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Maximal extensions of ordered fields

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The notion of maximal ordered fields was first introduced in [2] and [3] by the authors. The existence and the uniqueness for a given rank were mainly discussed there.

In this paper, we say that K is a *maximal extension* of an ordered field F if $\psi_{K/F}$ is bijective and K is a maximal ordered field. The aim of this paper is to develope serveral basic properties of maximal extensions. Namely, for maximal extensions K_i/F_i $(i=1, 2)$ and a given isomorphism σ: $F_1 \simeq F_2$ as ordered fields, there exists an extension σ': $\mathrm{K}_1 \simeq K_2$ of σ; we also show that there can be infinitely many such extensions. Moreover we show that, for any extension K/F_1 such that K is a maximal ordered field, there exists an F_1-embedding $K_1 \to K$.

§1. Maximal extensions

For an ordered field F, $A_0 := A(F, \mathrm{Q}) = \{a \in F; |a| < b$ for some $b \in \mathrm{Q}\}$ is the finest valuation ring, that is, every convex valuation ring of F is a localization of A_0 and conversely. The set $\mathscr{C}(F)$ of all convex valuation rings of F is a totally ordered set under the inclusion relation. Let G_0 be the value group of the finest valuation defined by A_0. It is clear that $\mathscr{C}(F)$ is isomorphic to the set $\mathscr{H}(F)$ of all convex subgroups of G_0 as totally ordered sets. If K/F is an extension of ordered fields, we have a surjection $\psi_{K/F}$: $\mathscr{C}(K) \to \mathscr{C}(F)$ defined by $\psi(B_i) = B_i \cap F$, $B_i \in \mathscr{C}(K)$ (cf. [2], §1). A pair (A, B) of subsets of F is called a cut of F if $A \cup B = F$ and $a < b$ for any $a \in A$ and $b \in B$, where A or B may be an empty set. It follows from [1], Theorem 1.2, that if F is real closed, then there is a one to one correspondence between the set of all cuts of F and the set of all orderings of $F(x)$, where $F(x)/F$ is a simple transcendental extension. For a subset C of F and an element a of F, we write $C < a$ if $c < a$ for any $c \in C$. We say that K is a maximal ordered field if $\psi_{L/K}$ is not bijective for any proper extension L/K of ordered fields (cf. [3], Definition 2.1).

DEFINITION 1.1. For an ordered field F, let K/F be an extension of ordered fields. When K is a maximal ordered field and $\psi_{K/F}$ is bijective, we say that K is a *maximal extension* of F.

For any ordered field F, there exists a maximal extension of F by [3], Theorem 3.3.

LEMMA 1.2. *Let F be a real closed field and K be a maximal extension of F. Let*

$F(x)/F$ be a simple transcendental extension of ordered fields such that $\psi_{F(x)/F}$ is bijective. Let G_K (resp. G_F) be the value group of the finest valuation of K (resp. F); we suppose that G_F is a subgroup of G_K. If $G_F = G_K$, then there exists $\alpha \in K \setminus F$ such that the isomorphism $F(x) \simeq F(\alpha) \subset K$, sending x to α, is an order preserving isomorphism.

PROOF. We put $A = \{a \in F;\ a < x \text{ in } F(x)\}$ and $B = \{b \in F;\ x < b \text{ in } F(x)\}$; then (A, B) is the cut of F which corresponds to the ordering σ of $F(x)$. We first show that there exists an element α of K satisfying $A < \alpha < B$.

If not, then (A_K, B_K) is a cut of K, where $A_K = \{a' \in K;\ a' \leqq a \text{ for some } a \in A\}$ and $B_K = \{b' \in K;\ b \leqq b' \text{ for some } b \in B\}$. Since K is real closed by [3], Proposition 2.3, we can take the ordering τ of the simple transcendental extension $K(x)/K$ which corresponds to the cut (A_K, B_K); it is easy to see that τ is an extension of σ (cf. [1], Theorem 1.2). Let $G_{K(x)}$ (resp. $G_{F(x)}$) be the value group of the finest valuation of $K(x)$ (resp. $F(x)$); here we suppose that $G_F = G_K \subset G_{F(x)} \subset G_{K(x)}$. The group G_K is isomorphic to a Hahn product $\mathrm{H}(\Gamma)$ for some totally ordered set Γ by [3], Proposition 2.4. Since $\psi_{F(x)/F}$ is bijective and G_F coincides with G_K, we see $G_F = G_{F(x)}$. On the other hand, the maximality of the ordered field K implies $G_K \neq G_{K(x)}$. Let v be the finest valuation with the value group $G_{K(x)}$. By applying the similar argument to the proof of [1], Lemma 2.2, we can show that there exists an element b of K such that $v(x-b) \in G_{K(x)} \setminus G_K$. We can also show that there exists $a \in F$ such that $0 < |x-a| < |x-b|$. In fact, if $x > b$, then $b \in A_K$ and $b < a$ for some $a \in A$. Therefore we have $0 < x - a < x - b$. If $x < b$, then we have $x - b < x - a < 0$ for some $a \in B$. Since v is compatible with τ and $v(x-b) \notin G_K = G_{F(x)}$, we have $v(x-a) > v(x-b)$. This shows that $v(x-b) = v(x-a-(x-b)) = v(b-a) \in G_K$, a contradiction. Therefore $A < \alpha < B$ for some $\alpha \in K$. It is clear that α is transcendental over F, and we can see that the F-isomorphism $F(x) \to F(\alpha)$, sending x to α, is an order preserving isomorphism by [1], Corollary 1.7, (1). Q.E.D.

A divisible ordered group can be embedded in the Hahn product determined by its skeleton (cf. [4], A, Théorème 2). However, the embedding map is not determined uniquely. For the convenience of readers, we first give the following lemma concerning the above embedding.

LEMMA 1.3. *Let $G \subset G'$ be divisible ordered groups and let ψ be the map from the set of convex subgroups of G' to that of G (cf. [3], §1). Suppose that ψ is bijective. Let (R_i) and (R_i'), $i \in \Gamma$ be skeletons of G and G' respectively. Then for a given embedding $G \to \mathrm{H}R_i$, we can find an embedding $G' \to \mathrm{H}R_i'$ so that the following diagram*

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{H}R_i \\ \downarrow & & \downarrow \\ G' & \longrightarrow & \mathrm{H}R_i' \end{array}$$

is commutative. (*The map* $\mathrm{H}R_i \to \mathrm{H}R'_i$ *is the embedding induced by the canonical embeddings* $R_i \to R'_i$.)

PROOF. For each $i \in \Gamma$, $R_i := H_i/H_i^*$ and $R'_i := H'_i/(H'_i)^*$ are isomorphic to subgroups of $\mathbf{R}$. Here H_i (resp. H'_i) is a non-zero principal convex subgroup of G (resp. G') and H_i^* (resp. $(H'_i)^*$) is a maximal convex subgroup properly contained in H_i (resp. H'_i) and $H'_i \cap G = H_i$, $(H'_i)^* \cap G = H_i^*$.

Now let V be a vector space over a field K and S be a non-empty set of subspaces of V. There exists a mapping γ from S to the set of subspaces of V satisfying the following properties (cf. [4], A, Lemme).

(1) $V = W \oplus \gamma(W)$ for any $W \in S$.

(2) If $W_1 \subset W_2$ in S, $\gamma(W_1) \supset \gamma(W_2)$.

The embedding $G \to \mathrm{H}R_i$ is given as follows. G is a vector space over $\mathbf{Q}$. Let S be the set of all convex subgroups of G and γ be a map defined on S which satisfies the above two properties. We denote by λ_i the composition of the projection $G = H_i \oplus \gamma(H_i) \to H_i$ and the canonical homomorphism $H_i \to H_i/(H_i)^* = R_i$. The embedding $G \to \mathrm{H}R_i$ is defined by sending $x \in G$ to $(\lambda_i(x))$, $i \in \Gamma$.

We put $S_1 := \{L \oplus \gamma(H); L \text{ is a convex subgroup of } G', H = L \cap G\}$. There exists γ_1 defined on S_1 which satisfies the properties (1) and (2). Let S' be the set of all convex subgroups of G'. For $L \in S'$ and $H = L \cap G \in S$, we have $G' = L \oplus \gamma(H) \oplus \gamma_1(L \oplus \gamma(H))$. We define γ' on S' by $\gamma'(L) = \gamma(H) \oplus \gamma_1(L \oplus \gamma(H))$. Then $G' = L \oplus \gamma'(L)$ for any $L \in S'$. For convex subgroups L and L' of G', we suppose $L \subset L'$. Then $H = L \cap G \subset H' = L' \cap G$, and $\gamma(H) \subset G \subset L' \oplus \gamma(H')$; therefore $L \oplus \gamma(H) \subset L' \oplus \gamma(H')$. Hence we have $\gamma'(L) \supset \gamma'(L')$. We have seen that γ' satisfies the properties (1) and (2). Clearly it also satisfies the following property (3).

(3) For any $L \in S'$, $\gamma'(L) \supset \gamma(H)$ where $H = L \cap G \in S$.

We now apply the embedding theorem to G' by using γ'. We put

$$\lambda'_i \colon G' = H'_i \oplus \gamma'(H'_i) \to H'_i \to H'_i/(H'_i)^* = R'_i.$$

By (3) the restriction of the projection $G' \to H'_i$ to G coincides with the projection $G \to H_i$, and so the following diagram commutes.

$$\begin{array}{ccccc} G & \longrightarrow & H_i & \longrightarrow & H_i/H_i^* = R_i \\ \downarrow & & \downarrow & & \downarrow \\ G' & \longrightarrow & H'_i & \longrightarrow & H'_i/(H'_i)^* = R'_i \end{array}$$

Thus the assertion is proved. Q.E.D.

THEOREM 1.4. *Let K_1 and K_2 be maximal extensions of ordered fields F_1 and F_2 respectively. If F_1 is isomorphic to F_2 as ordered fields, then there exists an order preserving isomorphism $K_1 \to K_2$ which is an extension of the isomorphism $F_1 \to F_2$.*

PROOF. It is sufficient to show that if $F_1 = F_2 = F$, then there exists an order preserving F-isomorphism $K_1 \to K_2$. We consider the set $S = \{(L, f);\ F \subset L \subset K_1, f$ is an order preserving F-injection $L \to K_2\}$. For (L_1, f_1), $(L_2, f_2) \in S$, we write $(L_1, f_1) \leqq (L_2, f_2)$ if $L_1 \subset L_2$ and $f_2|L_1 = f_1$. Then S is an inductive set. By Zorn's lemma, we can find a maximal element (L_0, f) of S. Let M_1 (resp. M_2) be the algebraic closure of L_0 (resp. $f(L_0)$) in K_1 (resp. K_2); then M_1 (resp. M_2) is a real closure of L_0 (resp. $f(L_0)$). Then, since f can be extended to $M_1 \to M_2$, we have $L_0 = M_1$; this implies that L_0 is real closed. Let v_1 (resp. v_2) be the finest valuation of K_1 (resp. K_2). We denote by G_1 (resp. G_2) the value group of v_1 (resp. v_2). Let G_0 be the value group of the finest valuation of L_0 and G_0' be that of $f(L_0)$. We may assume that $G_0 \subset G_1$ and $G_0' \subset G_2$.

First we show that $G_0 = G_1$. Suppose to the contrary that $G_0 \neq G_1$. Take $\alpha \in K_1$ so that $v_1(\alpha) \in G_1 \setminus G_0$ and $\alpha > 0$. Then we have a cut of G_0 determined by $v_1(\alpha)$. We claim that there exists an element $0 < \beta \in K_2 \setminus f(L_0)$ such that the cut of G_0' determined by $v_2(\beta)$ is the same as the cut by $v_1(\alpha)$ (here we identify G_0' with G_0). The claim is proved as follows.

For a totally ordered set Γ, we denote by $\mathrm{H}(\Gamma)$ the Hahn product $\mathrm{H}R_i'$, $i \in \Gamma$, where each R_i' is a copy of $\mathbf{R}$. By [3], Proposition 2.4, G_1 and G_2 are maximal ordered groups isomorphic to $\mathrm{H}(\Gamma)$ for some totally ordered set Γ. Take an embedding $G_0 \to \mathrm{H}R_i$, $i \in \Gamma$, where (R_i) is the skeleton of G_0. By Lemma 1.3, there exists an embedding $G_1 \to \mathrm{H}(\Gamma)$ such that the following diagram (a) commutes. Here note that the above embedding is an isomorphism because G_1 is maximal. Since $G_0 \simeq G_0'$, we can take the isomorphism $G_0' \to \mathrm{H}R_i$, which is the same as the above embedding $G_0 \to \mathrm{H}R_i$. Similarly there exists an isomorphism $G_2 \to \mathrm{H}(\Gamma)$ so that the following diagram (b) commutes.

$$
\text{(a)}\quad
\begin{array}{ccc}
G_0 & \longrightarrow & \mathrm{H}R_i \\
\downarrow & & \downarrow \\
G_1 & \longrightarrow & \mathrm{H}(\Gamma)
\end{array}
\qquad\qquad
\text{(b)}\quad
\begin{array}{ccc}
G_0' & \longrightarrow & \mathrm{H}R_i \\
\downarrow & & \downarrow \\
G_2 & \longrightarrow & \mathrm{H}(\Gamma)
\end{array}
$$

For the injections $j_i\colon R_i \to R_i' = \mathbf{R}$ which are used to determine $\mathrm{H}R_i \to \mathrm{H}(\Gamma)$ in (a) and $j_i'\colon R_i \to R_i' = \mathbf{R}$ which are used to determine $\mathrm{H}R_i \to \mathrm{H}(\Gamma)$ in (b), there exists an isomorphism $h\colon \mathbf{R} \to \mathbf{R}$ such that $hj_i' = j_i$. This shows that we may assume $j_i' = j_i$, and so the injection $\mathrm{H}R_i \to \mathrm{H}(\Gamma)$ in (a) coincides with the injection $\mathrm{H}R_i \to \mathrm{H}(\Gamma)$ in (b). Therefore we can take $v_2(\beta)$.

It is easy to show that α and β are transcendental over L_0 and $f(L_0)$, and they determine the same cuts of L_0 and $f(L_0)$ respectively. Thus the isomorphism $L_0(\alpha) \to f(L_0)(\beta)$, sending α to β, is order preserving, and so we have a contradiction.

Therefore we get $G_0 = G_1$, and so $G_0' = G_2$.

We now proceed to the proof of the statement $L_0 = K_1$. Suppose that $L_0 \neq K_1$. Then there exists an element $x \in K_1 \backslash L_0$. It is clear that x is transcendental over L_0 and $\psi_{L_0(x)/L_0}$ is bijective. Let (A, B) be the cut of L_0 corresponding to the ordering of $L_0(x) \subset K_1$. There is an ordering of $f(L_0)(y)$ which corresponds to the cut $(f(A), f(B))$ of $f(L_0)$ where y is a variable. Then the isomorphism $L_0(x) \to f(L_0)(y)$, sending x to y, is order preserving (cf. [1], Corollary 1.7). We may assume that $y \in K_2$ by Lemma 1.2. We can extend f to $L_0(x) \to f(L_0)(y)$, a contradiction. Therefore we conclude that $L_0 = K_1$ and $f(L_0) = K_2$. Q.E.D.

In Theorem 1.4, we showed that there exists an isomorphism $K_1 \to K_2$ which is an extension of the given isomorphism $F_1 \to F_2$. However such an extension is generally not unique; moreover there can be infinitely many extensions. We consider the case rank one in order to verify these situations.

EXAMPLE 1.5. Let $K = \mathbf{R}((x))^R$ and v be the canonical valuation of K. Then K is a maximal ordered field of rank 1 (cf. [2], Proposition 3.3), and moreover v is the finest valuation of K and the value group of v is $\mathbf{R}$. Let $\{g_i\}$, $i \in I$, be a basis of $\mathbf{R}$ as a vector space over $\mathbf{Q}$. For each g_i, we fix an element $\alpha_{g_i} \in K$ so that $v(\alpha_{g_i}) = g_i$ and $\alpha_{g_i} > 0$. Since K is real closed, $0 < (\alpha_{g_i})^{n/m}$ is determined uniquely for any integers m, n with $m > 0$. For a rational number $r = n/m \in Q$, we put $\alpha_{rg_i} = (\alpha_{g_i})^{n/m}$ and for a real number $s \in \mathbf{R}$, we put $\alpha_s = \prod \alpha_{r_i g_i}$ where $s = \sum r_i g_i$, $r_i \in \mathbf{Q}$. It is clear that $v(\alpha_s) = s$, and we can easily show that there exists an $\mathbf{R}$-isomorphism σ: $\mathbf{R}(x^s; s \in \mathbf{R}) \to \mathbf{R}(\alpha_s; s \in \mathbf{R})$, where $\sigma(x^s) = (\alpha_s)$. The map σ is an isomorphism as valued fields. It is clear that K is an immedaite extension of $\mathbf{R}(x^s; s \in \mathbf{R})$ and $\mathbf{R}(\alpha_s; s \in \mathbf{R})$ as valued fields. Since K is a maximal valued field, there exists an isomorphism σ': $K \to K$ which is an extension of σ (cf. [2], Proposition 3.2).

Now we write $1 = \sum r_i g_i$, $i = 1, \cdots, n$, $r_i \in \mathbf{Q}$, and we put $\alpha_{g_i} = x^{g_i}, i = 1, \cdots, n$. For $i \in I \backslash \{1, \cdots, n\}$, we fix an element $\alpha_{g_i} \in K$ so that $v(\alpha_{g_i}) = g_i$ and $\alpha_{g_i} > 0$. We have infinitely many such sets $\{\alpha_{g_i}\}$, $i \in I$. Then the automorphism σ': $K \to K$, determined by (α_{g_i}), $i \in I$, fix the element $x \in K$, and so the fixed field of σ' contains $\mathbf{R}(x)$. Thus there exist infinitely many $\mathbf{R}(x)$-automorphisms of K.

§2. Main theorem

In [1], we studied the theory of cuts of real closed fields under the assumption that a real closed field is of finite rank. We can get similar results for a real closed field of any rank.

Let F be a real closed field and let $\mathscr{C}(F) = \{A_i; i \in \triangle'(F)\}$ be the set of convex valuation rings of F, where the totally ordered set $\triangle'(F)$ has the initial element 0 (i.e. $A_0 = A(F, Q)$ and the final element λ (i.e. $A_\lambda = F$). Let X be the set of orderings of $F(x)$,

where $F(x)$ is a simple transcendental extension of F. Let C_F be the set of cuts of F. There exists a canonical bijection $g_F: X \to C_F$. We introduce an equivalence relation $\sim$; namely $\sigma \sim \tau$ in X if $(F(x), \sigma)$ is F-isomorphic to $(F(x), \tau)$ as ordered fields. Then $\sim$ induces an equivalence relation in C_F through the bijection g_F. Let X_1 be the set of orderings of $F(x)$ such that $\psi_{F(x)/F}$ is not bijective. For $\sigma \in X_1$, $\psi_{(F(x),\sigma)/F}$ is not bijective at some $j \in \triangle'(F)$ and we can define the map $N: X_1/\sim \to \triangle'(F)$ by $N(\sigma)=j$ (cf. [3], Proposition 1.2). By virtue of [1], Theorem 3.9, N is bijective. We write $X_1 = \bigcup Y_j$, $j \in \triangle'(F)$, where each Y_j is an equivalence class of X_1 satisfying $Y_j = N^{-1}(j)$.
Now for a cut $(C, D) \in C_F$, we put $M(C, D) := \{x \in F;\ \pm x \in C \text{ or } \pm x \in D\}$ and $\dot{M}(C, D) := M(C, D) \setminus \{0\}$. Let v_i be a valuation corresponding to A_i for $i \in \triangle'(F)$. It can be shown that the set $v_i(\dot{M}(C, D)) \cap v_i(\dot{F} \setminus \dot{M}(C, D))$ consists of at most one element (cf. [1], Proposition 3.2). For $i \in \triangle'(F) \setminus \{\lambda\}$, we put $T_i := \{(C, D)$, a proper cut of F; $v_i(\dot{M}(C, D)) \cap v_i(\dot{F} \setminus \mathring{M}(C, D)) = \phi$ and there exists $\min v_i(\dot{M}(C, D))$ or $\max v_i(\dot{F} \setminus \dot{M}(C, D))\}$. We also put $W_i := \{(C+a, D+a);\ (C, D) \in T_i,\ a \in F\}$ and denote by W_λ the set of non-proper cuts of F. (By [1], Proposition 1.4, a non-proper cut of F corresponds to an ordering of $F(x)$ such that F is not cofinal in $F(x)$.)

PROPOSITION 2.1. *For a real closed field F, the following statements hold.*

(1) *For any $j \in \triangle'(F)$, $g_F(Y_j) = W_j$.*

(2) *F is a maximal ordered field if and only if $C_F = \bigcup W_j$, $j \in \triangle'(F)$.*

The proof of Proposition 2.1 is similar to [1], Theorem 3.10 and Theorem 3.11, and so we omit it.

PROPOSITION 2.2 *Let K/F be an extension of ordered fields. Suppose that K is a maximal ordered field and for any L, $F \subsetneq L \subset K$, $\psi_{L/F}$ is not bijective. Then F is a maximal ordered field.*

PROOF. It is clear that F is real closed. We must show that $\psi_{(F(x),\sigma)/F}$ is not bijective for any $\sigma \in X$ (cf. [3], Proposition 2.3). For $\sigma \in X$, we put $(A, B) = g_F(\sigma)$. If (A, B) is a non-proper cut, F is not cofinal in $(F(x), \sigma)$. So $\psi_{(F(x),\sigma)/F}$ is not bijective at the final element $\lambda_F \in \triangle'(F)$. We now assume that (A, B) is a proper cut of F. We consider the case when $A < \alpha < \beta$ for some $\alpha \in K$. The element α is transcendental over F and the F-isomorphism $F(x) \to F(\alpha)$, sending x to α, is order preserving (cf. [1], Corollary 1.7, (1)). Since $\psi_{F(\alpha)/F}$ is not bijective by the assumption, so is $\psi_{(F(x),\sigma)/F}$. Suppose now that there exists no element $\alpha \in K$ such that $A < \alpha < B$. In this case, we have a cut (A_K, B_K) of K which is a unique extension of (A, B) similarly to the discussion in the proof of Lemma 1.2. Obviously (A_K, B_K) is a proper cut of K. The ordering τ of $K(x)$ corresponding to (A_K, B_K) is an extension of σ. By Proposition 2.1, (2), $(A_K, B_K) \in W_i$ for some $i \in \triangle'(K) \setminus \{\lambda_K\}$ and so $(A_K - \alpha, B_K - \alpha) \in T_i$ for some $\alpha \in K$. Let L be the algebraic closure of $F(\alpha)$ in K. Let v_i be the valuation with the valuation ring

A_i of K. We denote by w_j (resp. B_j) the restriction of v_i (resp. A_i) to L. We put $(C_K, D_K) := (A_K - \alpha, B_K - \alpha)$ and $(C_L, D_L) := (C_K \cap L, D_K \cap L)$. It follows from the fact $(C_K, D_K) \in T_i$ that $v_i(\dot{M}(C_K, D_K)) \cap v_i(\dot{K} \setminus \dot{M}(C_K, D_K)) = \phi$ and max $v_i(\dot{K} \setminus \dot{M}(C_K, D_K))$ or min $v_i(\dot{M}(C_K, D_K))$ exists. It is clear that $w_j(\dot{M}(C_L, D_L)) \cap w_j(\dot{L} \setminus \dot{M}(C_L, D_L)) = \phi$. Suppose $j = \lambda_L$. Then w_j is a trivial valuation of L and $w_j(\dot{L}) = \{0\}$. Therefore $w_j(\dot{M}(C_L, D_L)) = \phi$ and $w_j(\dot{L} \setminus \dot{M}(C_L, D_L)) = \{0\}$, or $w_j(\dot{M}(C_L, D_L)) = \{0\}$ and $w_j(\dot{L} \setminus \dot{M}(C_L, D_L)) = \phi$. The former case implies that max $C_L = 0$ or min $D_L = 0$ and the latter case implies that $(C_L, D_L) = (L, \phi)$ or (ϕ, L). In either case, we have a contradiction, since (C_L, D_L) is a proper cut. So we have $j \in \triangle'(L) \setminus \{\lambda_L\}$. Next we show $(C_L, D_L) \in T_j(L)$. We suppose, for example, that max $v_i(\dot{K} \setminus \dot{M}(C_K, D_K)) = g$ and $0 \in C_K$. Then there exists $y \in D_K$ such that $v_i(y) = g$, so we can write $y = b_K - \alpha$, $b_K \in B_K$. There exists $b \in B$ such that $b \leqq b_K$ and we put $z = b - \alpha$. Since $z \in L$, $0 < z \leqq y$ and $w_j(z) = g = \max w_j(\dot{L} \setminus \dot{M}(C_L, D_L))$, we have $(C_L, D_L) \in T_j(L)$. In other cases, we have the same conclusions. We now put $(A_L, B_L) := (A_K \cap L, B_K \cap L)$. Then $(A_L - \alpha, B_L - \alpha) = (C_L, D_L) \in T_j(L)$ and so $(A_L, B_L) \in W_j(L)$. It follows from Proposition 2.1, (1), $\psi_{L(x)/L}$ is not bijective. In the following diagram

$$\begin{array}{ccccccc} (F(x), \sigma) & \longrightarrow & F(\alpha, x) & \longrightarrow & L(x) & \longrightarrow & (K(x), \tau) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ F & \longrightarrow & F(\alpha) & \longrightarrow & L & \longrightarrow & K \end{array}$$

$\psi_{F(\alpha,x)/F(\alpha)}$ is not bijective since $L/F(\alpha)$ and $L(x)/F(\alpha, x)$ are algebraic extensions (cf. [2], Proposition 1.2). If $F \subsetneqq F(\alpha)$, then $\psi_{F(\alpha)/F}$ is not bijective by the assumption. So $\psi_{F(x)/F}$ is not bijective, since the transcendental degree of $F(\alpha, x)/F$ is two (cf. [3], Proposition 1.2). If $F(\alpha) = F$, then $\psi_{F(\alpha,x)/F(\alpha)} = \psi_{F(x)/F}$ and so $\psi_{F(x)/F}$ is not bijective.

Q.E.D.

COROLLARY 2.3. *Let K/F be an extension of ordered fields. Suppose that K is a maximal ordered field. Then there exists an intermediate field F', $F \subset F' \subset K$, which is a maximal extension of F.*

PROOF. We may assume that $\psi_{K/F}$ is not bijective. Put $S = \{F'; F \subset F' \subsetneqq K$ and $\psi_{F'/F}$ is bijective$\}$. It is easy to see that S is an inductive set; therefore there exists a maximal element F' of S by Zorn's lemma. Note that K/F' satisfies the assumption of Proposition 2.2, and so F' is a maximal ordered field and is a maximal extension of F.

Q.E.D.

The following Theorem 2.4 follows from Theorem 1.4 and Corollary 2.3.

THEOREM 2.4. *Let K_2/F be an extension of ordered fields and K_1 be a maximal extension of F. Suppose that K_2 is a maximal ordered field. Then there exists an F-embedding $K_1 \to K_2$.*

COROLLARY 2.5. *Let $\sigma: F \to L$ be an embedding of ordered fields. Let K_1 and K_2*

be maximal extensions of F and L respectively. Then there exists an order preserving embedding $K_1 \to K_2$ which is an extension of σ.

EXAMPLE 2.6. In [3], Example 1.4, we considered the extension $K/F := \mathbf{Q}(x_1, x_2, \cdots)/\mathbf{Q}(x_2, x_3, \cdots)$ of ordered fields. K/F is a proper extension for which rank K $=$ rank F but $\psi_{K/F}$ is not bijective.

Let L be a maximal ordered field. By the definition, there exists no proper extension M/L of ordered fields such that $\psi_{M/L}$ is bijective. However it is possible that there exists a proper extension M/L of ordered fields such that rank $M =$ rank L. We give such an example. Let L and M be maximal extensions of F and K respectively. There exists an F-embedding $L \to M$ by Corollary 2.5. Since $\psi_{M/L}$ is not bijective, the extension M/L is proper, and it is clear that rank $M =$ rank L

REMARK 2.7. Let B be an ordered field. Then the following statements are equivalent.

(a) B is a maximal ordered field.

(b) For any ordered field C for which rank $B =$ rank C, there is an order preserving injection $C \to B$ such that $\psi_{B/C}$ is bijective. The equivalence of (a) and (b) is clear by [3], Theorem 3.4.

We give an example which shows that the bijectivity of $\psi_{B/C}$ can not be dropped from the assumptions in (b). Similarly to Example 2.6, let L and M be maximal extensions of F and K respectively. By Corollary 2.5, We can regard L as a subfield of M. We put $B := L(x_1) \subset M$. Then rank $B =$ rank L and B is not maximal ordered field since B is not real closed. For any ordered field C for which rank $B =$ rank C, there is an order preserving injection $C \to B$ since $L\ (\subset B)$ is a maximal ordered field.

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