cup\mathcal{N}_n^+ && \text{for}\quad \sigma=-1 \text{ and } n \text{ even},\\
& \mathcal{N}=\mathcal{N}_1^+\cup\mathcal{N}_3^+\cup\cdots\cup\mathcal{N}_n^+\cup\mathcal{N}_0^- && \text{for}\quad \sigma=-1 \text{ and } n \text{ odd}.
\end{aligned}
$$

We note here that if $h(t)$ is either oscillatory or eventually negative, then (A) cannot possess a nonoscillatory solution $x(t)$ satisfying (2.5), so that in this case the class $\mathcal{N}_0^-$ should be removed form (2.6).

C) *Asymptotic behavior of nonoscillatory solutions.* From what were stated in the above subsections it follows that a nonoscillatory solution $x(t)$ of (A) falls into one of the following four cases:

$$\text{(I)}\qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^k}=\text{const}\neq 0\quad \text{for some}\quad k\in\{0, 1,\ldots, n-1\};$$

$$\text{(II)}\qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^l}=0,\qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^{l-1}}=\infty\quad\text{or}\quad -\infty$$

for some $l\in\{1, 2,\ldots, n-1\}$ with $(-1)^{n-l-1}\sigma=1$;

$$\text{(III)}\qquad \lim_{t\to\infty}\frac{x(t)-h(t)x(\tau(t))}{t^{n-1}}=\infty\quad\text{or}\quad -\infty;$$

$$\text{(IV)}\qquad \lim_{t\to\infty}[x(t)-h(t)x(\tau(t)]=0.$$

We will see how the asymptotic behavior of $x(t)-h(t)x(\tau(t))$ reflects on that of the solution $x(t)$ itself. It suffices to consider only the solutions $x(t)$ of (A) satisfying (2.4). Let $x(t)$ be one such solution. Then, $u(t)=x(t)-h(t)x(\tau(t))$ satisfies $(2.2)_l$ for some $l\in\{0, 1,\ldots, n\}$ with $(-1)^{n-l-1}\sigma=1$. Let $t_2>t_1$ be such that $\tau(t)\geqq t_1$ for $t\geqq t_2$. Using the relation

$$(2.7)\qquad x(t)=u(t)+h(t)x(\tau(t))$$

repeatedly, we find

$$x(t) = \sum_{j=0}^{n(t)-1} H_j(t)u(\tau^j(t)) + H_{n(t)}(t)x(\tau^{n(t)}(t)), \qquad t > t_2, \tag{2.8}$$

where $n(t)$ denotes the least positive integer such that $t_1 < \tau^{n(t)}(t) \leqq t_2$ and $H_m(t)$, $m = 0$, 1, 2,..., are defined by

$$H_0(t) = 1, \quad H_m(t) = \prod_{i=0}^{m-1} h(\tau^i(t)), \quad m = 1, 2, \ldots. \tag{2.9}$$

From (2.8) and the fact that $|H_m(t)| \leqq \lambda^m$ it follows that

$$|x(t)| \leqq \frac{|u(t)|}{1-\lambda} + \xi, \qquad t \geqq t_2, \tag{2.10}$$

if $l \geqq 1$, and

$$|x(t)| \leqq \frac{|u(t_1)|}{1-\lambda} + \xi, \qquad t \geqq t_2, \tag{2.11}$$

if $l = 0$, where $\xi > 0$ is a constant.

If $h(t)$ is eventually positive, then we have

$$|x(t)| \geqq |u(t)| \qquad \text{for all large} \quad t. \tag{2.12}$$

Otherwise, using (2.7) we get

$$x(t) = u(t) + h(t)u(\tau(t)) + h(t)h(\tau(t))x(\tau^2(t)),$$

which shows that if

$$h(t)h(\tau(t)) \geqq 0 \qquad \text{for all large} \quad t \tag{2.13}$$

and if the Kiguradze degree $l$ of $u(t)$ is positive, then

$$|x(t)| \geqq (1-\lambda)|u(t)| \qquad \text{for all large} \quad t. \tag{2.14}$$

In view of (2.10), (2.11), (2.12) and (2.14) we conclude that under the hypothesis (2.13) (which includes the case of one-signed $h(t)$) the following four types of asymptotic behavior are possible for nonoscillatory solutions $x(t)$ of equation (A):

(I) $\quad 0 < \liminf_{t \to \infty} \dfrac{|x(t)|}{t^k} \leqq \limsup_{t \to \infty} \dfrac{|x(t)|}{t^k} < \infty \quad$ for some $\quad k \in \{0, 1, \ldots, n-1\}$;

(II) $\quad \lim_{t \to \infty} \dfrac{x(t)}{t^l} = 0$ and $\lim_{t \to \infty} \dfrac{|x(t)|}{t^{l-1}} = \infty$

for some $l \in \{1, 2, \ldots, n-1\}$ with $\quad (-1)^{n-l-1}\sigma = 1$;

$$\text{(III)} \quad \lim_{t\to\infty} \frac{|x(t)|}{t^{n-1}} = \infty; \qquad \text{(IV)} \quad \lim_{t\to\infty} x(t) = 0.$$

## 3. Existence of nonoscillatory solutions

A) *Solutions of type* I. The objective of this section is to obtain criteria for equation (A) to have nonoscillatory solutions of types I and II described in Section 2. We start with type-I solutions, and show that such solutions can be completely characterized in case (2.13) is satisfied.

THEOREM 3.1. *Suppose that* (2.13) *holds. Equation* (A) *has a nonoscillatory solution* $x(t)$ *satisfying* (2.4) *and*

$$\lim_{t\to\infty} \frac{x(t)-h(t)x(\tau(t))}{t^k} = const \neq 0 \tag{3.1}$$

*for some* $k\in\{0, 1,\dots,n-1\}$ *if and only if*

$$\sum_{i=1}^{N} \int_a^\infty t^{n-k-1}[g_i(t)]^k p_i(t)dt < \infty. \tag{3.2}$$

PROOF. (The "only if" part) Let $x(t)$ be a solution of (A) satisfying (2.4) and (3.1). Since

$$\lim_{t\to\infty} \frac{d^i}{dt^i}[x(t)-h(t)x(\tau(t))] = 0, \qquad k+1 \leqq i \leqq n-1,$$

if $k<n-1$, repeated integration of (A) shows that

$$\sum_{i=1}^{N} \int_T^\infty t^{n-k-1} p_i(t)|x(g_i(t))|dt < \infty \tag{3.3}$$

provided $T>a$ is large enough. If $k=n-1$, an integration of (A) guarantees the truth of (3.3). On the other hand, from (2.13) and (3.1) we have

$$\liminf_{t\to\infty} \frac{|x(g_i(t))|}{[g_i(t)]^k} > 0, \qquad 1 \leqq i \leqq N. \tag{3.4}$$

The desired inequality (3.2) then follows from (3.3) and (3.4).

(The "if" part) Let $T>a$ be large enough so that

$$T_0 = \min\{\tau(T), \inf_{t\geqq T} g_1(t),\dots, \inf_{t\geqq T} g_N(t)\} \geqq a \tag{3.5}$$

and

$$\sum_{i=1}^{N}\int_{T}^{\infty} t^{n-k-1}[g_i(t)]^k p_i(t)dt \leqq \frac{(1-\lambda)^2}{2}. \tag{3.6}$$

We denote by $X_k$ the set of all functions $x\in C[T_0,\infty)$ such that $x(t)$ is nondecreasing and

$$\frac{c(t-T)^k}{k!} \leqq x(t) \leqq \frac{c(t-T)^k}{\lambda k!} \quad \text{for} \quad t \geqq T \quad \text{and } x(t)=x(T) \text{ for } t_0 \leqq t \leqq T,$$

where $c>0$ is an arbitrary but fixed constant. (Note that $x(t)=0$ for $T_0 \leqq t \leqq T$ if $k>1$.) Following Ruan [12], with each $x\in X_k$ we associate a function $\hat{x}$: $[T_0,\infty)\to \boldsymbol{R}$ defined by

$$\begin{cases} \hat{x}(t) = \displaystyle\sum_{j=0}^{n(t)-1} H_j(t)x(\tau^j(t)) + \frac{x(T)}{1-h(T)}H_{n(t)}(t), & t>T, \\ \hat{x}(t) = \dfrac{x(T)}{1-h(T)}, & T_0 \leqq t \leqq T, \end{cases} \tag{3.7}$$

where $n(t)$ denotes the least positive integer such that $T_0 < \tau^{n(t)}(t) \leqq T$ and $H_m(t)$, $m=0, 1, 2,\ldots$, are given by (2.9). It is easily verified that $\hat{x}\in C[T_0,\infty)$ and satisfies the functional equation

$$\hat{x}(t) - h(t)\hat{x}(\tau(t)) = x(t), \qquad t \geqq T. \tag{3.8}$$

Now, using the abbreviation

$$f(t, \hat{x}(g(t))) = \sum_{i=1}^{N} p_i(t)\hat{x}(g_i(t)), \tag{3.9}$$

we define the mapping $\mathscr{F}$: $X_k \to C[T_0,\infty)$ in the following manner: if $k\geqq 1$, then

$$\begin{cases} \mathscr{F}x(t) = \dfrac{\delta c(t-T)^k}{k!} \\ \quad +(-1)^{n-k-1}\sigma \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!}\int_s^{\infty}\frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, \hat{x}(g(r)))drds, & t \geqq T, \\ \mathscr{F}x(t) = 0, \qquad T_0 \leqq t \leqq T; \end{cases} \tag{3.10}$$

if $k=0$, then

$$
(3.11)\quad \begin{cases} \mathscr{F}x(t)=\delta c+(-1)^{n-1}\sigma\displaystyle\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}f(s,\hat{x}(g(s)))ds, & t\geqq T,\\ \mathscr{F}x(t)=\delta c+(-1)^{n-1}\sigma\displaystyle\int_T^\infty\frac{(s-T)^{n-1}}{(n-1)!}f(s,\hat{x}(g(s)))ds, & T_0\leqq t\leqq T,\end{cases}
$$

where $\delta=1$ if $(-1)^{n-k-1}\sigma=1$ and $\sigma=1/\lambda$ if $(-1)^{n-k-1}\sigma=-1$.

Let $x\in X_k$. Using (3.7), we have

$$
0\leqq\hat{x}(g_i(t))\leqq\sum_{j=0}^{n(g_i(t))-1}\lambda^j\frac{c[\tau^j(g_i(t))-T]^k}{\lambda k!}+\frac{\lambda^{n(g_i(t))}x(T)}{1-\lambda},
$$

which implies that

$$
0\leqq\hat{x}(g_i(t))\leqq\frac{c}{\lambda(1-\lambda)}[g_i(t)]^k,\qquad t\geqq T,\qquad\text{for}\quad k\geqq 1,
$$

and

$$
0\leqq\hat{x}(g_i(t))\leqq\frac{2c}{\lambda(1-\lambda)},\qquad t\geqq T,\qquad\text{for}\quad k=0.
$$

From the above inequalities and (3.6) we see that if $k\geqq 1$, then

$$
\begin{aligned}
0&\leqq\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)!}f(r,\hat{x}(g(r)))drds\\
&\leqq\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}ds\cdot\int_T^\infty r^{n-k-1}\frac{c}{\lambda(1-\lambda)}\sum_{i=1}^N p_i(r)\,[g_i(r)]^k dr\\
&\leqq\frac{1-\lambda}{2\lambda k!}c(t-T)^k,\qquad t\geqq T,
\end{aligned}
$$

and if $k=0$, then

$$
0\leqq\int_t^\infty\frac{(s-t)^{n-1}}{(n-1)!}f(s,\hat{x}(g(s)))ds\leqq\int_T^\infty s^{n-1}\frac{2c}{\lambda(1-\lambda)}\sum_{i=1}^N p_i(s)ds\leqq\frac{1-\lambda}{\lambda}c,\quad t\geqq T.
$$

Using these inequalities in (3.10) and (3.11), we conclude that $\mathscr{F}x\in X_k$, which implies that $\mathscr{F}$ maps $X_k$ into itself. It is not difficult to verify that $\mathscr{F}$ is continuous and $\mathscr{F}(X_k)$ is relatively compact in the topology of $C[T_0,\infty)$. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of an element $x^*\in X_k$ such that $x^*=\mathscr{F}x^*$, which is equivalent to the equation

$$
x^*(t)=\frac{\delta c(t-T)^k}{k!}+(-1)^{n-k-1}\sigma\int_T^t\frac{(t-s)^{k-1}}{(k-1)!}\int_s^\infty\frac{(r-s)^{n-k-1}}{(n-k-1)!}\cdot
$$

$$f(r, \hat{x}^*(g(r)))drds, \qquad t \geqq T \quad \text{for} \quad k \geqq 1,$$

$$x^*(t) = \delta c + (-1)^{n-1}\sigma \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \hat{x}^*(g(s)))ds, \qquad t \geqq T \quad \text{for} \quad k = 0.$$

Taking the relation (3.8) (with $x = x^*$) into account, we obtain

$$\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t)) = \frac{\delta c(t-T)^k}{k!}$$

$$+ (-1)^{n-k-1}\sigma \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f(r, \hat{x}^*(g(r)))drds,$$

$$t \geqq T \qquad \text{for} \quad k \geqq 1,$$

$$\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t)) = \delta c + (-1)^{n-1}\sigma \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} f(s, \hat{x}^*(g(s)))ds,$$

$$t \geqq T \qquad \text{for} \quad k = 0,$$

which implies by differentiation that $x^*(t)$ is a positive solution of (A) satisfying (2.4) and (3.1). This completes the proof.

B) *Solutions of type* II. We now consider nonoscillatory solutions of type II of equation (A), that is, those solutions $x(t)$ which satisfy (2.4) and

$$\text{(3.12)} \qquad \lim_{t\to\infty} \frac{x(t) - h(t)x(\tau(t))}{t^l} = 0, \quad \lim_{t\to\infty} \frac{x(t) - h(t)x(\tau(t))}{t^{l-1}} = \infty \quad \text{or} \quad -\infty$$

for some $l \in \{1, 2, \dots, n-1\}$ such that $(-1)^{n-l-1}\sigma = 1$.

If $x(t)$ is one such solution of (A), then integration of (A) gives

$$\sum_{i=1}^N \int_T^\infty t^{n-l-1} p_i(t)|x(g_i(t))|dt < \infty$$

and

$$\sum_{i=1}^N \int_T^\infty t^{n-l} p_i(t)|x(g_i(t))|dt = \infty$$

for some $T > a$ sufficiently large. Suppose that (2.13) holds. Combining these inequalities with the inequalities

$$|x(t)| \geqq \alpha t^{l-1} \quad \text{and} \quad |x(t)| \leqq \beta t^l \quad \text{for} \quad t \geqq T,$$

$\alpha$ and $\beta$ being arbitrary positive constants, which follow readily from (2.10), (2.13) and (3.12), we see that

$$\sum_{i=1}^{N}\int_a^\infty t^{n-l-1}[g_i(t)]^{l-1}p_i(t)dt < \infty \tag{3.13}$$

and

$$\sum_{i=1}^{N}\int_a^\infty t^{n-l}[g_i(t)]^{l}p_i(t)dt = \infty. \tag{3.14}$$

Thus, (3.13) and (3.14) are necessary conditions for the existence of a solution $x(t)$ satisfying (2.4) and (3.12) of equation (A) for which (2.13) is satisfied.

The following theorem provides sufficient conditions for the existence of such a solution of (A) in the case where $h(t) \geqq 0$ for $t \geqq a$. We have been unable to obtain a corresponding result for the case of negative $h(t)$.

THEOREM 3.2. *Suppose that* $h(t) \geqq 0$ *for* $t \geqq a$. *Let* $l \in \{1, 2, \ldots, n-1\}$ *be such that* $(-1)^{n-l-1}\sigma = 1$. *Equation* (A) *has a nonoscillatory solution* $x(t)$ *satisfying* (2.4) *and* (3.12) *if*

$$\sum_{i=1}^{N}\int_a^\infty t^{n-l-1}[g_i(t)]^{l}\, p_i(t)dt < \infty \tag{3.15}$$

*and*

$$\sum_{i=1}^{N}\int_a^\infty t^{n-l}[g_i(t)]^{l-1}p_i(t)dt = \infty. \tag{3.16}$$

PROOF. Choose $T > a$ so large that (3.5) holds and

$$\sum_{i=1}^{N}\int_T^\infty t^{n-l-1}[g_i(t)]^{l}p_i(t)dt \leqq 1-\lambda. \tag{3.17}$$

Let $c > 0$ be fixed and consider the set $X_l$ consistisng of all functions $x \in C[T_0, \infty)$ such that

$$\frac{c(t-T)^{l-1}}{(l-1)!} \leqq x(t) \leqq \frac{c(t-T)^{l-1}}{(l-1)!} + \frac{c(t-T)^{l}}{l!} \qquad \text{for} \quad t \geqq T \tag{3.18}$$

and

$$x(t) = x(T) \qquad \text{for} \quad T_0 \leqq t \leqq T.$$

If $x \in X_l$, then since $x(t) \leqq ct^l/(l-1)!$, $t \geqq T$, the function $\hat{x}(t)$ defined by (3.7) satisfies

$$\hat{x}(g_i(t)) \leqq \frac{2c}{1-\lambda}[g_i(t)]^l, \qquad t \geqq T, \quad 1 \leqq i \leqq N$$

and so the mapping $\mathscr{F}$ defined by

$$\mathscr{F}x(t) = \frac{c(t-T)^{l-1}}{(l-1)!} + \int_T^t \frac{(t-s)^{l-1}}{(l-1)!} \int_s^\infty \frac{(r-s)^{n-l-1}}{(n-l-1)!} f(r, \hat{x}(g(r)))drds, \qquad t \geqq T,$$

$$\mathscr{F}x(t) = 0, \quad T_0 \leqq t \leqq T, \qquad \text{for} \quad l > 1, \tag{3.19}$$

$$\mathscr{F}x(t) = c, \quad T_0 \leqq t \leqq T, \qquad \text{for} \quad l = 1,$$

maps $X_l$ into itself. Since the continuity of $\mathscr{F}$ and the relative compactness of $\mathscr{F}(X_l)$ can be proved without difficulty, there exists an element $x^* \in X_l$ such that $x^* = \mathscr{F}x^*$. As in the proof of Theorem 3.1, the $\hat{x}^*(t)$ associated with $x^*(t)$ via (3.7) satisfies the integral equation

$$\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t)) = \frac{c(t-T)^{l-1}}{(l-1)!}$$

$$+ \int_T^t \frac{(t-s)^{l-1}}{(l-1)!} \int_s^\infty \frac{(r-s)^{n-l-1}}{(n-l-1)!} f(r, \hat{x}^*(g(r)))drds, \quad t \geqq T. \tag{3.20}$$

That $\hat{x}^*(t)$ is a solution of equation (A) follows from differentiation of (3.20). To show that $\hat{x}^*(t)$ has the desired asymptotic behavior, we note that

$$\frac{d^{l-1}}{dt^{l-1}}[\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t))] = c + \int_T^t \int_s^\infty \frac{(r-s)^{n-l-1}}{(n-l-1)!} f(r, \hat{x}^*(g(r)))drds \tag{3.21}$$

and

$$\frac{d^l}{dt^l}[\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t))] = \int_t^\infty \frac{(s-t)^{n-l-1}}{(n-l-1)!} f(s, \hat{x}^*(g(s)))ds \tag{3.22}$$

for $t \geqq T$. In view of (3.8) with $x = x^*$ and (2.14), we have

$$\hat{x}^*(t) = x^*(t) + h(t)\hat{x}^*(\tau(t))$$

$$\geqq (1-\lambda)x^*(t) \geqq (1-\lambda)c(t-T)^{l-1}/(l-1)! \tag{3.23}$$

for all large $t$. Combining (3.23) with the inequality

$$\frac{d^{l-1}}{dt^{l-1}}[\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t))] \geqq c + \int_T^t \frac{(r-T)^{n-l}}{(n-l)!} f(r, \hat{x}^*(g(r)))dr$$

which is a consequence of (3.21), we obtain

$$\lim_{t \to \infty} \frac{d^{l-1}}{dt^{l-1}}[\hat{x}^*(t) - h(t)\hat{x}^*(\tau(t))] = \infty.$$

On the other hand, (3.22) implies that

$$\lim_{t\to\infty}\frac{d^l}{dt^l}[\hat{x}^*(t)-h(t)\hat{x}^*(\tau(t))]=0.$$

It follows that $\hat{x}^*(t)$ satisfies (2.4) and (3.12). This completes the proof.

C) *Remarks and examples.* Consider the special case of (A) with $N=1$:

$$\frac{d^n}{dt^n}[x(t)-h(t)x(\tau(t))]+\sigma p(t)x(g(t))=0. \tag{3.24}$$

In addition to (a)–(d) assume that (2.13) is satisfied. Conditions (3.2), (3.15) and (3.16) for this equation reduce to

$$\int_a^\infty t^{n-k-1}[g(t)]^k p(t)dt<\infty, \tag{3.25}$$

$$\int_a^\infty t^{n-l-1}[g(t)]^l p(t)dt<\infty, \tag{3.26}$$

and

$$\int_a^\infty t^{n-l}[g(t)]^{l-1} p(t)dt=\infty, \tag{3.27}$$

respectively. Suppose that $g(t)$ satisfies

$$0<\liminf_{t\to\infty}\frac{g(t)}{t}\leqq\limsup_{t\to\infty}\frac{g(t)}{t}<\infty. \tag{3.28}$$

(Examples of such $g(t)$ are

$$g(t)=t\pm\delta,\quad g(t)=\mu t,\quad g(t)=t+\sin t,$$

where $\delta$ and $\mu$ are positive constants.) Then, the set of (3.25) for all $k=0, 1,\dots, n-1$ reduces to a single condition

$$\int_a^\infty t^{n-1}p(t)dt<\infty. \tag{3.29}$$

From Theorem 3.1 it follows that if (3.29) holds, then (3.24) has a solution $x(t)$ satisfying (3.1) for every $k\in\{0, 1,\dots, n-1\}$, and that if

$$\int_a^\infty t^{n-1}p(t)dt=\infty, \tag{3.30}$$

then (3.24) cannot have a solution $x(t)$ satisfying (3.1) for any $k\in\{0, 1,\dots, n-1\}$. We note that Theorem 3.2 is not applicable to equation (3.24) subject to (3.28), since in this case conditions (3.26) and (3.27) are not consistent for any $l$.

Next, suppose that $g(t)=t^{\theta}$, where $\theta\in(0, 1)$ is a constant. Then, (3.26) and (3.27) become

$$\int_a^{\infty} t^{n-l-1+\theta l}p(t)dt < \infty \tag{3.31}$$

and

$$\int_a^{\infty} t^{n-l+\theta(l-1)}p(t) = \infty, \tag{3.32}$$

which may hold simultaneously; for example, the function $p(t)=t^{\gamma}$, $\gamma$ being a constant, satisfies both (3.31) and (3.32) if $a>0$ and $(1-\theta)(l-1)-n\leqq\gamma<(1-\theta)l-n$. According to Theorem 3.2, conditions (3.31) and (3.32) for some $l\in\{1, 2,\dots, n-1\}$ with $(-1)^{n-l-1}\sigma=1$ guarantee the existence of a solution $x(t)$ of (3.24) which has the asymptotic behavior (3.12).

EXAMPLE 3.1. Consider the equation

$$\frac{d^2}{dt^2}[x(t)-\lambda x(t-\rho)]+(\lambda e^{\rho}-1)e^{(\nu-1)t}x(\nu t)=0, \quad t\geqq 0, \tag{3.33}$$

where $0<\lambda<1$, $\rho>0$ and $\nu>0$.

(i) Suppose that $\lambda e^{\rho}>1$. Then, (3.33) is a special case of (3.24) in which $n=2$, $\sigma=1$, $h(t)=\lambda$, $\tau(t)=t-\rho$, $p(t)=(\lambda e^{\rho}-1)e^{(\nu-1)t}$ and $g(t)=\nu t$. From (2.6) we have $\mathcal{N}=\mathcal{N}_1^+\bigcup\mathcal{N}_0^-$ for (3.33). Note that $\mathcal{N}_0^-\neq\phi$, since (3.33) has a solution $x(t)=e^{-t}$ belonging to this class. The possible asymptotic behaviors of the members $x(t)$ of $\mathcal{N}_1^+$ are:

$$\lim_{t\to\infty}\frac{x(t)-\lambda x(t-\rho)}{t}=\text{const}\neq 0, \tag{3.34}$$

$$\lim_{t\to\infty}[x(t)-\lambda x(t-\rho)]=\text{const}\neq 0, \tag{3.35}$$

and

$$\lim_{t\to\infty}\frac{x(t)-\lambda x(t-\rho)}{t}=0, \quad \lim_{t\to\infty}[x(t)-\lambda x(t-\rho)]=\infty \quad \text{or} \quad -\infty. \tag{3.36}$$

If $\nu<1$, then (3.29) $(n=2)$ holds, and so (3.33) has a solution satisfying (3.34) as well as a solution satisfying (3.35). However, there is no solution of (3.33) which has the asymptotic property (3.36), because the condition (3.14) which is necessary for the existence of such a solution is violated for equation (3.33).

If $\nu\geqq 1$, then (3.30) $(n=2)$ holds, so that (3.33) has neither a solution satisfying

(3.34) nor a solution satisfying (3.35). Since (3.13) is not satisfied, (3.33) does not admit a solution with the property (3.36).

(ii) Suppose that $\lambda e^{\rho} < 1$. Then, (3.33) is a special case of (3.24) in which $n = 2$, $\sigma = -1$, $h(t) = \lambda$, $\tau(t) = t - \rho$, $p(t) = (1 - \lambda e^{\rho})e^{(\nu-1)t}$ and $g(t) = \nu t$. The classification (2.6) then reduces to $\mathcal{N} = \mathcal{N}_0^+ \bigcup \mathcal{N}_2^+$, and the possible types of asymptotic behavior of nonoscillatory solutions $x(t)$ of (3.33) are (3.34), (3.35),

$$\lim_{t\to\infty} [x(t) - \lambda x(t-\rho)] = 0 \tag{3.37}$$

and

$$\lim_{t\to\infty} \frac{x(t) - \lambda x(t-\rho)}{t} = \infty \quad \text{or} \quad -\infty. \tag{3.38}$$

Exactly the same statements as in (i) hold for solutions which satisfy (3.34) and (3.35), depending on whether $\nu < 1$ or $\nu \geqq 1$. Equation (3.33) has a solution $x(t) = e^{-t}$ satisfying (3.37). No information can be drawn about the solutions $x(t)$ of (3.33) which satisfy (3.38).

EXAMPLE 3.2. Consider the equation

$$\frac{d^2}{dt^2}[x(t) - \lambda x(t-\rho)] - d(1 - \lambda e^{-\rho})e^{t - t^{\theta}} x(t^{\theta}) = 0, \quad t \geqq 0, \tag{3.39}$$

where $0 < \lambda < 1$, $\rho > 1$ and $\theta > 0$. The classification and the asymptotic behavior of nonoscillatory solutions of (3.39) are the same as in (ii) of Example 3.1. This equation possesses a solution $x(t) = e^t$ satisfying (3.38). It is not known if there is a solution of (3.39) satisfying (3.37). It is easy to see that if $\theta > 1$, then (3.39) has two types of solutions with asymptotic behaviors (3.34) and (3.35), and that if $\theta \leqq 1$, then (3.39) admits neither of these two types of nonoscillatory solutions.

## 4. Oscillation of all solutions

A) *Lemmas.* We are interested in the situation in which all proper solutions of equation (A) are oscillatory. Since this situation is equivalent to the nonexistence of nonoscillatory solutions of (A), the problem is to obtain conditions under which none of the solution classes appearing in the classification relation (2.6) has a member. The derivation of the desired results are based on the following lemmas due to Kitamura [8, p. 487] which provide oscillation criteria for "ordinary" functional differential inequalities of the form

$$\{\sigma u^{(n)}(t) + p(t)u(g(t))\} \operatorname{sgn} u(t) \leqq 0, \tag{4.1}$$

where $n \geqq 2$, $\sigma = \pm 1$, $p$: $[a, \infty) \to (0, \infty)$ is continuous, $g$: $[a, \infty) \to (0, \infty)$ is continuous, and $\lim_{t\to\infty} g(t) = \infty$. We use the notation:

$$(4.2)\qquad \begin{aligned} g^*(t) &= \max\{g(t), t\}, &\qquad \alpha[g]\,(t) &= \min_{s \geqq t} g^*(s), \\ g_*(t) &= \min\{g(t), t\}, &\qquad \rho[g]\,(t) &= \max_{a \leqq s \leqq t} g_*(s). \end{aligned}$$

LEMMA 4.1. *Let $\sigma = 1$ and $n$ be even. There is no nonoscillatory solution of* (4.1) *if*

$$(4.3)\qquad \int_a^\infty [g_*(t)]^{n-1}[g(t)]^{-\varepsilon} p(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0.$$

LEMMA 4.2. *Let $\sigma = 1$ and $n$ be odd. If*

$$(4.4)\qquad \int_a^\infty [g_*(t)]^{n-2}[g(t)]^{1-\varepsilon} p(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0,$$

*then all possible nonoscillatory solutions of* (4.1) *are of degree* 0. *Such nonoscillatory solutions are precluded if*

$$(4.5)\qquad \limsup_{t\to\infty} \int_{\rho[g](t)}^t \frac{\{s-\rho[g](t)\}^{n-i-1}}{(ni-1)!} \frac{\{\rho[g](t)-g(s)\}^i}{i!} p(s)ds > 1$$

*for some $i \in \{0, 1,\ldots, n-1\}$.*

LEMMA 4.3. *Let $\sigma = -1$ and $n$ be odd. If*

$$(4.6)\qquad \int_a^\infty t[g_*(t)]^{n-2}[g(t)]^{-\varepsilon} p(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0,$$

*then all possible nonoscillatory solutions are of degree $n$. Such nonoscillatory solutions are precluded if*

$$(4.7)\qquad \limsup_{t\to\infty} \int_t^{\alpha[g](t)} \frac{\{g(s)-\alpha[g](t)\}^{n-j-1}}{(n-j-1)!} \frac{\{\alpha[g](t)-s\}^j}{j!} p(s)ds > 1$$

*for some $j \in \{0, 1,\ldots, n-1\}$.*

LEMMA 4.4. *Let $\sigma = -1$ and $n$ be even. If*

$$(4.8)\qquad \int_a^\infty t[g_*(t)]^{n-3}[g(t)]^{1-\varepsilon} p(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0,$$

*then every nonoscillatory solution of* (4.1) *is either of degree* 0 *or of degree $n$. Solutions of degree* 0 [*resp. degree $n$*] *are precluded provided* (4.5) *holds for some $i \in \{0, 1,\ldots, n$*

$-1\}$ [*resp.* (4.7) *holds for some* $j \in \{0, 1, \ldots, n-1\}$].

B) *Oscillation criteria.* First we examine the case where $h(t)$ is eventually positive. Let $x(t)$ be a nonoscillatory solution of equation (A). Then, the function $x(t)[x(t)-h(t)x(\tau(t))]$ is either eventually positive or eventually negative.

Consider the case where $x(t)[x(t)-h(t)(\tau(t))]>0$ for large $t$. Put $v(t)=x(t)-h(t)x(\tau(t))$. Since, in this case, $|v(t)| \leqq |x(t)|$ for large $t$ (cf. (2.12)), we see from (A) that

$$\{\sigma v^{(n)}(t)+\sum_{i=1}^{N} p_i(t)v(g_i(t))\} \operatorname{sgn} v(t) \leqq 0$$

provided $t$ is large enough. It follows that $v(t)$ is a nonoscillatory solution of each of the differential inequalities

$$(4.9)_i \qquad \{\sigma v^{(n)}(t)+p_i(t)v(g_i(t))\} \operatorname{sgn} v(t) \leqq 0, \qquad 1 \leqq i \leqq N,$$

for all sufficiently large $t$, and that $x(t)$ is a member of $\mathcal{N}_l^+$ (cf. (2.6)) if and only if $v(t)$ is a solution of degree $l$ of $(4.9)_i$, $1 \leqq i \leqq N$.

Next consider the case where $x(t)[x(t)-h(t)x(\tau(t))]$ is eventually negative. Put $w(t)=h(t)x(\tau(t))-x(t)$. Since $|w(t)| \leqq h(t)|x(\tau(t))| \leqq \lambda |x(\tau(t))|$, we find $|w(\tau^{-1}(t))|/\lambda \leqq |x(t)|$, which combined with (A), yields

$$\{(-\sigma)w^{(n)}(t)+\lambda^{-1}\sum_{i=1}^{N} p_i(t)w(\tau^{-1}(g_i(t)))\} \operatorname{sgn} w(t) \leqq 0.$$

It follows that

$$(4.10)_i \qquad \{(-\sigma)w^{(n)}(t)+\lambda^{-1}p_i(t)w(\tau^{-1}(g_i(t)))\} \operatorname{sgn} w(t) \leqq 0, \qquad 1 \leqq i \leqq N,$$

for all sufficiently large $t$, which shows that $x(t)$ is a member of $\mathcal{N}_0^-$ if and only if $w(t)$ is a solution of degree 0 of $(4.10)_i$ for each $i$, $1 \leqq i \leqq N$.

Let us now turn to the case where $h(t)$ is eventually negative and the case where $h(t)$ is oscillatory and such that

$$(4.11) \qquad h(t)h(\tau(t)) \geqq 0 \qquad \text{for all large} \quad t.$$

In these cases, as was remarked in Section 2–B, there is no solution of (A) satisfying $x(t)[x(t)-h(t)x(\tau(t))]<0$ for large $t$, and, for a solution $x(t)$ of (A) such that $x(t)[x(t)-h(t)x(\tau(t))]>0$ for large $t$, the function $v(t)=x(t)-h(t)x(\tau(t))$ satisfies

$$(4.12) \qquad (1-\lambda)|v(t)| \leqq |x(t)| \qquad \text{for all large} \quad t,$$

provided the Kiguradze degree of $x(t)$ is positive. From (A) and (4.12) we see that $v(t)$ satisfies

$$(4.13)_i \qquad \{\sigma v^{(n)}(t)+(1-\lambda)p_i(t)v(g_i(t))\} \operatorname{sgn} v(t) \leqq 0, \qquad 1 \leqq i \leqq N,$$

for all sufficiently large $t$, and that $x \in \mathcal{N}_l^+$ with $l \geqq 1$ if and only if $v(t)$ is a solution of degree $l$ of $(4.13)_i$, $1 \leqq i \leqq N$.

To derive oscillation criteria for equation (A) it suffices to obtain conditions which preclude all the possible solution classes $\mathcal{N}_l^+$, $0 \leqq l \leqq n$, and $\mathcal{N}_0^-$ appearing in the classification (2.6). That this is indeed possible can be seen from the above observations combined with Lemmas 4.1–4.4 which apply directly to the functional differential inequalities $(4.9)_i$, $(4.10)_i$ and $(4.13)_i$, $1 \leqq i \leqq N$. The following theorems follow in this manner.

THEOREM 4.1. *Let $\sigma = 1$ and $n$ be even.*

(i) *Suppose that $h(t)$ is eventually negative or that $h(t)$ is oscillatory and satisfies* (4.11). *All proper solutions of* (A) *are oscillatory if there is $i \in \{1, \dots, N\}$ such that*

$$\int_a^\infty [g_{i*}(t)]^{n-1}[g_i(t)]^{-\varepsilon} p_i(t)dt = \infty \quad \textit{for some} \quad \varepsilon > 0, \tag{4.14}$$

*where $g_{i*}(t) = \min\{g_i(t), t\}$.*

(ii) *Suppose that $h(t)$ is eventually positive. All proper solutions of* (A) *are oscillatory if there are $i, j \in \{1, \dots, N\}$ such that* (4.14) *holds and*

$$\limsup_{t \to \infty} \int_{\rho[\tau^{-1} \circ g_j](t)}^t \frac{\{s - \rho[\tau^{-1} \circ g_j](t)\}^k}{k!} \cdot \frac{\{\rho[\tau^{-1} \circ g_j](t) - \tau^{-1} \circ g_j(s)\}^{n-k-1}}{(n-k-1)!} p_j(s)ds > \lambda \tag{4.15}$$

*for some $k \in \{0, 1, \dots, n-1\}$, where $\rho[\tau^{-1} \circ g_j]\ (t) = \max_{a \leq s \leq t} [\tau^{-1} \circ g_i]_*(s)$.*

THEOREM 4.2. *Let $\sigma = 1$ and $n$ be odd.*

(i) *Suppose that $h(t)$ is eventually positive. All proper solutions of* (A) *are oscillatory if there are $i, j \in \{1, \dots, N\}$ such that*

$$\int_a^\infty [g_{i*}(t)]^{n-2}[g_i(t)]^{1-\varepsilon} p_i(t)dt = \infty \qquad \textit{for some} \quad \varepsilon > 0 \tag{4.16}$$

*and*

$$\limsup_{t \to \infty} \int_{\rho[g_j](t)}^t \frac{\{s - \rho[g_j](t)\}^k}{k!} \frac{\{\rho[g_j](t) - g_j(s)\}^{n-k-1}}{(n-k-1)!} p_j(s)ds > 1 \tag{4.17}$$

*for some $k \in \{0, 1, \dots, n-1\}$.*

(ii) *Suppose that $h(t)$ is eventually negative or that $h(t)$ is oscillatory and satisfies* (4.11). *If* (4.16) *holds, then every proper solution of* (A) *is either oscillatory or belongs to class $\mathcal{N}_0^+$.*

THEOREM 4.3. *Let $\sigma=-1$ and $n$ be odd.*

(i) *Suppose that $h(t)$ is eventually negative or that $h(t)$ is oscillatory and satisfies* (4.11). *All proper solutions of* (A) *are oscillatory if there are $i, j\in\{1,\dots, N\}$ such that*

$$\int_a^\infty t[g_{i*}(t)]^{n-2}[g_i(t)]^{-\varepsilon}p_i(t)dt=\infty \quad \textit{for some} \quad \varepsilon>0 \tag{4.18}$$

*and*

$$\limsup_{t\to\infty}\int_t^{\alpha[g_j](t)}\frac{\{g_j(s)-\alpha[g_j](t)\}^\mu}{\mu!}\frac{\{\alpha[g_j](t)-s\}^{n-\mu-1}}{(n-\mu-1)!}p_j(s)ds>\frac{1}{1-\lambda} \tag{4.19}$$

*for some $\mu\in\{0, 1,\dots, n-1\}$.*

(ii) *Suppose that $h(t)$ is eventually positive. All proper solutions of* (A) *are oscillatory if there are $i, j, k\in\{1,\dots, N\}$ such that* (4.18) *holds,*

$$\limsup_{t\to\infty}\int_t^{\alpha[g_j](t)}\frac{\{g_j(s)-\alpha[g_j](t)\}^\mu}{\mu!}\frac{\{\alpha[g_j](t)-s\}^{n-\mu-1}}{(n-\mu-1)!}p_j(s)ds>1 \tag{4.20}$$

*for some $\mu\in\{0, 1,\dots, n-1\}$, and*

$$\limsup_{t\to\infty}\int_{\rho[\tau^{-1}\circ g_k](t)}^t\frac{\{s-\rho[\tau^{-1}\circ g_k](t)\}^\nu}{\nu!}\cdot\frac{\{\rho[\tau^{-1}\circ g_k](t)-\tau^{-1}\circ g_k(s)\}^{n-\nu-1}}{(n-\nu-1)!}p_k(s)ds>\lambda \tag{4.21}$$

*for some $\nu\in\{0, 1,\dots, n-1\}$.*

THEOREM 4.4. *Let $\sigma=-1$ and $n$ be even.*

(i) *Suppose that $h(t)$ is eventually positive. All proper solutions of* (A) *are oscillatory if there are $i, j, k\in\{1, \dots, N\}$ such that*

$$\int_a^\infty t[g_{i*}(t)]^{n-3}[g_i(t)]^{1-\varepsilon}p_i(t)dt=\infty \quad \textit{for some} \quad \varepsilon>0, \tag{4.22}$$

$$\limsup_{t\to\infty}\int_{\rho[g_j](t)}^t\frac{\{s-\rho[g_j](t)\}^\mu}{\mu!}\frac{\{\rho[g_j](t)-g_j(s)\}^{n-\mu-1}}{(n-\mu-1)!}p_j(s)ds>1 \tag{4.23}$$

*for some $\mu\in\{0, 1,\dots, n-1\}$, and*

*for some $\mu\in\{0, 1,\dots, n-1\}$, and*

$$\limsup_{t\to\infty}\int_t^{\alpha[g_k](t)}\frac{\{g_k(s)-\alpha[g_k](t)\}^\nu}{\nu!}\frac{\{\alpha[g_k](t)-s\}^{n-\nu-1}}{(n-\nu-1)!}p_k(s)ds>1 \tag{4.24}$$

*for some $\nu\in\{0, 1,\dots, n-1\}$.*

(ii) *Suppose that $h(t)$ is eventually negative or that $h(t)$ is oscillatory and satisfies* (4.11). *Every proper solution of* (A) *is either oscillatory or belongs to class $\mathcal{N}_0^+$ if there are $i, j \in \{1, \ldots, N\}$ such that* (4.22) *holds and*

$$(4.25) \qquad \limsup_{t \to \infty} \int_t^{\alpha[g_j](t)} \frac{\{g_j(s) - \alpha[g_j](t)\}^\mu}{\mu!} \frac{\{\alpha[g_j](t) - s\}^{n-\mu-1}}{(n-\mu-1)!} p_j(s) ds > \frac{1}{1-\lambda}$$

*for some $\mu \in \{0, 1, \ldots, n-1\}$.*

Proof of Theorem 4.1. According to (2.6), $\mathcal{N}_l^+$, $l \in \{1, 3, \ldots, n-1\}$, and $\mathcal{N}_0^-$ are the possible classes of nonoscillatory solutions of (A) with $\sigma = 1$ and even $n$. Our task is, therefore, to show that all of these solution classes are empty if the hypotheses of the theorem are satisfied.

(i) In this case $\mathcal{N}_0^-$ is necessarily empty. Suppose that $\mathcal{N}_l^+ \neq \phi$ for some $l \in \{1, 3, \ldots, n-1\}$. Then, each of the inequalities in (4.13) possesses a nonoscillatory solution of degree $l$. However, this is impossible, because from Lemma 4.1 applied to $(4.13)_i$ it follows that (4.14) prevents $(4.13)_i$ from having a nonoscillatory solution of any kind. Thus we must have $\mathcal{N}_l^+ = \phi$ for all $l \in \{1, 3, \ldots, n-1\}$.

(ii) If $\mathcal{N}_l^+ \neq \phi$ for some $l \in \{1, 3, \ldots, n-1\}$, then all inequalities in (4.9) have nonoscillatory solutions of degree $l$. On the other hand, applying Lemma 4.1, we see that, because of (4.14), $(4.9)_i$ has no nonoscillatory solution. This contradiction shows that $\mathcal{N}_l^+ = \phi$ for $l \in \{1, 3, \ldots, n-1\}$. If $\mathcal{N}_0^- \neq \phi$, then each of the inequalities in (4.10) admits a nonoscillatory solution of degree 0. However, from (4.15) and the second statement of Lemma 4.4 it follows that $(4.10)_j$ cannot have a solution of degree 0. Therefore, $\mathcal{N}_0^- = \phi$ as well.

Proof of Theorem 4.2. (i) If $\mathcal{N}_l^+ \neq \phi$ for some $l \in \{0, 2, \ldots, n-1\}$, then the inequalities in (4.9) must possess nonoscillatory solutions of degree $l$. However, this is impossible, because the first statement of Lemma 4.2 implies that under (4.16), $(4.9)_i$ cannot possess a nonoscillatory solution of degree $l \in \{2, 4, \ldots, n-1\}$, while the second statement of Lemma 4.2 implies that, under (4.17), $(4.9)_j$ does not admit a solution of degree 0. It follows in view of (2.6) that all proper solutions of (A) are oscillatory.

(ii) In this case $\mathcal{N}_l^+ = \phi$ for all $l \in \{2, 4, \ldots, n-1\}$, since, by Lemma 4.2, (4.16) ensures the nonexistence of solutions of degree $l$ for each of the inequalities $(4.13)_i$. However, the possibility $\mathcal{N}_0^+ \neq \phi$ cannot be excluded.

Proof of Theorem 4.3. (i) It is obvious that $\mathcal{N}_0^- = \phi$. Lemma 4.3 (the first part) applied to $(4.13)_i$ shows that $\mathcal{N}_l^+ = \phi$ for $l \in \{1, 3, \ldots, n-2\}$, and Lemma 4.3 (the second part) applied to $(4.13)_j$ shows that $\mathcal{N}_n^+ = \phi$

(ii) Apply the first and second parts of Lemma 4.3 to $(4.9)_i$ and $(4.9)_j$, respectively. Then, we see that $\mathcal{N}_l^+ = \phi$, $l \in \{1, 3, \ldots, n-2\}$, and $\mathcal{N}_n^+ = \phi$, respectively. That $\mathcal{N}_0^- = \phi$ follows from the second part of Lemma 4.2 applied to $(4.10)_k$. The conclusion of the theorem then follows from (2.6).

PROOF OF THEOREM 4.4. (i) It suffices to prove that $\mathcal{N}_l^+ = \phi$ for all $l \in \{0, 2, \dots, n\}$. First, from (4.22) and Lemma 4.4 (the first part) we see that $(4.9)_i$ has no nonoscillatory solution of degree $l \in \{2, 4, \dots, n-2\}$, which implies that $\mathcal{N}_l^+ = \phi$, $l \in \{2, 4, \dots, n-2\}$, for (A). Then, applying the second part of Lemma 4.4 to $(4.9)_j$ and $(4.9)_k$, we conclude that (4.23) implies $\mathcal{N}_0^+ = \phi$ and (4.24) implies $\mathcal{N}_n^+ = \phi$.

(ii) Lemma 4.4 applied to $(4.13)_i$ and $(4.13)_j$ shows, in view of (4.22) and (4.25), that $\mathcal{N}_l^+ = \phi$ for $l \in \{2, 4, \dots, n\}$. This completes the proof.

C) *Examples.* Consider the equations

$$(4.26)_\pm \qquad \frac{d^n}{dt^n}[x(t) \pm \lambda x(\alpha t)] + p(t)x(\beta t) = 0,$$

where $0 < \lambda < 1$, $0 < \alpha < 1$, $\beta > 0$ and $p$: $[a, \infty) \to (0, \infty)$ is continuous, $a > 0$. This is a special case of (A) in which $\sigma = 1$, $N = 1$, $h(t) = \mp \lambda$, $\tau(t) = \alpha t$, $p_1(t) = p(t)$ and $g_1(t) = \beta t$. Noting that

$$g_{1*}(t) = \min\{1, \beta\}t, \ \rho[\tau^{-1} \circ g_1](t) = (\beta/\alpha)t \quad \text{if} \quad \alpha > \beta, \ \rho[g_1] = \beta t \quad \text{if} \quad \beta < 1,$$

we have the following oscillation criteria for $(4.26)_\pm$ from Theorems 4.1 and 4.2.

(i) Let $n$ be even. All proper solutions of $(4.26)_+$ are oscillatory if

$$(4.27) \qquad \int_a^\infty t^{n-1-\varepsilon} p(t)dt = \infty \qquad \text{for some} \quad \varepsilon > 0.$$

All proper solutions of $(4.26)_-$ are oscillatory if, in addition to (4.27), $0 < \beta < \alpha$ and

$$(4.28) \qquad \limsup_{t \to \infty} \frac{(\beta/\alpha)^{n-i-1}}{i!(n-i-1)!} \int_{(\beta/\alpha)t}^t \left(s - \frac{\beta}{\alpha}t\right)^i (t-s)^{n-i-1} p(s)ds > \lambda$$

for some $i \in \{0, 1, \dots, n-1\}$.

(ii) Let $n$ be odd. All proper solutions of $(4.26)_-$ are oscillatory if, in addition to (4.27), $0 < \beta < 1$ and

$$(4.29) \qquad \limsup_{t \to \infty} \frac{\beta^{n-j-1}}{j!(n-j-1)!} \int_{\beta t}^t (s - \beta t)^j (t-s)^{n-j-1} p(s)ds > 1$$

for some $j \in \{0, 1, \dots, n-1\}$. If (4.27) holds, then every solution of $(4.26)_+$ is either oscillatory or tends to zero as $t \to \infty$.

Conditions (4.27), (4.28) and (4.29) are actually satisfied if $p(t) = t^\delta$ with $\delta > -n$.

Next consider the equations

$$(4.30)_{\pm} \qquad \frac{d^n}{dt^n}[x(t) \pm \lambda x(\alpha t)] - p(t)x(\beta t) - q(t)x(\gamma t) = 0,$$

where $\lambda$, $\alpha$, $\beta$ and $p(t)$ are as in (4.26)**o**, $\gamma > 0$ and $q$: $[a, \infty) \to (0, \infty)$ is continuous. This is a special case of (A) in which $\sigma = -1$, $N = 2$, $h(t) = \mp \lambda$, $\tau(t) = \alpha t$, $p_1(t) = p(t)$, $p_2(t) = q(t)$, $g_1(t) = \beta t$ and $g_2(t) = \gamma t$, and Theorems 4.3 and 4.4 give the following oscillation criteria for (4.30)**o**.

(iii) Let $n$ be odd. All proper solutions of $(4.30)_+$ are oscillatory if the following conditions (4.31) and (4.32) are satisfied:

$$(4.31) \qquad \text{either } \int_a^\infty t^{n-1-\varepsilon} p(t)dt = \infty \qquad \text{for some} \quad \varepsilon > 0, \quad \text{or}$$

$$\int_a^\infty t^{n-1-\varepsilon} q(t)dt = \infty \qquad \text{for some} \quad \varepsilon$$

(4.32) either $\beta > 1$ and

$$\limsup_{t\to\infty} \frac{\beta^i}{i!(n-i-1)!} \int_t^{\beta t} (s-t)^i (\beta t - s)^{n-i-1} q(s)ds > \frac{1}{1-\lambda}$$

for some $i \in \{0, 1, \ldots, n-1\}$, or $\gamma > 1$ and

$$\limsup_{t\to\infty} \frac{\gamma^i}{i!(n-i-1)!} \int_t^{\gamma t} (s-t)^i (\gamma t - s)^{n-i-1} q(s)ds > \frac{1}{1-\lambda}$$

for some $i \in \{0, 1, \ldots, n-1\}$.

All proper solutions of $(4.30)_-$ are oscillatory if, in addition to (4.31), $\beta > 1$, $0 < \gamma < \alpha$,

$$(4.33) \qquad \limsup_{t\to\infty} \frac{\beta^i}{i!(n-i-1)!} \int_t^{\beta t} (s-t)^i (\beta t - s)^{n-i-1} p(s)ds > 1$$

for some $i \in \{0, 1, \ldots, n-1\}$, and

$$(4.34) \qquad \limsup_{t\to\infty} \frac{(\gamma/\alpha)^{n-j-1}}{j!(n-j-1)!} \int_{(\gamma/\alpha)t}^{t} \left(s - \frac{\gamma}{\alpha} t\right)^j (t-s)^{n-j-1} q(s)ds > \lambda$$

for some $j \in \{0, 1, \ldots, n-1\}$

(iv) Let $n$ be even. All proper solutions of $(4.30)_-$ are oscillatory if, in addition to (4.31), $\beta > 1$ and (4.33) holds for some $i \in \{0, 1, \ldots, n-1\}$, and $0 < \gamma < 1$ and

$$(4.35) \qquad \limsup_{t\to\infty} \frac{\gamma^{n-k-1}}{k!(n-k-1)!} \int_{\gamma t}^{t} (s - \gamma t)^k (t-s)^{n-k-1} q(s)ds > 1$$

for some $k \in \{0, 1, \ldots, n-1\}$. Every proper solution of $(4.30)_+$ is either oscillatory or tends to zero as $t \to \infty$ if (4.31) holds, $\beta > 1$ and

$$\limsup_{t\to\infty}\frac{\beta^i}{i!(n-i-1)!}\int_t^{\beta t}(s-t)^i(\beta t-s)^{n-i-1}p(s)ds>\frac{1}{1-\lambda}\tag{4.36}$$

for some $i\in\{0, 1,\dots, n-1\}$, or if (4.31) holds, $\gamma>1$ and

$$\limsup_{t\to\infty}\frac{\gamma^i}{i!(n-i-1)!}\int_t^{\gamma t}(s-t)^i(\gamma t-s)^{n-i-1}q(s)ds>\frac{1}{1-\lambda}\tag{4.37}$$

for some $i\in\{0, 1,\dots, n-1\}$.

It is easy to see that the functions $p(t)=q(t)=t^\delta$, $\delta>-n$, satisfy all the integral conditions in (4.31)–(4.37).

Our final example is

$$\frac{d^n}{dt^n}[x(t)+\lambda\sin t\cdot x(t-2\pi)]\pm px(t^\theta)=0,\tag*{(4.38)$_\pm$}$$

where $0<\lambda<1$, $p>0$ and $\theta>0$. Since the function $h(t)=-\lambda\sin t$ satisfies (4.11), we conclude from Theorem 4.1 [resp. Theorem 4.3] that all proper solutions of $(4.38)_+$ [resp. $(4.38)_-$] are oscillatory if $n$ is even and $0<\theta<1$ [resp. $n$ is odd and $\theta>1$].

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