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Reductions of graded rings and pseudo-flat graded modules

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Introduction

The aim of this paper is to develop a theory of *reductions* for graded rings and to study graded modules using this theory. In particular, we introduce a certain class of graded modules which we call *pseudo-flat* graded modules and examine some of their properties. Our theory of reductions of graded rings is a natural generalization of the theory of reductions of ideals due to Northcott and Rees [16], and the techniques used are similar to the ones in the case of ideals. But the viewpoint of general graded rings greatly clarifies the situations and is useful even in the case of ideals.

In §1 and §2, we define the *analytic spread* and the pseudo-flatness of graded modules, and prove some elementary facts about them.

In §3, we introduce the notion of reductions of homogeneous graded rings with respect to finitely generated graded modules. Then we prove a fundamental theorem in the theory of reductions, namely, the existence of minimal reductions and the characterization of minimal reductions by the analytic spread (cf. Theorem 3.3). By this theorem, we can give the structure theorem for pseudo-flat graded modules (cf. Theorem 3.4).

In §4, using this structure theorem, we examine some properties of pseudo-flat graded modules.

In §5, making use of minimal reductions, we introduce a numerical invariant of a graded module which we call the *reduction exponent*, and study properties of graded modules by this invariant. Especially, we compare the reduction exponent with *Castelnuovo's regularity* which the author introduced in [17].

Notation and terminology: Throughout this paper, all rings are commutative noetherian rings. Any graded ring $A=\oplus_{n\in Z}A_n$ is positively graded (i.e., $A_n=0$ for all $n<0$), and is generated over $A_0=R$ by elements of degree one. Then we say that A is a *homogeneous* R-algebra. We put $A_+=\oplus_{n>0}A_n$. Let R be a ring, I an ideal of R and E an R-module. $\mathrm{Min}_R(E)$ denotes the set of minimal elements in $\mathrm{Supp}_R(E)$. $\mu(E)$ denotes the smallest number of generators of E. For a homogeneous R-algebra A, put $\mathrm{emb}(A)=\mu(A_1)$ (the *embedding dimension* of A). If A is a homogeneous algebra over a field and M is a finitely generated graded A-module, then $\mathrm{e}(M)$ denotes the multiplicity of M. We put $R(I, E)=\oplus_{n\geqq 0}I^nE$, $G(I, E)=\oplus_{n\geqq 0}I^nE/I^{n+1}E$, $R(I)=R(I,$

R) and $G(I)=G(I, R)$. $S_R(E)$ denotes the symmetric algebra of E. If the residue field of the local ring $R_{\mathfrak{p}}$ is infinite for any $\mathfrak{p}\in\mathrm{Spec}(R)$, then we say that all residue fields of R are infinite. Note that for any R, the ring $R(X)$ has this property.

§1. Analytic spread of graded modules

Throughout this paper, $A=\oplus_{n\geqq 0}A_n$ denotes a homogeneous algebra over a ring R and $M=\oplus_{n\in Z}M_n$ is a finitely generated graded A-module. A mapping f defined on $\boldsymbol{Z}$ is said to be *stable* if $f(n)$ is constant for all sufficiently large n.

PROPOSITION 1.1. *The invariants* $\mathrm{ann}_R(M_n)$, $\mathrm{Supp}_R(M_n)$, $\dim_R(M_n)$, $\mathrm{grade}_R(M_n)$ *and* $\mathrm{Ass}_R(M_n)$ *are stable. When R is local,* $\mathrm{depth}_I(M_n)$, $\mathrm{hd}_R(M_n)$ *and* $\mathrm{id}_R(M_n)$ *are also stable, where I is an ideal of R.* ($\mathrm{hd}_R(M_n)$ *and* $\mathrm{id}_R(M_n)$ *denote the projective dimension and the injective dimension of* M_n *respectively.*)

PROOF. Since M is finitely generated, we have $A_1M_n=M_{n+1}$ for all $n\gg 0$. Hence $\mathrm{ann}_R(M_n)\subset\mathrm{ann}_R(A_1M_n)=\mathrm{ann}_R(M_{n+1})$ for all $n\gg 0$. This implies that $\mathrm{ann}_R(M_n)$ is stable, and the assertions for $\mathrm{Supp}_R(M_n)$, $\dim_R(M_n)$ and $\mathrm{grade}_R(M_n)$ follow from this. For $\mathrm{Ass}_R(M_n)$, see [15]. (We have $\mathrm{Ass}_R(M_n)=\{\mathfrak{P}\cap R\,|\,\mathfrak{P}\in\mathrm{Ass}_A(M), \mathfrak{P}\not\supset A_+\}$ for all $n\gg 0$.) We show the stability of $\mathrm{depth}_I(M_n)$ by induction on $d=\dim_R(M_n)$ for all $n\gg 0$. Put $Z=Z_R(M_n)$, the set of zero-divisors of M_n, for all $n\gg 0$. If $I\subset Z$ (the case $d=0$ is included in this case), then $\mathrm{depth}_I(M_n)=0$ for all $n\gg 0$. If $I\not\subset Z$, i.e., there is an element $a\in I$ which is M_n-regular for all $n\gg 0$, then using the induction hypothesis on M/aM, we get our assertion. The stability of $\mathrm{hd}_R(M_n)$: If $\mathrm{hd}_R(M_n)<\infty$ for all $n\gg 0$, then $\mathrm{hd}_R(M_n)=\mathrm{depth}(R)-\mathrm{depth}_R(M_n)$ is stable. Otherwise, for infinitely many $n\geqq 0$, we have $\mathrm{hd}_R(M_n)=\infty$, i.e., $\mathrm{Tor}_r^R(M_n, k)\neq 0$, $r=\mathrm{depth}(R)+1$ where k is the residue field of R. Since $\mathrm{Tor}_r^R(M, k)=\oplus_n\mathrm{Tor}_r^R(M_n, k)$ is a finitely generated graded A-module, we have $\mathrm{Tor}_r^R(M_n, k)\neq 0$, i.e., $\mathrm{hd}_R(M_n)=\infty$ for all $n\gg 0$. The stability of $\mathrm{id}_R(M_n)$: We may assume that $\dim_R(M_n)=d$ (constant) for all $n\geqq 0$. If $\mathrm{id}_R(M_n)=\infty$ for all $n\gg 0$, the assertion is clear. Otherwise, for infinitely many $n\geqq 0$, we have $\mathrm{id}_R(M_n)<\infty$, or equivalently, $\mathrm{Ext}_R^i(k, M_n)=0$ for all i such that $r\leqq i\leqq r+d$, $r=\mathrm{depth}(R)+1$. Since $\mathrm{Ext}_R^i(k, M)=\oplus_n\mathrm{Ext}_R^i(k, M_n)$ is a finitely generated graded A-module, for all $n\gg 0$, we have $\mathrm{Ext}_R^i(k, M_n)=0$, $r\leqq i\leqq r+d$, i.e., $\mathrm{id}_R(M_n)<\infty$. Therefore $\mathrm{id}_R(M_n)=\mathrm{depth}(R)$ for all $n\gg 0$. Q.E.D.

Let $\boldsymbol{P}$ be a property for finitely generated R-modules. Then M is said to be *asymptotically* $\boldsymbol{P}$ if M_n is $\boldsymbol{P}$ for all $n\gg 0$. $\boldsymbol{P}$ is said to be an *asymptotic property* if the following condition holds: For any A and M, if M_n is $\boldsymbol{P}$ for infinitely many $n\geqq 0$, then M is asymptotically $\boldsymbol{P}$.

COROLLARY 1.2. *The following properties are asymptotic: a zero module, a faithful module, a torsion module. When R is local, the following properties are also* *asymptotic: a free module, an injective module, a Cohen-Macaulay module, a perfect module, a Gorenstein module.*

PROOF. The assertions follow from Proposition 1.1, because the above properties are characterized as follows: $\mathrm{ann}_R(E)=R$, $\mathrm{ann}_R(E)=0$, $\mathrm{grade}_R(E)>0$, $\mathrm{hd}_R(E)=0$, $\mathrm{id}_R(E)=0$, $\dim_R(E)=\mathrm{depth}_R(E)$, $\mathrm{hd}_R(E)=\mathrm{grade}_R(E)$, $\mathrm{id}_R(E)=\mathrm{depth}_R(E)$. Q.E.D.

In the rest of this section, we assume that $(R, \mathfrak{m}, k)$ is a local ring. We define the *analytic spread* $\ell(M)$ of M by $\ell(M)=\dim_{A\otimes_R k}(M\otimes_R k)$, the Krull dimension of the $A\otimes_R k$-module $M\otimes_R k$. If I is an ideal of R, then $\ell(R(I))=\ell(G(I))$ coincides with the analytic spread $\ell(I)$ of I introduced by Northcott and Rees [16]. We have $\ell(M)=\ell(A/\mathrm{ann}_A(M))$. If $\dim_R(M)>0$ and $\mathfrak{P}\cap R\in\mathrm{Min}_R(M)$ for any $\mathfrak{P}\in\mathrm{Min}_A(M)$, then $\ell(M)\leqq\dim_A(M)-1$. As the example $A=M=R[X]/(\mathfrak{m}X)$ with R a DVR shows, the second condition cannot be deleted. We have $\dim(A)=\mathrm{ht}(\mathfrak{M})$, where $\mathfrak{M}=\mathfrak{m}\oplus A_+$ (cf. [7]). To state the following proposition, we recall the definitions of some invariants of ideals (cf. [18]). For an ideal I of a (not necessarily graded) ring A, put $\mathrm{alt}(I)=\max\{\mathrm{ht}(\mathfrak{p})|\mathfrak{p}$ is a minimal prime ideal of $I\}$, $\mathrm{cora}(I)=\max\{n|H_I^n(A)\neq 0\}$, and $\mathrm{ara}(I)=\min\{n|\mathrm{rad}(I)=\mathrm{rad}(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n\in I\}$ (when A is a graded ring and I is a homogeneous ideal, we assume that each a_i is a homogeneous element). If M is a finitely generated A-module, then we write $\mathrm{ht}(I, M)$, $\mathrm{alt}(I, M)$, $\mathrm{cora}(I, M)$ and $\mathrm{ara}(I, M)$ instead of $\mathrm{ht}(J)$, $\mathrm{alt}(J)$, $\mathrm{cora}(J)$ and $\mathrm{ara}(J)$ respectively, where $J=(I+\mathrm{ann}_A(M))/\mathrm{ann}_A(M)$.

PROPOSITION 1.3. (1) $\mathrm{alt}(A_+, M)\leqq\mathrm{cora}(A_+, M)\leqq\mathrm{ara}(A_+, M)\leqq\ell(M)\leqq\mathrm{emb}(A)$ *and* $\mathrm{ht}(A_+, M)\leqq\dim_A(M)-\dim_R(M)\leqq\ell(M)\leqq\dim_A(M)$.

(2) *The function* $\mathfrak{p}\mapsto\ell(M_\mathfrak{p})=\dim(M\otimes_R k(\mathfrak{p}))$ *defined on* $\mathrm{Spec}(R)$ *is upper semicontinuous.*

(3) $\ell(M)=\mathrm{ht}(A_+,M)$ *if and only if* $\dim(M\otimes_R k(\mathfrak{p}))$ *is constant for all* $\mathfrak{p}\in\mathrm{Supp}_R(M)$.

(4) $\dim_A(M)-\dim_R(M_n)\leqq\ell(M)\leqq\dim_A(M)-\mathrm{depth}_R(M_n)$ *for all* $n\gg 0$ *(cf.* [2]).

PROOF. (1) We may assume that $M=A$. The inequalities $\ell(A)\leqq\mathrm{emb}(A)$, $\ell(A)\leqq\dim(A)$, $\mathrm{ht}(A_+)\leqq\dim(A)-\dim(R)$ are clear. For the inequalities $\mathrm{alt}(A_+)\leqq\mathrm{cora}(A_+)\leqq\mathrm{ara}(A_+)$, see [18]. We show the inequality $\mathrm{ara}(A_+)\leqq\ell(A)$. If R is a field and $\dim(A)=d$, then by Noether's normalization theorem, there exist homogeneous elements $a_1,\ldots,a_d$ in A such that A is integral over $R[a_1,\ldots,a_d]$. It is easy to see that $\mathrm{rad}(A_+)=\mathrm{rad}(a_1,\ldots,a_d)$. General case: Put $\bar{A}=A/\mathfrak{m}A$ and $\ell=\ell(A)$. Then there exist homogeneous elements $a_1,\ldots,a_\ell$ in A such that $\mathrm{rad}(\bar{A}_+)=\mathrm{rad}(\bar{a}_1,\ldots,\bar{a}_\ell)$. Hence $\bar{A}_+{}^n\subset(\bar{a}_1,\ldots,\bar{a}_\ell)$ for some n and this implies that $A_+{}^n\subset(a_1,\ldots,a_\ell)+\mathfrak{m}A$, i.e., $A_+{}^n\subset(a_1,\ldots,a_\ell)+\mathfrak{m}A_+{}^n$. By Nakayama's lemma, we have $A_+{}^n$

$\subset(a_1,\ldots,a_\ell)$. Hence $\mathrm{rad}(A_+)=\mathrm{rad}(a_1,\ldots,a_\ell)$. Next we prove the inequality $\dim(A)-\dim(R)\leqq \ell(A)$. Put $\mathfrak{M}=\mathfrak{m}\oplus A_+$ and apply [14], Theorem 13.B to the local homomorphism $R\to A_{\mathfrak{M}}$. Then we get $\dim(A)=\mathrm{ht}(\mathfrak{M})\leqq\dim(R)+\dim(A_{\mathfrak{M}}\otimes_R k)=\dim(R)+\ell(A)$. (2) follows from [9], 28, (13.1.5). (3) follows from the equality $\mathrm{ht}(A_+, M)=\min\{\dim(M\otimes_R k(\mathfrak{p}))|\mathfrak{p}\in\mathrm{Supp}_R(M)\}$. (4) Since $\ell(M)=\ell(M_{\geqq n})$, where $M_{\geqq n}=\oplus_{m\geqq n}M_m$, we may assume that $\dim_R(M_n)=\dim_R(M)$ for all $n\gg 0$. Then (1) implies the first inequality. We prove the second inequality by induction on $r=\mathrm{depth}_R(M_n)$ for all $n\gg 0$. If $r=0$, then the assertion is clear. If $r>0$, then there exists $a\in\mathfrak{m}$ such that a is M_n-regular for all $n\gg 0$. Therefore $\mathrm{depth}_R(M_n/aM_n)=\mathrm{depth}_R(M_n)-1$ for all $n\gg 0$, and by the induction hypothesis (note that $M/aM\otimes_R k=M\otimes_R k$), we get $\ell(M)=\ell(M/aM)\leqq\dim_A(M/aM)-\mathrm{depth}_R(M_n/aM_n)=\dim_A(M)-\mathrm{depth}_R(M_n)$ for all $n\gg 0$. Q.E.D.

For an ideal I of R, we have the following (in)equalities: $\mathrm{ht}(I)=\mathrm{ht}(G(I)_+)$, $\mathrm{alt}(I)=\mathrm{alt}(G(I)_+)$, $\mathrm{ara}(I)\leqq\mathrm{ara}(G(I)_+)$, $\mathrm{ht}(I)\leqq\dim(R)-\dim(R/I)\leqq\ell(I)$ and $\mathrm{alt}(I)\leqq\mathrm{cora}(I)\leqq\mathrm{ara}(I)\leqq\ell(I)\leqq\mu(I)$

§2. Pseudo-flat graded modules

Let A be a homogeneous algebra over a ring R and M a finitely generated graded A-module. We say that M is *pseudo-flat* if $\dim_{A\otimes_R k(\mathfrak{p})}(M\otimes_R k(\mathfrak{p}))$ is constant for all $\mathfrak{p}\in\mathrm{Spec}(R)$. M is said to be *locally pseudo-flat* if $M_\mathfrak{p}$ is a pseudo-flat $A_\mathfrak{p}$-module for all $\mathfrak{p}\in\mathrm{Spec}(R)$.

Let f: $R\to R'$ be a ring homomorphism. If M is (locally) pseudo-flat, then so is the $A\otimes_R R'$-module $M\otimes_R R'$, and the converse holds if ${}^a f$: $\mathrm{Spec}(R')\to\mathrm{Spec}(R)$ is surjective. If M is pseudo-flat, then M is locally pseudo-flat, and the converse holds if $\mathrm{Spec}(R)$ is connected (cf. [23], Corollary 3.7). If R is local and $M\neq 0$, then by Proposition 1.3, (3), M is pseudo-flat if and only if $\ell(M)=\mathrm{ht}(A_+, M)$ and $\mathrm{Supp}_R(M)=\mathrm{Spec}(R)$.

EXAMPLE 2.1. (1) If M_n is a flat R-module for all $n\gg 0$ and $\mathrm{Spec}(R)$ is connected, then M is pseudo-flat and $\mathrm{e}(M\otimes_R k(\mathfrak{p}))$ is constant for all $\mathfrak{p}\in\mathrm{Spec}(R)$, because the Hilbert polynomial of $M\otimes_R k(\mathfrak{p})$ is constant for all $\mathfrak{p}\in\mathrm{Spec}(R)$. The example $A=R[X, Y, Z]/(\mathfrak{m}X^2, XY, XZ)$ with $(R, \mathfrak{m})$ a DVR shows that the converse is not true.

(2) If I is an ideal of a local ring R, then $G(I)$ is a pseudo-flat R/I-algebra if and only if $\ell(I)=\mathrm{ht}(I)$, namely, R is *normally pseudo-flat* along I in the sense of [12] (see also [10], [11], [13]).

(3) For a finitely generated R-module E, $S_R(E)$ is locally pseudo-flat if and only if $E\otimes_R R_{\mathrm{red}}$ is a flat R_{red}-module. If $(R, \mathfrak{m})$ is a local ring, then $S_R(\mathfrak{m})$ is pseudo-flat if and only if R is artinian or a DVR.

(4) For an ideal I of an equidimensional local ring R, $R(I)$ is pseudo-flat if and

only if I is nilpotent or $\ell(I)=\mathrm{ht}(I)=1$. $R(\mathfrak{m})$ is pseudo-flat if and only if $\dim(R)\leqq 1$.

(5) *If* R is local, $\dim_A(M)=\dim_R(M)+\mathrm{ht}(A_+, M)$ and $\mathrm{depth}(M_n)=\dim_R(M)$ for all $n\gg 0$, then M is pseudo-flat by Proposition 1.3, (4).

(6) Let A be a normal homogeneous domain with torsion class group. Then each A_n is a reflexive R-module (cf. [6], Proposition 6.8, Theorem 10.8). Hence, if $\dim(R)\leqq 2$, then A is pseudo-flat by (5). Moreover, it is easy to see that if R is a UFD and $\mathrm{ht}(A_+)=1$ (resp. R is a regular local ring with $\dim(R)\leqq 2$ and A is a UFD with $\mathrm{ht}(A_+)\leqq 2$), then $A\cong R[X]$ (resp. $A\cong S_R(A_1)$ and A_1 is a projective R-module).

(7) Let A be a homogeneous algebra over a local ring R. If either $\dim(R)=1$ and A is an integral domain or $\dim(R)=2$ and A is a UFD, then A is pseudo-flat by (5) and (6). For any DVR $(R, \mathfrak{m})$, the R-algebra $R[X]/(\mathfrak{m}X)$ is reduced but is not pseudo-flat. For any regular local ring $(R, \mathfrak{m})$ with $\dim(R)\geqq 2$, the R-algebra $R(\mathfrak{m})$ is normal but is not pseudo-flat. For any regular local ring $(R, \mathfrak{m})$ with $\dim(R)\geqq 3$, let $\{a_1,\dots,a_n\}$ be a minimal basis of $\mathfrak{m}$ and put $E=\oplus_{i=1}^n Re_i/(\sum_{i=1}^n a_ie_i)R$. Then $S_R(E)\cong R[X_1,\dots,X_n]/(\sum_{i=1}^n a_iX_i)$ is a UFD but is not pseudo-flat because E is not free (cf. [21]).

(8) Let A be a homogeneous integral domain. If either $\dim(A)\leqq 2$ or $\dim(A)\leqq 4$ and A is a UFD, then A is pseudo-flat. To see this, by (7), we may assume that R is local, A is a UFD, $\dim(R)=3$ and $\dim(A)=4$. Then the assertion follows from (6). Even if either $\dim(A)=3$ and A is normal or $\dim(A)=5$ and A is a UFD, A is not necessarily pseudo-flat.

§3. Reductions of graded rings

Let A be a homogeneous algebra over a ring R. We say that A is *of the principal class* if the equality $\mathrm{emb}(A)=\mathrm{ht}(A_+)$ holds. If $(R, \mathfrak{m}, k)$ is a local ring, then A is of the principal class if and only if A is pseudo-flat and $A/\mathfrak{m}A$ is a polynomial k-algebra, and in this case $A\otimes_R k(\mathfrak{p})$ is a polynomial $k(\mathfrak{p})$-algebra for every $\mathfrak{p}\in\mathrm{Spec}(R)$. If A is of the principal class, then so are A_S, A/IA and A_{red}, where S and I are a multiplicative set and an ideal of R respectively.

PROPOSITION 3.1. *The following conditions are equivalent*:

(1) *A is of the principal class.*

(2) *A is isomorphic to $R[X_1,\dots,X_v]/I$, where $v=\mathrm{emb}(A)$ and I is a nilpotent ideal of $R[X_1,\dots,X_v]$.*

(3) *A_{red} is isomorphic to $R_{\mathrm{red}}[X_1,\dots,X_v]$ with $v=\mathrm{emb}(A)$.*

PROOF. The equivalence of (2) and (3) is clear. (1) implies (2): Put $I=\mathrm{Ker}(R[X_1,\dots,X_v]\to A)$. Then, for any $\mathfrak{p}\in\mathrm{Min}(R)$, we have $I_\mathfrak{p}\subset\mathfrak{p}R_\mathfrak{p}[X_1,\dots,X_v]$ because $\dim(A_\mathfrak{p})=\dim(R_\mathfrak{p}[X_1,\dots,X_v])=v$. Therefore we have $I\subset\bigcap\{\mathfrak{p}[X_1,\dots,X_v]|\mathfrak{p}\in\mathrm{Min}(R)\}=\mathrm{nil}(R[X_1,\dots,X_v])$. (3) implies (1): We have $\mathrm{ht}(A_+)=\mathrm{ht}(A_{\mathrm{red}})_+$, and $\mathrm{emb}(A)=\mathrm{emb}(A_{\mathrm{red}})=\mathrm{ht}(A_{\mathrm{red}})_+$ by the assumption. Therefore

$\mathrm{emb}(A)=\mathrm{ht}(A_+)$. Q.E.D.

COROLLARY 3.2. (1) *If R is reduced, then A is of the principal class if and only if A is a polynomial R-algebra.*

(2) *If A is of the principal class, then for any radical ideal I of R, we have $A/IA \cong R/I[X_1,\dots,X_v]$ with $v=\mathrm{emb}(A)$.*

Classically, an ideal I of R is said to be *of the principal class* if the equality $\mu(I) = \mathrm{ht}(I)$ holds. This is equivalent to the condition that $G(I)$ is an R/I-algebra of the principal class and $\mu(I)=\mu(I/I^2)$. The latter condition is always satisfied if R is a local ring. By Proposition 3.1, we are able to give very simple proofs for a few fundamental facts about ideals of the principal class in [3] and [4]. But we omit the details.

A homogeneous sub R-algebra B of A is said to be a *reduction* of A with respect to a graded A-module M, or simply an *M-reduction* of A, if M is finitely generated B-module. An M-reduction of A which is minimal with respect to inclusion relation is called a *minimal M-reduction* of A. An A-reduction (resp. a minimal A-reduction) of A is called a *reduction* (resp. a *minimal reduction*) of A. A homogeneous sub R-algebra B of A is an M-reduction of A if and only if $B_1M_n=M_{n+1}$ for all $n \gg 0$. If $J \subset I$ are ideals of R and E is a finitely generated R-module, then $R(J)$ is an $R(I, E)$-reduction of $R(I)$ if and only if $JI^nE=I^{n+1}E$ for some n (we say that J is an E-reduction of I in this case). Therefore $R(J)$ is a reduction (resp. a minimal reduction) of $R(I)$ if and only if J is a reduction (resp. a minimal reduction) of I in the sense of [16].

The following is a fundamental theorem in the theory of reductions. Though it can be proved in the similar way as in the case of ideals, we give a proof for completeness.

THEOREM 3.3. *Assume that $(R, \mathfrak{m}, k)$ is a local ring. Then, for any M-reduction B of A, there is a minimal M-reduction C of A which is contained in B and we have $\mathfrak{m}C_1 = \mathfrak{m}A_1 \cap C_1$ (in particular, $\ell(M) \leqq \mathrm{emb}(C) \leqq \mathrm{emb}(B)$). If k is an infinite field and B is an M-reduction of A, then the following conditions are equivalent:*

(1) *B is a minimal M-reduction of A.*

(2) *$B/\mathfrak{m}B$ is regular (i.e., a polynomial k-algebra) and $M/\mathfrak{m}M$ is a faithful $B/\mathfrak{m}B$-module.*

(3) $\mathrm{emb}(B)=\ell(M)$.

The assertions (2) $\Leftrightarrow$ (3) $\Rightarrow$ (1) *are also true even if k is a finite field.*

PROOF. We denote by $\bar{a}$ and $\bar{x}$ the images of $a \in R$ and $x \in A_1$ in $R/\mathfrak{m}$ and $A_1/\mathfrak{m}A_1$ respectively. Among the M-reductions of A contained in B, take an M-reduction D of A such that $\dim_k(D_1+\mathfrak{m}A_1/\mathfrak{m}A_1)$ is minimal. Let $x_1,\dots,x_r \in D_1$ be such that $\bar{x}_1,\dots,\bar{x}_r$ is a basis of $D_1+\mathfrak{m}A_1/\mathfrak{m}A_1$, and put $C=R[x_1,\dots,x_r]$. Since C_1

$+\mathfrak{m}A_1=D_1+\mathfrak{m}A_1$ and $D_1M_n=M_{n+1}$ for all $n\gg 0$, we have $C_1M_n+\mathfrak{m}M_{n+1}=C_1+\mathfrak{m}A_1M_n=(C_1+\mathfrak{m}A_1)M_n=(D_1+\mathfrak{m}A_1)M_n=D_1M_n+\mathfrak{m}A_1M_n=M_{n+1}$ for all $n\gg 0$. Hence $C_1M_n=M_{n+1}$ for all $n\gg 0$, i.e., C is an M-reduction of A. We show the equality $\mathfrak{m}A_1\cap C_1=\mathfrak{m}C_1$. If $x=\sum a_ix_i$ is in $\mathfrak{m}A_1\cap C_1$, then $\sum\bar{a}_i\bar{x}_i=0$ in $D_1+\mathfrak{m}A_1/\mathfrak{m}A_1$. Hence $\bar{a}_i=0$, i.e., a_i is in $\mathfrak{m}$ for all i, and we have $x=\sum a_ix_i\in\mathfrak{m}C_1$. Let $E\subset C$ be an M-reduction of A. Then $E_1+\mathfrak{m}A_1\subset C_1+\mathfrak{m}A_1$ and we have $E_1+\mathfrak{m}A_1=C_1+\mathfrak{m}A_1$ by the choice of D. For any element x of C_1, put $x=y+z$, $y\in E_1$, $z\in\mathfrak{m}A_1$. Then $z=x-y\in\mathfrak{m}A_1\cap C_1=\mathfrak{m}C_1$. Hence $x=y+z\in E_1+\mathfrak{m}C_1$. Thus $C_1+\mathfrak{m}A_1=E_1+\mathfrak{m}A_1$, and we have $E_1=C_1$, i.e., $E=C$. Therefore C is a minimal M-reduction of A. Next we prove the second assertion. The equivalence of (2) and (3): Note that if $\mathrm{emb}(B)=\ell(M)$, then $\mathrm{emb}(B)=\dim_{A/\mathfrak{m}A}(M/\mathfrak{m}M)=\dim_{B/\mathfrak{m}B}(M/\mathfrak{m}M)\leqq\dim(B/\mathfrak{m}B)$. Hence $B/\mathfrak{m}B$ is regular. When $B/\mathfrak{m}B$ is regular, $M/\mathfrak{m}M$ is a faithful $B/\mathfrak{m}B$-module $\Leftrightarrow$ $\dim(M/\mathfrak{m}M)=\dim(B/\mathfrak{m}B)$ $\Leftrightarrow$ $\ell(M)=\mathrm{emb}(B)$. (1) implies (2): Put $\mu(B_1)=r$ and let $x_1,\dots,x_r$ be a minimal basis of B_1. We consider M as a graded module over $S=R[X_1,\dots,X_r]$ by $X_im=x_im$ ($m\in M$). If $M/\mathfrak{m}M$ is a faithful $S/\mathfrak{m}S$-module, then $S/\mathfrak{m}S\to B/\mathfrak{m}B$ is an isomorphism. Let $f(X_1,\dots,X_r)\in S$ be a homogeneous polynomial of degree $t\geqq 1$ and assume that $\bar{f}(M/\mathfrak{m}M)=0$, where $\bar{f}$ is the image of f in $S/\mathfrak{m}S$. Then $fM_n\subset\mathfrak{m}M_{n+t}$ for all n. It is enough to show that $f\in\mathfrak{m}[X_1,\dots,X_r]$. If the coefficient of X_1^t is a unit, then $x_1^tM_n\subset(x_2,\dots,x_r)^{t-1}B_1M_n+\mathfrak{m}M_{n+t}$ for all n. Hence $B_tM_n\subset(x_2,\dots,x_r)B_{t-1}M_n+\mathfrak{m}M_{n+t}$ for all n. Since B is an M-reduction of A, we have $B_1M_s=M_{s+1}$ for all $s\gg 0$. Therefore $M_{s+t}=B_tM_s\subset(x_2,\dots,x_r)B_{t-1}M_s+\mathfrak{m}M_{s+t}=(x_2,\dots,x_r)M_{s+t-1}+\mathfrak{m}M_{s+t}$ for all $s\gg 0$. Hence $M_{s+t}=(x_2,\dots,x_r)M_{s+t-1}$ for all $s\gg 0$. This implies that $R[x_2,\dots,x_r]$ ($\subsetneqq B$) is an M-reduction of A which contradicts with our assumption. Hence the coefficient of X_1^t is in $\mathfrak{m}$. Next, we show that if not all of $a_{11},a_{21},\dots,a_{r1}\in R$ are in $\mathfrak{m}$, then $f(a_{11},a_{21},\dots,a_{r1})\equiv 0 \bmod \mathfrak{m}$. Then since k is an infinite field, we have $f\in\mathfrak{m}[X_1,\dots,X_r]$. Since a_{11}, $a_{21},\dots,a_{r1}$ is a unimodular sequence, there exist $a_{ij}\in R$ ($1\leqq i\leqq r$, $2\leqq j\leqq r$) such that if $A=(a_{ij})$, then $\det(A)$ is not in $\mathfrak{m}$. Define a minimal basis $y_1,\dots,y_r$ of B_1 by $x_i=\sum_{j=1}^{n}a_{ij}y_j$. Then $f(Ay)M_n=f(\underline{x})M_n\equiv 0 \bmod \mathfrak{m}M_{n+t}$ for all n. By what we showed above, the coefficient of Y_1^t in $f(A\underline{Y})=f(\sum_{j=1}^{r}a_{1j}Y_j,\dots,\sum_{j=1}^{r}a_{rj}Y_j)$ is in $\mathfrak{m}$. Putting $Y_1=1$, $Y_2=\cdots=Y_r=0$, we get $f(a_{11},\dots,a_{r1})\in\mathfrak{m}$ as desired. (3) implies (1): If k is an infinite field and C is a minimal M-reduction contained in B, then $C/\mathfrak{m}C\subset B/\mathfrak{m}B$ are both polynomial rings of the same dimension. Hence $C/\mathfrak{m}C=B/\mathfrak{m}B$ and we get $C=B$. If k is a finite field and C is an M-reduction of A contained in B, then $C\otimes_R R(X)$ is an $M\otimes_R R(X)$-reduction of $A\otimes_R R(X)$ and $B\otimes_R R(X)$ is a minimal $M\otimes_R R(X)$-reduction of $A\otimes_R R(X)$. Hence $C\otimes_R R(X)=B\otimes_R R(X)$ and we get $C=B$. Q.E.D.

Assume that R is a local ring. For an M-reduction B of A, we have $\ell_A(M)=\ell_B(M)$ and $\mathrm{ht}(A_+,M)=\mathrm{ht}(B_+,M)$. Hence M is a pseudo-flat A-module if and only if M is a pseudo-flat B-module. When the residue field of R is infinite, if M is pseudo-flat and B is a minimal M-reduction of A, then $B_\mathfrak{p}$ is a minimal $M_\mathfrak{p}$-reduction of $A_\mathfrak{p}$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Minimal reductions of a given homogeneous algebra are not necessarily isomorphic. For example, let $(R, \mathfrak{m})$ be a DVR and put $A = R[X, Y]/(\mathfrak{m}X, X^2 + Y^2, \mathfrak{m}Y^2) = R[x, y]$. Then, since $x(x, y) = y(x, y) = (x, y)^2$, $B_1 = R[x] = R[X]/(\mathfrak{m}X)$ and $B_2 = R[y] = R[Y]/(\mathfrak{m}Y^2)$ are minimal reductions of A, but B_1 and B_2 are not isomorphic.

The following theorem gives an important structure theorem for pseudo-flat graded modules.

THEOREM 3.4. *Assume that R is a reduced local ring with infinite residue field. Then M is pseudo-flat if and only if there is a polynomial sub R-algebra B of A such that M is a finitely generated faithful B-module. The 'if' part is also valid even if the residue field of R is a finite field.*

In particular, A is pseudo-flat if and only if A is a finite extension of a polynomial R-algebra.

PROOF. Suppose that M is pseudo-flat and let B be a minimal M-reduction of A. Then, for all $\mathfrak{p} \in \mathrm{Min}(R)$, we have $\dim(B_\mathfrak{p}) \geqq \dim(M_\mathfrak{p}) = \ell(M_\mathfrak{p}) = \ell(M) = \mathrm{emb}(B)$. Hence $\mathrm{ht}(B_+) = \min\{\dim(B_\mathfrak{p}) | \mathfrak{p} \in \mathrm{Min}(R)\} \geqq \mathrm{emb}(B)$, i.e., $\mathrm{emb}(B) = \mathrm{ht}(B_+)$. Thus B is an R-algebra of the principal class, and by Corollary 3.2, B is isomorphic to a polynomial R-algebra $R[X_1, \dots, X_n]$ with $n = \ell(M)$. For all $\mathfrak{p} \in \mathrm{Min}(R)$, since $\dim(M_\mathfrak{p}) = \dim(B_\mathfrak{p}) = n$ and $B_\mathfrak{p} \cong R_\mathfrak{p}[\underline{X}]$ with $R_\mathfrak{p}$ a field, we have $\mathrm{ann}_{B_\mathfrak{p}}(M_\mathfrak{p}) = 0$. Therefore $\mathrm{ann}_B(M) \subset \bigcap_{\mathfrak{p} \in \mathrm{Min}(R)} \mathfrak{p}(\underline{X}) = 0$, i.e., M is a faithful B-module.

Conversely, assume that some M-reduction B of A is a polynomial R-algebra and M is a faithful B-module. Then, for all $\mathfrak{p} \in \mathrm{Min}(R)$, we have $\ell(B) = \dim(B_\mathfrak{p}) = \dim(M_\mathfrak{p}) = \ell(M_\mathfrak{p}) \leqq \ell(M) \leqq \mathrm{emb}(B) = \ell(B)$. Therefore we have $\ell(M_\mathfrak{p}) = \ell(M)$ for all $\mathfrak{p} \in \mathrm{Min}(R)$, i.e., M is a pseudo-flat A-module. Q.E.D.

PROPOSITION 3.5. *Let I be an ideal of a local ring R. Put $\bar{A} = A/IA$, $\bar{M} = M/IM$ and $\bar{B} = B + IA/IA$ for a homogeneous sub R-algebra B of A. Then B is an M-reduction of A if and only if $\bar{B}$ is an $\bar{M}$-reduction of $\bar{A}$. Assume that the residue field of R is infinite. If B is a minimal M-reduction of A, then $\bar{B}$ is a minimal $\bar{M}$-reduction of $\bar{A}$, and any minimal $\bar{M}$-reduction of $\bar{A}$ can be obtained in this way.*

PROOF. The first assertion follows from Nakayama's lemma. If B is a minimal M-reduction of A, then $\bar{B}$ is an $\bar{M}$-reduction of $\bar{A}$ and $\bar{B}/\mathfrak{m}\bar{B} = B + IA/\mathfrak{m}B + IA = B/(\mathfrak{m}B + IA) \cap B = B/\mathfrak{m}B$ (since $\mathfrak{m}A \cap B = \mathfrak{m}B$). Therefore $\mathrm{emb}(\bar{B}) = \mathrm{emb}(B) = \ell(M) = \ell(\bar{M})$ and this implies that $\bar{B}$ is a minimal $\bar{M}$-reduction of $\bar{A}$. Conversely, let B^* be a minimal $\bar{M}$-reduction of $\bar{A}$. Then there is an M-reduction B of A such that $\bar{B} = B^*$. Take a minimal M-reduction C of A contained in B. Then $\bar{C}$ is an $\bar{M}$-reduction of $\bar{A}$ such that $\bar{C} \subset \bar{B} = B^*$. Thus we have $\bar{C} = B^*$. Q.E.D.

COROLLARY 3.6. *Let R be a local ring with infinite residue field.*

(1) *If B is a minimal M-reduction of A, then $B/\mathfrak{m}B$ is a minimal $M/\mathfrak{m}M$-reduction of $A/\mathfrak{m}A$, and any minimal $M/\mathfrak{m}M$-reduction of $A/\mathfrak{m}A$ can be obtained in this way.*

(2) *Let I be an ideal of R and E a finitely generated R-module. If J is a minimal E-reduction of I, i.e., $R(J)$ is a minimal $R(I, E)$-reduction of $R(I)$, then $\oplus_{n\geqq 0}J^n + I^{n+1}/I^{n+1} = \mathrm{Image}(G(J)\otimes_R R/I \to G(I))$ is a minimal $G(I, E)$-reduction of $G(I)$, and any minimal $G(I, E)$-reduction of $G(I)$ can be obtained in this way.*

§4. Some properties of pseudo-flat graded modules

Let A be a homogeneous algebra over a ring R and M a finitely generated graded A-module. The following lemma follows easily from Theorem 3.4. We omit the proof.

LEMMA 4.1. *Let $\mathfrak{p}$ be a prime ideal of R such that $R_{\mathfrak{p}}$ is a reduced local ring with infinite residue field. If $M_{\mathfrak{p}}$ is pseudo-flat, then there is an element $f\in R-\mathfrak{p}$ such that A_f has a polynomial sub R_f-algebra, over which M_f is a finitely generated faithful module.* (*Hence $M_{\mathfrak{q}}$ is pseudo-flat for all $\mathfrak{q}\in D(f)$.*)

PROPOSITION 4.2. *The sets $U=\{\mathfrak{p}\in \mathrm{Spec}(R)|M_{\mathfrak{p}}$ is a pseudo-flat $A_{\mathfrak{p}}$-module$\}$ and $V=\{\mathfrak{p}\in \mathrm{Spec}(R)|A_{\mathfrak{p}}$ is of the principal class$\}$ are open in* Spec(R). *If A is locally pseudo-flat, then the set $W=\{\mathfrak{p}\in \mathrm{Spec}(R)|A\otimes_R k(\mathfrak{p})$ is a polynomial $k(\mathfrak{p})$-algebra$\}$ is open in* Spec(R).

PROOF. By the base change $R\to R(X)_{\mathrm{red}}$, we may assume that R is reduced and all residue fields of R are infinite. (Note that the canonical mapping $\mathrm{Spec}(R(X)) \to \mathrm{Spec}(R)$ is an open mapping.) The openness of U follows from Lemma 4.1. To prove the openness of W, by localization, we may assume that A is a finite extension of a polynomial R-algebra B. Then, since $W=\mathrm{Spec}(R)-\mathrm{Supp}_R(A/B)$, the assertion is clear. Finally, these facts implies the openness of $V=\{\mathfrak{p}\in \mathrm{Spec}(R)|A_{\mathfrak{p}}$ is pseudo-flat and $A\otimes_R k(\mathfrak{p})$ is a polynomial $k(\mathfrak{p})$-algebra$\}$. Q.E.D.

For a DVR $(R, \mathfrak{m})$ put $A=R[X, Y]/\mathfrak{m}X(X, Y)$ and $B=R[X, Y]/X(\mathfrak{m}X, Y)$. Then A is not pseudo-flat, B is pseudo-flat, and the set $\{\mathfrak{p}\in \mathrm{Spec}(R)|A\otimes_R k(\mathfrak{p})$ is a polynomial $k(\mathfrak{p})$-algebra$\}=\{\mathfrak{p}\in \mathrm{Spec}(R)|B\otimes_R k(\mathfrak{p})$ is Cohen-Macaulay (or Gorenstein)$\}=\{\mathfrak{m}\}$ is not open in Spec(R).

PROPOSITION 4.3. *If M is locally pseudo-flat, then the function $\mathfrak{p}\mapsto \mathrm{e}(M\otimes_R k(\mathfrak{p}))$ defined on* Spec(R) *is upper semicontinuous.*

PROOF. We have to show that for any $n\in \boldsymbol{Z}$, the set $M_n(M) =\{\mathfrak{p}\in \mathrm{Spec}(R)|\mathrm{e}(M\otimes_R k(\mathfrak{p}))\leqq n\}$ is open in Spec(R). By the base change R

$\rightarrow R(X)_{red}$, we may assume that R is reduced and all residue fields of R are infinite. By localization, we may assume that there is a polynomial sub R-algebra $B = R[X_1,\dots,X_v]$ of A such that M is a finitely generated faithful B-module. Fix a prime ideal $\mathfrak{p}$ of R and put $\mathfrak{P}=\mathfrak{p}B$. Then we have $e(M\otimes_R k(\mathfrak{p})) = \operatorname{rank}_{B\otimes_R k(\mathfrak{p})}(M\otimes_R k(\mathfrak{p})) = \dim_{k(\mathfrak{P})}(M\otimes_B k(\mathfrak{P})) = \mu_{B_\mathfrak{P}}(M_\mathfrak{P})$ (note that $k(\mathfrak{P}) = k(\mathfrak{p})(X)$ is the quotient field of $B\otimes_R k(\mathfrak{p}) = k(\mathfrak{p})[X]$). We show that $M_n(M) = \{\mathfrak{Q}\cap R \mid \mathfrak{Q}\in \operatorname{Spec}(B), \mu_{B_\mathfrak{Q}}(M_\mathfrak{Q}) \leqq n\}$ for any $n\in \boldsymbol{Z}$. Indeed, if $\mathfrak{p}\in M_n(M)$ and $\mathfrak{P}=\mathfrak{p}B$, then $\mathfrak{P}\cap R=\mathfrak{p}$ and $\mu_{B_\mathfrak{P}}(M_\mathfrak{P}) = e(M\otimes_R k(\mathfrak{p})) \leqq n$. Conversely, assume that $\mathfrak{Q}\in \operatorname{Spec}(B)$, $\mathfrak{Q}\cap R=\mathfrak{p}$ and $\mu_{B_\mathfrak{P}}(M_\mathfrak{P}) \leqq n$. Put $\mathfrak{P}=\mathfrak{p}B$. Then we have $\mathfrak{P}\subset\mathfrak{Q}$ and $e(M\otimes_R k(\mathfrak{p})) = \mu_{B_\mathfrak{P}}(M_\mathfrak{P}) \leqq \mu_{B_\mathfrak{Q}}(M_\mathfrak{Q}) \leqq n$. Since $\operatorname{Spec}(B)\rightarrow \operatorname{Spec}(R)$ is an open mapping, this implies our assertion. Q.E.D.

THEOREM 4.4. *Assume that R is a reduced local ring and $M/\mathfrak{m}M$ is Cohen-Macaulay. Then the following conditions are equivalent*:

(1) *M is pseudo-flat and* $e(M\otimes_R k(\mathfrak{p}))$ *is constant for all* $\mathfrak{p}\in \operatorname{Spec}(R)$.

(2) *M is R-free.*

PROOF. We show that (1) implies (2). If M is pseudo-flat, then $e(M\otimes_R k(\mathfrak{p}))$ is constant for all $\mathfrak{p}\in \operatorname{Spec}(R)$ if and only if $e(M\otimes_R k) = e(M\otimes_R k(\mathfrak{p}))$ for all $\mathfrak{p}\in \operatorname{Min}(R)$. Hence by the base change $R\rightarrow R(X)$, we may assume that the residue field of R is infinite. Let $B\cong R[X_1,\dots,X_n]$ be a minimal M-reduction of A. Then since $M/\mathfrak{m}M$ is a Cohen-Macaulay $B/\mathfrak{m}B$-module with $\dim(M/\mathfrak{m}M) = \dim(B/\mathfrak{m}B)$, $M/\mathfrak{m}M$ is $B/\mathfrak{m}B$-free. Therefore $e(M/\mathfrak{m}M) = \operatorname{rank}_{B/\mathfrak{m}B}(M/\mathfrak{m}M) = \mu_{B/\mathfrak{m}B}(M/\mathfrak{m}M)$ and $e(M_\mathfrak{p}) = \operatorname{rank}_{B_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{rank}_{B_\mathfrak{P}}(M_\mathfrak{P}) = \mu_{B_\mathfrak{P}}(M_\mathfrak{P})$ for all $\mathfrak{p}\in \operatorname{Min}(R)$, $\mathfrak{P}=\mathfrak{p}B$. By the assumption, we have $\mu_{B_\mathfrak{P}}(M_\mathfrak{P}) = \mu_{B/\mathfrak{m}B}(M/\mathfrak{m}M) = \mu_{B_\mathfrak{N}}(M_\mathfrak{N})$ for all $\mathfrak{P}\in \operatorname{Min}(B)$, where $\mathfrak{N}=\mathfrak{m}\oplus B_+$, Therefore $M_\mathfrak{N}$ is $B_\mathfrak{N}$-free, which implies that M is B-free.

Q.E.D.

THEOREM 4.5. *Assume that* $(R, \mathfrak{m})$ *is a regular local ring. Then the following conditions are equivalent*:

(1) *M is pseudo-flat and is a Cohen-Macaulay* (*resp. Gorenstein*) *A-module.*

(2) *M is R-free and $M/\mathfrak{m}M$ is a Cohen-Macaulay* (*resp. Gorenstein*) *$A/\mathfrak{m}A$-module.*

PROOF. By Lemma 4.6 below, it is enough to show the assertion for the Cohen-Macaulay case because for the Gorenstein case, A is R-free under the each condition. We may assume that $R/\mathfrak{m}$ is an infinite field. Consider the following condition:

(3) There is a polynomial sub R-algebra $B\cong R[X_1,\dots,X_n]$ of A such that M is a finitely generated free B-module. (1) implies (3): Let $B\cong R[X_1,\dots,X_n]$ be a minimal M-reduction of A and put $\mathfrak{N}=\mathfrak{m}\oplus B_+$. Then, by the assumption, $M_\mathfrak{N}$ is a Cohen-Macaulay module over a regular local ring $B_\mathfrak{N}$ with $\dim(M_\mathfrak{N}) = \dim(B_\mathfrak{N})$. Hence $M_\mathfrak{N}$ is a free $B_\mathfrak{N}$-module and this implies that M is B-free. (3) implies (2): Since M is

$B \cong R[X_1,\dots,X_n]$-free, M is R-free. Since $M/\mathfrak{m}M$ is a finitely generated free $B/\mathfrak{m}B \cong R/\mathfrak{m}[X_1,\dots,X_n]$-module, $M/\mathfrak{m}M$ is a Cohen-Macaulay $B/\mathfrak{m}B$-module. Hence $M/\mathfrak{m}M$ is a Cohen-Macaulay $A/\mathfrak{m}A$-module. The fact that (2) implies (1) follows from Lemma 4.6 below. Q.E.D.

LEMMA 4.6 (cf. [9], 24, (6.3.3), [8]). *Let $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local homomorphism of noetherian local rings, and let M (resp. N) be a non-zero finitely generated A-module (resp. B-module). Assume that N is A-flat. Then $M \otimes_A N$ is a Cohen-Macaulay (resp. Gorenstein) B-module if and only if M is a Cohen-Macaulay (resp. Gorenstein) A-module and $N/\mathfrak{m}N$ is a Cohen-Macaulay (resp. Gorenstein) $B/\mathfrak{m}B$-module. For the Gorenstein case, we assume that B is A-flat.*

PROPOSITION 4.7. *Let A be a pseudo-flat normal homogeneous domain over R. Then each A_n is a reflexive R-module, the Going-down theorem holds for the extension $R \subset A$, and there is an injection $C\ell(R) \to C\ell(A)$ between the ideal class groups. Moreover, if $\mathrm{ht}(A_+) = 1$, then $A \cong S_R(A_1)$ and A_1 is an invertible R-module.*

PROOF. For the first assertion, we have only to show the Going-down theorem for $R \subset A$ (cf. [6], Proposition 10.7). By localization and the base change $R \to R(X)$, we may assume that R is a local ring with infinite residue field. Then A is a finite extension of a polynomial R-algebra which is a normal domain. Hence the assertion is clear. For the second assertion, assume that R is as above and let K be the quotient field of R. Since $A \otimes_R K$ is a one-dimensional homogeneous K-algebra, $A \otimes_R K \cong K[X]$. Hence $\mathrm{rank}_R(A_n) = 1$ for all n. Since A is pseudo-flat, there is an element $x \in A_1$ such that $xA_n = A_{n+1}$ for some n. Then, for each $y \in A_1$, we have $yA_n \subset A_{n+1} = xA_n$. Hence $y/x \in (A_n : A_n)_K = R$, namely $y \in Rx$. Therefore $A_1 = Rx$. Considering the generic fibers, the canonical surjection $R[X] \to R[x]$ is an isomorphism. Q.E.D.

§5. Reduction exponents of graded modules

Throughout this section, let $(R, \mathfrak{m}, k)$ be a local ring with infinite residue field, A a homogeneous R-algebra and M a finitely generated graded A-module. We define the *reduction exponent* $\delta_A(M)$ of M by $\delta_A(M) = \min\{n \in \boldsymbol{Z} \mid \text{there is a minimal } M\text{-reduction } B \text{ of } A \text{ such that } B_1 M_m = M_{m+1} \text{ for all } m \geqq n\}$. We denote $\delta_A(A)$ by $\delta(A)$. For a finitely generated R-module E and an ideal I of R, put $\delta_I(E) = \delta_{R(I)}(R(I, E))$ and $\delta(I) = \delta(R(I))$. Then $\delta_I(E) = \min\{n \mid \text{there is a minimal } E\text{-reduction } J \text{ of } I \text{ such that } JI^nE = I^{n+1}E\}$. Hence our $\delta(I)$ coincides with the reduction exponent of I which was introduced by Sally.

THEOREM 5.1. (1) *We have $\delta_A(M) \leqq \mathrm{reg}_A(M) := \min\{n \in \boldsymbol{Z} \mid [H_P^i(M)]_j = 0 \text{ if } i + j > n\}$, where $P = A_+$. (For $\mathrm{reg}_A(M)$, see [17].) If M is pseudo-flat and $\dim_A(M) = d$,* *then* $[H_P^d(M)]_n = 0$ *for all* $n > \delta_A(M) - d$. *Hence, if M is pseudo-flat and Cohen-Macaulay, then* $\delta_A(M) = \mathrm{reg}_A(M)$.

(2) *If R is a field and M is Buchsbaum, then* $\delta_A(M) = \mathrm{reg}_A(M)$.

(3) *We have* $\delta_{A/IA}(M/IM) = \delta_A(M)$ *for any ideal I of R.*

(4) *If* $A/\mathfrak{m}A$ *is Cohen-Macaulay, then* $\delta(A) \leqq \mathrm{f}(A) + \ell(A) - \mathrm{emb}(A)$, *where we put* $\mathrm{f}(A) = \mathrm{e}(A/\mathfrak{m}A)$.

PROOF. (1) Put $\gamma_A(M) = \min\{n \in Z \mid A_1 M_m = M_{m+1}$ for all $m \geqq n\}$. Take a minimal M-reduction B of A such that $\delta_A(M) = \gamma_B(M)$. Then, by [17], Theorem 2, we have $\gamma_B(M) \leqq \mathrm{reg}_B(M) = \mathrm{reg}_A(M)$. We show that if $\dim_A(M) = \dim(A) = d$, then $[H_P^d(M)]_n = 0$ for all $n > \mathrm{reg}(A) + \gamma_A(M) - d$. In fact, since there is a surjective homomorphism $\oplus_{i=1}^r A(-a_i) \to M$ with $a_i \leqq \gamma_A(M)$, the induced homomorphism $\oplus_{i=1}^r [H_P^d(A)]_{n-a_i} \to [H_P^d(M)]_n$ is surjective, and $[H_P^d(A)]_{n-a_i} = 0$ if $n - a_i + d > \mathrm{reg}(A)$. Therefore, for all $n > \mathrm{reg}(A) + \gamma_A(M) - d$, we have $[H_P^d(M)]_n = 0$. If M is pseudo-flat, then $B \otimes_R R_{\mathrm{red}} \cong R_{\mathrm{red}}[X_1, \dots, X_v]$ with $v = \ell(M)$. Put $C = R[X_1, \dots, X_v]$. Hence $[H_P^d(M)]_n = 0$ for all $n > \mathrm{reg}(C) + \gamma_C(M) - d = \delta_A(M) - d$. (2) Let $B \cong R[X_1, \dots, X_d]$ be a minimal M-reduction of A. Then by [17], Proposition 18, we have $\mathrm{reg}_A(M) = \mathrm{reg}(M/(X_1, \dots, X_d)M) = \mathrm{reg}(M/B_1 M) = \gamma_B(M)$. Hence $\delta_A(M) = \mathrm{reg}_A(M)$. (Note that in this case $\gamma_B(M)$ does not depend on the choice of B.) (3) Let B be a minimal M-reduction of A. Then $\bar{B} = B + IA/IA$ is a minimal $\bar{M} = M/IM$-reduction of $\bar{A} = A/IA$ and we have $\gamma_B(M) = \gamma_{\bar{B}}(\bar{M})$ by Nakayama's lemma. Hence our assertion follows from Proposition 3.5. (4) follows from (2), (3) and [17], Proposition 13. Q.E.D.

By Theorem 5.1, (3), we have $\delta_I(E) = \delta_{G(I)}(G(I, E))$ for any finitely generated R-module E. Since $\delta_A(M) = \delta_{A/\mathfrak{m}A}(M/\mathfrak{m}M)$, some problems about reduction exponents reduce to the case when R is a field. If $\ell(A) = d$ and $\mu(A_n) < \binom{n+d}{d}$ for some n, then $\delta(A) < n$, and if $\ell(A) > 0$ and $A/\mathfrak{m}A$ is Cohen-Macaulay, then $\delta(A) \leqq \ell(A)!\mathrm{f}(A) - 1$ (see [5], [20]).

PROPOSITION 5.2. *Assume that R is a (not necessarily local) reduced ring whose residue fields are all infinite. If M is locally pseudo-flat, then the function* $\mathfrak{p} \mapsto \delta_{A_\mathfrak{p}}(M_\mathfrak{p})$ *is upper semicontinuous.*

PROOF. We show that the set $U = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \delta_{A_\mathfrak{p}}(M_\mathfrak{p}) \leqq r\}$ is open for all r. For any $\mathfrak{p} \in U$, take a homogeneous sub R-algebra B of A and an element $f \in R - \mathfrak{p}$ such that B_f is a polynomial R_f-algebra and M_f is a finitely generated faithful B_f-module. Then $B_\mathfrak{q}$ is a minimal $M_\mathfrak{q}$-reduction of $A_\mathfrak{q}$ for all $\mathfrak{q} \in D(f)$. By the assumption, we may assume that $(B_1)_\mathfrak{p}(M_n)_\mathfrak{p} = (M_{n+1})_\mathfrak{p}$ for all $n \geqq r$. Hencethere is an element $g \in R - \mathfrak{p}$ such that $D(g) \subset D(f)$ and $(B_1)_g(M_n)_g = (M_{n+1})_g$ for all $n \geqq r$. Therefore we have $\mathfrak{p} \in D(g) \subset U$. This completes the proof. Q.E.D.

COROLLARY 5.3. (*R is not necessarily local.*) *If M is locally pseudo-flat and*

$M\otimes_R k(\mathfrak{p})$ is Buchsbaum for all $\mathfrak{p}\in\mathrm{Spec}(R)$. Then the function $\mathfrak{p}\mapsto \mathrm{reg}(M\otimes_R k(\mathfrak{p}))$ is upper semicontinuous.

PROOF. By the base change $R\to R(X)_{\mathrm{red}}$, we may assume that R is a reduced ring whose residue fields are all infinite. Then by the assumption, we have $\mathrm{reg}(M\otimes_R k(\mathfrak{p}))=\delta(M\otimes_R k(\mathfrak{p}))=\delta(M_{\mathfrak{p}})$. Hence the assertion follows from Proposition 5.2. Q.E.D.

In the rest of this section, we assume that A is a homogeneous algebra over an infinite field k. Recall that if $A=k[X_1,\dots,X_v]/I$ with $\mathrm{emb}(A)=v$, then the *initial degree* $i(A)$ of A is defined by $i(A)=\min\{n|I_n\neq 0\}=\min\{n|\dim_k(A_n)\neq\binom{v+n-1}{n}\}$.

THEOREM 5.4. *Assume that A is not regular.*

(1) *We have $\delta(A)\geqq i(A)-1$.*

(2) *Suppose that the equality $\delta(A)=i(A)-1$ holds. Put $v=\mathrm{emb}(A)$, $d=\dim(A)$ and $m=\delta(A)$. Then we have $\mathrm{e}(A)\leqq\binom{v+m}{m}-d\binom{v+m-1}{m-1}$.*

(3) *In the case (2), the equality $\mathrm{e}(A)=\binom{v+m}{m}-d\binom{v+m-1}{m-1}$ holds if and only if A is Cohen-Macaulay. Moreover, in this case A is an extremal Cohen-Macaulay algebra in the sense of* [22], *i.e., A has a linear resolution.*

PROOF. (1) Put $A=S/I$, $S=k[X_1,\dots,X_v]$, $v=\mathrm{emb}(A)$, and $r=i(A)-1$. Then the canonical mapping $S_i\to A_i$ is an isomorphism if $0\leqq i\leqq r$. Take a minimal reduction B of A. Then, since A is not regular, we have $B_1\subsetneqq A_1$. Therefore $B_1A_{r-1}=B_1\otimes_k A_{r-1}\subsetneqq A_1\otimes_k A_{r-1}\subset A_r$. Hence $B_1A_{r-1}\neq A_r$. This implies that $\gamma_B(A)\geqq r$. Therefore $\delta(A)\geqq r$.

(2) Let B be a minimal reduction of A such that $\delta(A)=\gamma_B(A)$. Then $B_1A_i=A_{i+1}$ for all $i\geqq m$. Hence $A/B_1A=k\oplus(A_1/B_1)\oplus(A_2/B_1A_1)\oplus\cdots\oplus(A_m/B_1A_{m-1})$. Since $\delta(A)\leqq i(A)-1$, we have $A_1\otimes_k A_i=A_1A_i$ and $B_1\otimes_k A_i=B_1A_i$ for $1\leqq m-1$. Put $\mathfrak{N}=B_+$. Then $\mu_{B_{\mathfrak{N}}}(A_{\mathfrak{N}})=\dim_k(A/B_1A)=\sum_{i=0}^m\binom{v+i-1}{i}-d\sum_{i=0}^{m-1}\binom{v+i-1}{i}=\binom{v+m}{m}-d\binom{v+m-1}{m-1}$. Since $\mathrm{e}(A)=\mathrm{rank}_B(A)\leqq\mu_{B_{\mathfrak{N}}}(A_{\mathfrak{N}})$, we get the desired inequality.

(3) The equality holds in (2)$\Leftrightarrow\mathrm{rank}_{B_{\mathfrak{N}}}(A_{\mathfrak{N}})=\mu_{B_{\mathfrak{N}}}(A_{\mathfrak{N}})\Leftrightarrow A_{\mathfrak{N}}$ is $B_{\mathfrak{N}}$-free $\Leftrightarrow A_{\mathfrak{N}}$ is Cohen-Macaulay $\Leftrightarrow A$ is Cohen-Macaulay. Moreover, in this case, we have $\mathrm{reg}(A)=\delta(A)=i(A)-1$, i.e., A is an extremal Cohen-Macaulay algebra (cf. [17]). Q.E.D.

COROLLARY 5.5. *Assume that $\delta(A)=1$. Then we have* $\mathrm{emb}(A)\geqq\mathrm{e}(A)+\dim(A)$

-1, *and the equality* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-1$ *holds if and only if A is Cohen-Macaulay. If A is Buchsbaum, then we have* $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)-1+\mathrm{I}(A)$.

Let R be a Cohen-Macaulay local ring with $\mathrm{emb}(R)=\mathrm{e}(R)+\dim(R)-1$. Then $\delta(\mathfrak{m})\leqq 1$ and $G(\mathfrak{m})$ is Cohen-Macaulay by Corollary 5.5 (cf. [19]). If A is a homogeneous integral domain over an algebraically closed field, then $\delta(A)\leqq 1$ if and only if $\mathrm{reg}(A)\leqq 1$, and in this case A is Cohen-Macaulay (cf. [1]).

EXAMPLE 5.6. (1) Put $A=k[X_1,\dots,X_v]/((X_1,\dots,X_v)^r\cap(X_v))=k[x_1,\dots,x_v]$ ($v\geqq 2, r\geqq 2$). Then $k[x_1,\dots,x_{v-1}]$ is a minimal reduction of A and we have $\mathrm{emb}(A)=v$, $\dim(A)=v-1$, $\mathrm{e}(A)=1$, $\mathrm{depth}(A)=0$ and $\delta(A)=\mathrm{reg}(A)=r-1$. In particular, if $A=k[X, Y]/Y(X, Y)$, then $\delta(A)=\mathrm{reg}(A)=1$ and $\mathrm{emb}(A)=\mathrm{e}(A)+\dim(A)$.

(2) Assume that $\dim(A)>0$ and $r>\mathrm{reg}(A)$. Put $M=(A/A_+^{r+1})_+=A_1\oplus\cdots\oplus A_r$ and $A'=A\times M$. Then A' is a homogeneous k-algebra with $\delta(A')=\delta(A)$ and $\mathrm{reg}(A')=r$. In particular, if A is one-dimensional Cohen-Macaulay algebra with $\delta(A)=1$ and let $r\geqq 2$ be an integer, then we have $\delta(A')=1<r=\mathrm{reg}(A')$.

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