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Lie structures on differential algebras

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1. Introduction

Let L be a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of characteristic zero, A a commutative associative algebra over $\mathfrak{k}$ with an identity, and $L \subseteq A$. In this paper, we first extend the Lie structure on L to A by means of some derivations of A. After presenting examples of such Lie algebras and showing a way to give a Lie structure on a localization of A, we study the Lie structures on the formal power series ring and some factor algebras of polynomial algebras.

2. Notations and preliminaries

Poisson Lie structure (Berezin [1]): Let L be a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of characteristic zero and c_k^{ij} the structure constants with respect to a basis $\{x_1,\dots,x_n\}$ of L. Let $C^\infty(\boldsymbol{R}^n)$ be the set of all C^∞ function on $\boldsymbol{R}^n$. Then the Poisson Lie structures on $C^\infty(\boldsymbol{R}^n)$ is given by

$$[f, g] = \sum_{i,j,k} c_k^{ij} x_k (\partial f/\partial x_i)(\partial g/\partial x_j) \qquad \text{for} \quad f, g \in C^\infty(\boldsymbol{R}^n).$$

Let $U(L)$ be the universal enveloping algebra of L and $U_n(L)$ the vector space spanned by the products $y_1 \dots y_p$, where $y_1,\dots,y_p \in L$ and $p \leqslant n$. Let $S(L)$ be the symmetric algebra of the vector space L and $S^n(L)$ the set of elements of $S(L)$ which are homogeneous of degree n. By making use of the canonical mapping π_n of $U_n(L)$ onto $S^n(L)$, we can obtain a Lie structure on $S(L)$ as follows: Let $p \in S^m(L)$ and $q \in S^n(L)$, and take elements $\tilde{p} \in U_m(L)$ and $\tilde{q} \in U_n(L)$ such that $\pi_m(\tilde{p}) = p$ and $\pi_n(\tilde{q}) = q$. The Poisson bracket $[p, q]$ of p and q is defined to be $\pi_{m+n-1}([\tilde{p}, \tilde{q}])$ ([3, p. 97]). By a simple computation we can see that this Lie structure on $S(L)$ is the same as the Poisson Lie structure on the polynomial algebra $\mathfrak{k}[x_1,\dots, x_n]$.

Profinite Lie algebra (Christdoulou [2]): Let A_m ($m \in \boldsymbol{N}$) be a finite-dimensional Lie algebra and f_{mn} a homomorphism of A_m into A_n for $m \geqslant n$. Let A be the inverse limit $\varprojlim \{A_m; f_{mn}\}$. Then A is a profinite Lie algebra in the following sense: Let f_m be a canonical homomorphism of A onto A_m and $K_m = \mathrm{Ker} f_m$. Then the set $\{K_m: m \in \boldsymbol{N}\}$

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satisfies (i) A/K_m is a finite-dimensional Lie algebra for each $m \in \boldsymbol{N}$, (ii) $\bigcap_{m \in N} K_m = 0$, (iii) for each m, $n \in \boldsymbol{N}$ there exists $p \in \boldsymbol{N}$ such that $K_p \subseteq K_m \cap K_n$. If we give a topology on A taking as a closed subbase the set $\{x+U\colon x \in A, U \text{ is a subspace of } A \text{ such that } K_m \subseteq U \text{ for some } m \in N\}$, then A is compact with this topology. If each of A_m is nilpotent, solvable, then A is called to be pro-nilpotent, pro-solvable respectively.

3. Definition

Let L be a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of charcteristic zero with a basis $\{x_1, \ldots, x_n\}$. Let A be a commutative associative algebra over $\mathfrak{k}$ with an identity 1 and having derivations $d_1, \ldots, d_n$ satisfying the following conditions; for $i, j = 1, \ldots, n$,

$$L \subseteq A,\ d_i(x_j) = \delta_{ij},\ d_i d_j = d_j d_i.$$

For any elements a, $b \in A$, we define the product of them by

$$\begin{aligned}[a, b] &= \textstyle\sum_{i,j} [x_i, x_j] d_i(a) d_j(b) \\ &= \textstyle\sum_{i,j,k} c_k^{ij} x_k d_i(a) d_j(b)\end{aligned}$$

where c_k^{ij} are the structure constants with respect to a basis $\{x_1, \ldots, x_n\}$ of L. By a slightly longer computation we can see that A is a Lie algebra with this product. We denote this Lie algebra by

$$\mathrm{L}(L;\ A,\ \{d_i\}).$$

EXAMPLE 1. The Lie structure of $\mathrm{L}(L;\ C^\infty(\boldsymbol{R}^n),\ \{\partial/\partial x_i\})$ is the same as the Poisson Lie structure on $C^\infty(\boldsymbol{R}^n)$.

EXAMPLE 2. Let A be a commutative associative algebra over $\mathfrak{k}$ with an identity 1 and have derivations $d_1, \ldots, d_n$ satisfying the conditions; $L \subseteq A$, $d_i(x_j) = \delta_{ij}$, $d_i d_j = d_j d_i$ $(i, j = 1, \ldots, n)$. Let B be a commutative associative algebra and d a derivation of B. Consider the tensor product $A \otimes_k B$ of the associative algebras A and B over $\mathfrak{k}$. Let us define the derivations $D_1, \ldots, D_n$ of $A \otimes_k B$ by

$$D_i = d_i \otimes 1_B + 1_A \otimes d \quad (i = 1, \ldots, n).$$

Then we consider a Lie algebra $\mathrm{L}(L;\ A \otimes_k B,\ \{D_i\})$. For elements $a \otimes b$, $e \otimes f$ of this Lie algebra, the product of them is given by

$$[a \otimes b,\ e \otimes f] = [a,\ e] \otimes bf + [a,\ x] e \otimes b d(f) - a[e,\ x] \otimes d(b) f,$$

where $x = \sum_{i=1}^n x_i$, and the products $[a, e]$, $[a, x]$ and $[e, x]$ are calculated in $\mathrm{L}(L;\ A,\ \{d_i\})$.

REMARK. For a Lie algebra $L(L; A, \{d_i\})$, we can see that a Lie structure on the associative subalgebra L_0 generated associatively by L is independent of a choice of a basis of L. Let $\{y_1,\dots, y_n\}$ be another basis of L and $v_1,\dots, v_n$ derivations such that $v_i(y_j)=\delta_{ij}$, $v_iv_j=v_jv_i$ $(i, j=1, \dots, n)$. Set $y_i=\sum_{p=1}^{n} a_{ip}x_p$, $x_i=\sum_{q=1}^{n} b_{iq}y_q$ $(a_{ip}, b_{iq}\in \mathfrak{k})$. It is easy to see that $v_i|_L=(\sum_{s=1}^{n} b_{si}d_s)|_L$. Hence we have $v_i=\sum_{s=1}^{n} b_{si}d_s$ on L_0. Therefore we have, for $a, b\in L_0$,

$$\begin{aligned}
&\sum_{i,j}[y_i, y_j]v_i(a)v_j(b)\\
&=\sum_{i,j,p,q,s,t}a_{ip}a_{jq}b_{si}b_{tj}[x_p, x_q]d_s(a)d_t(b)\\
&=\sum_{p,q}[x_p, x_q]\{\sum_{j,s,t}(\sum_i b_{si}a_{ip})b_{tj}a_{jq}d_s(a)d_t(b)\}\\
&=\sum_{p,q}[x_p, x_q]\ (\sum_{j,t}b_{tj}a_{jt}d_p(a)d_t(b))\\
&=\sum_{p,q}[x_p, x_q]d_p(a)d_q(b).
\end{aligned}$$

4. Localization

Let A be a commutative associative algebra over a field $\mathfrak{k}$ of characteristic zero with an identity 1. Assume that A is an integral domain and has a Lie structure whose Lie product [,] satisfies the condition: $[ab, c]=[a, c]b+a[b, c]$ $(a, b, c\in A)$. Considering A as an associative algebra, we take a multiplicatively closed subset S of A containing 1, and denote a localization of A by $S^{-1}A$. We can extend a Lie structure on A to $S^{-1}A$ as follow.

PROPOSITION 1. *Let A, S be given above. Then the localization $S^{-1}A$ is a Lie algebra with the product*

$$[f/s, g/t]=([f, g]st+[g, s]tf+[s, t]fg+[t, f]gs)/(s^2t^2)$$

$(f, g\in A,\ s, t\in S)$.

PROOF. We verify that this rule gives the same result for $[fu/su, g/t]$ $(u\in S)$. By a slightly longer computation we can see that the Jacobi identity holds. Q.E.D.

Let $G=L(L; A, \{d_i\})$ and S a multiplicatively closed subset of an associative algebra A containing 1. Assume that A is an integral domain. We extend the derivations d_i to the localization $S^{-1}A$ by

$$D_i(f/s)=(d_i(f)s-fd_i(s))/s^2$$

where $f\in A$, $s\in S$ (Kaplansky [4; Theorem 1.1]). Let $L^*=L(L; S^{-1}A, \{D_i\})$, and f/s, g/t any two elements of L^*. Then the product of them is

$$[f/s, g/t]=\sum_{i,j}[x_i, x_j]D_i(f/s)D_j(g/t)$$

$$= \sum_{i,j}[x_i, x_j] ((d_i(f)s - fd_i(s))/s^2) ((d_j(g)t - gd_j(t))/t^2)$$
$$= ([f, g]st - [s, g]ft - [f, t]gs + [s, t]fg)/(s^2t^2).$$

Therefore the Lie structure of L^* is the same as that of the localization $S^{-1}A$ given in Proposition 1.

5. Lie structures of $\mathrm{L}(L; \mathfrak{k}[[x_1,\dots,x_n]], \{\partial/\partial x_i\})$

Let L be a Lie algebra over $\mathfrak{k}$ with a basis $\{x_1,\dots,x_n\}$ and G the Lie algebra $\mathrm{L}(L; \mathfrak{k}[[x_1,\cdots,x_n]], \{\partial/\partial x_i\})$, where $\mathfrak{k}[[x_1,\dots, x_n]]$ is the formal power series ring. For $m\in N$, let K_m be the ideal of G spanned by $\{\sum_{k_1\cdots k_n}\mathfrak{k}x_1^{k_1}\cdots x_n^{k_n}: k_1+\cdots+k_n\geqslant m\}$, G_m a finite-dimensional Lie algebra G/K_m and π_{mn} the canonical homomorphism of G_m onto G_n $(m\geqslant n)$. Then G is isomorphic to the inverse limit $\varprojlim\{G_m; \pi_{mn}\}$, in other words, G is a profinite Lie algebra.

If L is nilpotent or solvable, then a structure of G_m is deduced as follows.

LEMMA 2. *If L is nilpotent, then G_m is nilpotent for any $m\in N$.*

PROOF. Since $[K_q, {}_pK_2]\subseteq K_{q+p}$, we have $[G, {}_{m-1}K_2]\subseteq K_m$ and so $[G_m, {}_{m-1}(K_2+K)/K_m]=0$. On the other hand, since L is nilpotent, there exists an integer n such that $[G_{m,n}(L+K_m)/K_m]=0$. Therefore $(G_m)^r=0$ for $r=mn+1$. Q.E.D.

By induction on n, we see $K_2^{(n)}\subseteq K_{2n+1}$. From this we immediately have

LEMMA 3. *If L is solvable, then G_m is solvable for any $m\in N$.*

Summing up these results we have

PROPOSITION 4. *Let G be a Lie algebra given above. Then G is a profinite Lie algebra and if L is nilpotent or solvable, then G is pro-nilpotent or pro-solvable respectively.*

6. Lie structures on $\mathfrak{k}[x_1,\dots, x_n, y]/(y^2-2\alpha y+\beta)$

Let L be a finite-dimensional Lie algebra over $\mathfrak{k}$ with a basis $\{x_1,\dots,x_n\}$ and R be the polynomial algebra $\mathfrak{k}[x_1,\dots, x_n]$. Consider the polynomial algebra $R[y]$ over R and take a polynomial $T(y)=y^2-2\alpha y+\beta$ $(\alpha, \beta\in R)$. Let $A(T(y))$ be a commutative associative factor algebra $R[y]/(T(y))$, where $(T(y))$ is the ideal of $R[y]$ generated by $T(y)$.

We first extend a derivation $\partial/\partial x_i$ on R to $A(T(y))$.

LEMMA 5. *There exist derivations* $d_1,\dots, d_n$ *of* $A(T(y))$ *such that* $d_id_j = d_jd_i$ *and* $d_i|_R = \partial/\partial x_i$ $(i, j = 1,\dots, n)$ *if and only if* $\beta - \alpha^2 \in \mathfrak{k}$ *and there exists an element* b *of* R *such that for* $i = 1,\dots, n$,

$$d_i(y) = d_i(\alpha) - \alpha d_i(b) + d_i(b)y$$

$$= (\partial\alpha/\partial x_i) - \alpha(\partial b/\partial x_i) + (\partial b/\partial x_i)y.$$

PROOF. Let $d_1,\dots, d_n$ be the derivations of $A(T(y))$ satisfying the conditions given above. We set $d_i(y) = a_i + b_iy$ $(a_i, b_i \in R)$. Since $d_i(T(y)) = 0$ and $d_id_j(y) = d_jd_i(y)$ in $A(T(y))$ $(i, j = 1,\dots, n)$, we have the equivalent conditions; for $i, j = 1,\dots, n$,

$$(1)\quad a_i + \alpha b_i = d_i(\alpha), \qquad (2)\quad 2\alpha a_i + 2\beta b_i = d_i(\beta),$$

$$(3)\quad d_i(b_j) = d_j(b_i), \qquad (4)\quad d_i(a_j) + a_ib_j = d_j(a_i) + a_jb_i.$$

If $\beta - \alpha^2 \neq 0$, then by (1), (2) we have $2(\alpha^2 - \beta)b_i = d_i(\alpha^2 - \beta)$. Therefore $b_i = 0$ $(i = 1,\dots, n)$ and $\beta - \alpha^2 \in \mathfrak{k}$. In this case a derivation d_i, defined by $d_i(y) = d_i(\alpha)$, satisfies all conditions given in the proposition.

Assume that $\beta = \alpha^2$. Then by (3) there exists an element b in R such that, for $i = 1,\dots, n$,

$$b_i = d_i(b) = \partial b/\partial x_i.$$

Hence by (1), $a_i = d_i(\alpha) - \alpha d_i(b)$. In this case the derivation d_i, defined by $d_i(y) = d_i(\alpha) - \alpha d_i(b) + d_i(b)y$, satisfies all conditions given above.

LEMMA 6. *Let* A, B *be commutative associative algebras over* $\mathfrak{k}$ *and* ϕ *be an associative isomorphism of* A *onto* B. *Assume that* A *contains a Lie algebra* L *and derivations* $d_1,\dots, d_n$ of A *and derivations* $D_1,\dots, D_n$ *of* B *satisfy the conditions*; $d_id_j = d_jd_i$, $D_iD_j = D_jD_i$, $d_i(x_j) = \delta_{ij}$, $D_i\phi = \phi d_i$ $(i, j = 1,\dots, n)$. *Then the map* ϕ *is a Lie isomorphism of a Lie algebra* $\mathrm{L}(L; A, \{d_i\})$ *onto a Lie lgebra* $\mathrm{L}(\phi(L); B, \{D_i\})$.

We denote by $d_i^{\alpha,b}(y) = d_i(\alpha) - \alpha d_i(b) + d_i(b)y$. Now we set about proving the following results.

THEOREM 7. *Let* $c \in \mathfrak{k}$. *Every Lie algebra* $\mathrm{L}(L; A((y-\alpha)^2 + c), \{d_i^{\alpha,b}\})$ *is isomorphic to the Lie algebra* $\mathrm{L}(L; A(y^2 + c), \{d_i^{0,b}\})$, *where* b *is taken as* 0 *in the case that* $c \neq 0$. *The Lie product of any two elements of the Lie algebra* $\mathrm{L}(L; A(y^2 + c), \{d_i^{0,b}\})$ *is given by, for* $p, q, s, t \in R$,

$$[p + qy, s + ty] = [p, s] - c[q, t] + ([q, s] + [p, t])y \qquad (c \neq 0),$$

$$[p + qy, s + ty] = [p, s] + ([p, t] + [q, s] + q[b, s] + [p, b])y \qquad (c = 0).$$

PROOF. Let ϕ be a linear map of $A(y^2 + c)$ onto $A((y-\alpha)^2 + c)$ defined by

$\phi(p+qy)=p+q(y-\alpha)$ $(p,\ q\in R)$. Then the map ϕ makes sense. By simple computation we can see that $d_i^{\alpha,b}\phi=\phi d_i^{0,b}(y)$ $(i=1,\dots,n)$. Therefore the map ϕ is a differential isomorphism of $A(y^2+c)$ onto $A((y-\alpha)^2+c)$. Hence by Lemma 6 and the remark given in the section of Definition we have the first assertion.

The second assertion follows from the formula

$$\begin{aligned}[r,\ y] &= \textstyle\sum_{i,j}[x_i,\ x_j]d_i^{0,b}(r)d_j^{0,b}(y)\\ &= \textstyle\sum_j[r,\ x_j]d_j(b)y\\ &= [r,\ b]y \qquad (r\in R).\end{aligned}$$

Q.E.D.

COROLLARY 8. *The Lie algebra* $\mathrm{L}(L;\ A(y^2),\ \{d_i^{0,b}\})$ *is a split extension* $Ry\dotplus_d R$ *of the abelian Lie algebra* Ry *by* R, *where* $d_r(qy)=([q,\ r]+q[b,\ r])y$ $(q\in R)$. *If* $c\neq 0$, *then the Lie algebra* $\mathrm{L}(L;\ A(y^2+c),\ \{d_i^{0,0}\})$ *is isomorphic to the Lie algebra* $R\times R$ *with* $[(p,\ 0),\ (0,\ t)]=([p,\ t],\ 0)$, $[(p,\ 0),\ (t,\ 0)]=(0,\ -c[p,\ t])$ $(p,\ t\in R)$.

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