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Any group is represented by an outerautomorphism group

Dedicated to Professor Hirosi Toda on his 60th birthday

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1. Introduction

Recently S. Kojima showed in [5] that any finite group G is isomorphic to the outerautomorphism (class) group $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ of some discrete subgroup π of $PSL(2, \boldsymbol{C})$ (cf. §5). The purpose of this paper is to show the following theorem.

THEOREM. *For any group G there is a group π such that G is isomorphic to the outerautomorphism group* $\mathrm{Out}(\pi)$ *of* π.

In terms of the homotopy theory our theorem says that G is isomorphic to the group $\mathscr{E}(K(\pi, 1))$ of free homotopy classes of homotopy self-equivalences, with multiplication by the composition, of the Eilenberg-MacLane space $K(\pi, 1)$ (Corollary 4.2). In [3] J. de Groot showed that any group G is isomorphic to the homeomorphism group $\mathrm{Homeo}(X)$ of some metric space X. This implies also that G is isomorphic to the (outer)automorphism group of the ring of real-valued continuous functions on X. Moreover, we can see that $\mathrm{Homeo}(X) = \mathscr{E}(X)$ in his specific example (cf. §4).

It is easy to see that there is no group π whose automorphism group $\mathrm{Aut}(\pi)$ is the cyclic group of odd order $\neq 1$ (cf. [3]). In contrast with this we may ask if there is any based space X such that G is isomorphic to the group $\mathscr{E}_0(X)$ of based homotopy classes of based homotopy self-equivalences of X. In [9] S. Oka showed that $\mathscr{E}_0(X_0)$ is a cyclic group of order n for some 1-connected finite CW complex X_0 unless $n \equiv 8 \bmod 16$.

This paper is an outcome of the talk presented on the occasion of the Oka memorial symposium held at Kyushu University on October 29–31, 1986. The author would like to express his hearty thanks to Professor Toru Maeda at Kansai University for informing him of de Groot's paper and helping him to correct and improve the first version of the result by teaching about the fundamental group of a graph of groups. The author is also deeply grateful to Professor Eiichi Bannai at Ohio State University who taught him a proof of the

key lemma (Proposition 3.2) by a personal letter. Without these helps this paper could not be completed.

2. Directed graph and fundamental group of a graph of groups

A graph Γ consists of the set $V(\Gamma)$ of vertices and the set $E(\Gamma)$ of edges with maps $^{-}$: $E(\Gamma) \to E(\Gamma)$ and (ι, τ): $E(\Gamma) \to V(\Gamma) \times V(\Gamma)$ satisfying $\bar{\bar{e}} = e$, $\iota(e) = \tau(\bar{e})$ and $\tau(e) = \iota(\bar{e})$. An orientation of Γ is given by a subset $E_+(\Gamma)$ of $E(\Gamma)$ with $E(\Gamma) = E_+(\Gamma) \cup \bar{E}_+(\Gamma)$ (disjoint union). A graph with orientation is called a directed graph. An automorphism of the directed graph is a pair of bijections of $V(\Gamma)$ and $E_+(\Gamma)$ to themselves commuting with the map $(\iota, \tau)|E_+(\Gamma)$. To prove our theorem we quote the following result of de Groot with a rough proof.

THEOREM 2.1 (Corollary to Theorem 6 of de Groot [3, p. 96]). *Any group G is isomorphic to the automorphism group of some connected directed graph Γ.*

PROOF. We take first the Cayley graph $\Gamma(G)$ associated to a system $\{g_\alpha\}_{\alpha \in A}$ of generators of G: $V(\Gamma(G)) = G$ and $E_+(\Gamma(G)) = \{(g, g_\alpha); g \in G, \alpha \in A\}$ with $\iota(g, g_\alpha) = g$ and $\tau(g, g_\alpha) = gg_\alpha$. Then, the edges of the Cayley graph have colors corresponding to the generators and it is easy to see that G is isomorphic to the color and orientation preserving automorphism group of $\Gamma(G)$. So, it suffices to replace the directed colored edges with the distinct rigid graphs color by color. A rigid graph is constructed from a directed edge as follows and its (orientation disregarding) automorphism group is the identity. Let $e = [a, b]$, which means $\iota(e) = a$ and $\tau(e) = b$, be the given edge. Take three edges $[a, p]$, $[p, q]$ and $[q, b]$ with the new vertices p and q instead of the edge $[a, b]$. At the first step we take a cardinal $\mathbf{m} > |A|$. From q we draw new $\mathbf{m}$ directed edges $[q, q_{\alpha_1}]$. At the n-th step we draw new $\mathbf{m}_{\alpha_1 \dots \alpha_n}$ directed edges $[q_{\alpha_1 \dots \alpha_n}, q_{\alpha_1 \dots \alpha_n \alpha_{n+1}}]$ from each $q_{\alpha_1 \dots \alpha_n}$. We need that the cardinals $\mathbf{m}_{\alpha_1 \dots \alpha_n}$ are different from each other and from all the cardinals previously defined. The rigid graph is defined after a countable number of steps. The rigidity of the resulting graph is easily deduced. Repeat this construction for the other edges with different cardinals when the color is different. q.e.d.

Now we review the notion of a graph of groups and its fundamental group owing to Bass-Serre. (See [10].) Let Γ be a graph. A graph of groups $(\mathscr{G}, \Gamma)$ consists of the vertex groups $\mathscr{G}_v$ for $v \in V(\Gamma)$ and the edge groups $\mathscr{G}_e = \mathscr{G}_{\bar{e}}$ for $e \in E(\Gamma)$ with the monomorphisms $\mathscr{G}_e \to \mathscr{G}_{\tau(e)}$ denoted by $a \to a^e$. Group $F(\mathscr{G}, \Gamma)$ is defined by the group generated by the groups $\mathscr{G}_v$ ($v \in V(\Gamma)$) and the elements e ($e \in E(\Gamma)$) with relations $\bar{e} = e^{-1}$ and $ea^e e^{-1} = a^{\bar{e}}$ if $e \in E(\Gamma)$ and $a \in \mathscr{G}_e$. Let $v_0 \in V(\Gamma)$. Then, the fundamental group $\pi_1(\mathscr{G}, \Gamma, v_0)$ is defined by the subgroup of $F(\mathscr{G}, \Gamma)$ generated by (the image of) the elements

$$a_0 e_1 a_1 e_2 \dots e_n a_n \quad \text{with} \quad v_n = v_0\,, \quad v_i = \iota(e_{i+1}) = \tau(e_i) \quad \text{and} \quad a_i \in \mathscr{G}_{v_i}\,.$$

Let T be a maximal tree of Γ and $\pi_1(\mathscr{G}, \Gamma, T)$ is the group generated by the groups $\mathscr{G}_v$ and elements t_e with relations $t_{\bar{e}} = t_e^{-1}$, $t_e a^e t_e^{-1} = a^{\bar{e}}$ $(e \in E(\Gamma)$, $a \in \mathscr{G}_e)$ and $t_e = 1$ $(e \in E(T))$. Note that the canonical projection $p\colon F(\mathscr{G}, \Gamma) \to \pi_1(\mathscr{G}, \Gamma, T)$, defined by $p(e) = t_e$, induces an isomorphism of $\pi_1(\mathscr{G}, \Gamma, v_0)$ onto $\pi_1(\mathscr{G}, \Gamma, T)$ for any maximal tree T (Proposition 20 of [10, p. 63]). In particular, the isomorphism class of the fundamental group does not depend on the choice of the vertex v_0 when Γ is connected. Moreover, the natural maps $\mathscr{G}_v \to \pi_1(\mathscr{G}, \Gamma, T)$ are monomorphisms. When Γ is a finite graph, this is so because $\pi_1(\mathscr{G}, \Gamma, T)$ is a generalized HNN extension, that is, a group constructed by a repeat of either HNN extension or free product with amalgamation. We shall use the same notation $\mathscr{G}_v$ for its image in $\pi_1(\mathscr{G}, \Gamma, T)$.

Recall that the fundamental group is called an HNN extension or a free product with amalgamation when the graph is the loop $\{v, e\}$ with $\iota(e) = \tau(e) = v$ or the segment $\{v_1, e, v_2\}$ with $\iota(e) = v_1$ and $\tau(e) = v_2$ respectively. A sequence g_0, t^{ε_1}, g_1, $\dots$, t^{ε_n}, g_n $(n \geqq 0,\ g_i \in \mathscr{G}_v$ and $\varepsilon_i = \pm 1)$ in the HNN extension $\langle \mathscr{G}_v, t = t_e;\ t a^e t^{-1} = a^{\bar{e}}$ for $a \in \mathscr{G}_e\rangle$, with a factor $\mathscr{G}_v$, is called reduced if there is no consecutive subsequence t, g_i, t^{-1} with $g_i = a_i^e$ and $a_i \in \mathscr{G}_e$ or t^{-1}, g_i, t with $g_i = b_i^{\bar{e}}$ and $b_i \in \mathscr{G}_e$. A reduced sequence is called cyclically reduced if t^{ε_n}, $g_n g_0$, t^{ε_1} is also reduced. Similarly, a sequence $g_0, g_1, \dots, g_n$ $(n \geqq 0,\ g_i \in \mathscr{G}_{v_1}$ or $\in \mathscr{G}_{v_2})$ in the free product with amalgamation $\langle \mathscr{G}_{v_1}, \mathscr{G}_{v_2};\ a^e = a^{\bar{e}}$ for $a \in \mathscr{G}_e\rangle$, with factors $\mathscr{G}_{v_1}$ and $\mathscr{G}_{v_2}$, is called reduced if no successive g_i and g_{i+1} are contained in the same factor. A reduced sequence is called cylically reduced if g_n, g_0 is also reduced. In both cases we know that the product $g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n$ or $g_0 g_1 \cdots g_n$ of the elements in a reduced sequence is not trivial in the fundamental group if $n \geqq 1$ (Britton's lemma cf. [6]). Any element can be expressed as a product of the elements in a reduced sequence. An element which is a product of the elements in a cyclically reduced sequence will be called a cyclically reduced element.

LEMMA 2.2. *Let $\mathscr{G}^*$ be either an* HNN *extension $\langle G, t;\ tat^{-1} = \varphi(a)$ for $a \in H\rangle$ with $G \supset H$ and $\varphi\colon H \hookrightarrow G$ or a free product with amalgamation $\langle G_1$, $G_2;\ a = \varphi(a)$* for *$a \in H\rangle$ with $G_1 \supset H$ and $\varphi\colon H \hookrightarrow G_2$. The image of H (or $\varphi(H)$) in $\mathscr{G}^*$ is also denoted by H (or $\varphi(H)$ resp.).*

(1) *Any finite subgroup F of $\mathscr{G}^*$ is contained in a conjugate of some factor.*

(2) *If a cyclically reduced element g of $\mathscr{G}^*$ is conjugate to an element h of H, then g is in some factor and there is a sequence $h, h_1, \dots, h_k, g$ where every h_i is in H (or possibly $\varphi(H)$ in the case of* HNN *extension) and consecutive terms of the sequence are conjugate in a factor (or possibly by $t^{\pm 1}$ in the case of* HNN

extension). Moreover, if $H_0 = c^{-1}Xc \subset H$ *for a subset* X *of* $\mathscr{G}^*$ *consisting of cyclically reduced elements, then there is a sequence of subsets* $H_0, H_1, \dots, H_n$, X *where* $H_i \subset H$ *(or possibly* $H_i \subset \varphi(H)$ *in the case of* HNN *extension) and consecutive terms of the sequence are conjugate in a factor (or possibly by* $t^{\pm}$ *in the case of* HNN *extension) and* c *is a product of the elements which give such conjugations. In particular,* X *itself is contained in a factor.*

(3) *If a cyclically reduced element* g *of* $\mathscr{G}^*$ *is conjugate to an element* g' *of some factor by an element* c*, but not in a conjugate of* H*, then* g, g' *and* c *are in the same factor.*

Before giving a proof of Lemma 2.2 we state a corollary applied to a generalized HNN extension.

LEMMA 2.3. *Let* $(\mathscr{G}, \Gamma)$ *be a graph of groups and consider the fundamental group* $\pi = \pi_1(\mathscr{G}, \Gamma, T)$ *of* $(\mathscr{G}, \Gamma)$ *associated to a fixed maximal tree* T *of* Γ.

(1) *Any finite subgroup* F *of* π *is contained in some conjugate of* $\mathscr{G}_v$ *for some* $v \in V(\Gamma)$.

(2) *Assume that the union* U_v *of the conjugates of* $\bigcup_{\tau(e)=v} \mathscr{G}_e$ *in* $\mathscr{G}_v$ *is not full in* $\mathscr{G}_v$*. Then, any element of* $\mathscr{G}_v - U_v$ *is not contained in any conjugate of* $\mathscr{G}_{v'}$ *for* $v'\ (\neq v) \in V(\Gamma)$.

PROOF. Since F is contained in a finite generalized HNN extension, we have (1) by applying (1) of Lemma 2.2 repeatedly. The statement (2) is also a consequence of (3) of Lemma 2.2. q.e.d.

PROOF OF LEMMA 2.2. Any element of finite order is contained in a conjugate of some factor by Britton's lemma ([6], [7]). The Britton's lemma shows also that the product of two elements of conjugates of different factors or different conjugates of the same factor cannot be in a conjugate of a factor (assuming that any factor is not contained in another factor). This shows (1). The first half of the statement (2) and the statement (3) for the free product with amalgamation are Theorem 4.6 (i) and (ii) of Magnus-Karrass-Solitar [7, p. 212]. We will give a proof in the case of HNN extension. To prove (2) suppose that $g_0, t^{\varepsilon_1}, g_1, \dots, t^{\varepsilon_n}, g_n$ is a reduced sequence for c and note that $X \subset cHc^{-1}$. If $n = 0$, then $c = g_0$ and the sequence $H_0 = g_0^{-1}Xg_0$, X is of the required type, since $X \subset g_0 H g_0^{-1} \subset G$. Let $n \geqq 1$. We use the notation $\varphi_m(H)$ which stands for H if $\varepsilon_m = 1$ and $\varphi(H)$ if $\varepsilon_m = -1$ for $m > 0$ and $\varphi_0(H) = H$ and put $c_m = g_m t^{\varepsilon_{m+1}} \dots t^{\varepsilon_n} g_n$ for $0 \leqq m \leqq n$ with $c_0 = c$. Suppose there is a largest integer $q \geqq 0$ such that $c_q H_0 c_q^{-1}$ is not contained in $\varphi_q(H)$. Then, either $q = n$ or $c_j H_0 c_j^{-1} \subset \varphi_j(H)$ for $j > q$. If now $q > 0$, then some element

$$g = chc^{-1} = g_0 \dots t^{\varepsilon_q}(c_q h c_q^{-1})t^{-\varepsilon_q} \dots g_0^{-1}\,,$$

with $h \in H_0$, cannot be a cyclically reduced element. Hence, $q = 0$ or there is no q. In both cases the sequence

$$H_0,\ g_n H_0 g_n^{-1} = c_n H_0 c_n^{-1}, \ldots, c_m H_0 c_m^{-1},\ t^{\varepsilon_m} c_m H_0 c_m^{-1} t^{-\varepsilon_m}, \ldots, cH_0 c^{-1} = X$$

is of the required type and $X \subset G$. Since a proof of the second statement for the free product with amalgamation can be carried out in the same way, we have (2). To prove (3) note that $g = cg'c^{-1}$ with $g' \in G$ and suppose that g_0, t^{ε_1}, g_1, ..., t^{ε_n}, g_n is a reduced sequence for c. If $n = 0$, then $c = g_0$, g' and $g = g_0 g' g_0^{-1}$ are contained in G. Suppose now $n \geqq 1$. Then, in order that $g = g_0 \ldots t^{\varepsilon_n} g_n g' g_n^{-1} t^{-\varepsilon_n} \ldots g_0^{-1}$ be a cyclically reduced element, $g_n g' g_n^{-1}$ should be contained in $\varphi_n(H)$. But this contradicts the assumption. Hence, $n = 0$ and we complete the proof. q.e.d.

3. Proof of Theorem in the introduction

We shall prove Theorem 3.1 and Proposition 3.2 which imply the theorem stated in the introduction.

Now let Γ be a directed graph. We take a barycentric subdivision Γ' of the directed graph Γ. More precisely we denote:

(1) $V(\Gamma') = V(\Gamma) \cup V_E(\Gamma')$ with $V_E(\Gamma') = E_+(\Gamma)$,

(2) $E_+(\Gamma') = E_1(\Gamma') \cup E_2(\Gamma')$ (disjoint union) with bijections $E_1(\Gamma') \cong E_2(\Gamma') \cong E_+(\Gamma)$ and,

(3) for $e \in E_+(\Gamma)$ we have $e_i \in E_i(\Gamma')$ $(i = 1, 2)$ with $\iota(e_1) = \iota(e)$, $\tau(e_1) = \iota(e_2) = e$ ($= \bar{e}$ as an element of $V_E(\Gamma')$) and $\tau(e_2) = \tau(e)$.

Let K_1 and K_2 are common subgroups of groups K and $\tilde{K}$. Then, a graph of groups $(\mathscr{G}, \Gamma')$ is defined by

$$\mathscr{G}_v = \begin{cases} K & \text{if} \quad v \in V(\Gamma) \\ \tilde{K} & \text{if} \quad v \in V_E(\Gamma') \end{cases}, \qquad \mathscr{G}_e = \begin{cases} K_1 & \text{if} \quad e \in E_1(\Gamma') \\ K_2 & \text{if} \quad e \in E_2(\Gamma') \end{cases}$$

and the monomorphisms $\mathscr{G}_e \to \mathscr{G}_{\tau(e)}$ are the inclusions.

THEOREM 3.1. *Let Γ be a connected directed graph and Γ' its barycentric subdivision. Let π denote the fundamental group $\pi_1(\mathscr{G}, \Gamma', v_0)$ of a graph of groups $(\mathscr{G}, \Gamma')$ defined above. Then, the outerautomorphism group* Out (π) *of the group π is isomorphic to the automorphism group* Aut (Γ) *of the directed graph Γ, provided that K, $\tilde{K}$, K_1 and K_2 satisfy the following conditions*:

(a) *K and $\tilde{K}$ are finite groups and they are not isomorphic to each other.*

(b) Out $(K) = \{id\}$.

(c) *$N(K_i, K) = K_i$ and $N(K_i, \tilde{K}) = K_i$ for $i = 1$ and 2, where $N(H, G)$ is the normalizer of H in G.*

(d) *Some element of* K_1 *(or* K_2*) is not conjugate to any element of* K_2 *(or* K_1 *resp.) in* K *and in* $\tilde{K}$.

(e) *The union of conjugates of* K_1 *and* K_2 *is not full either in* K *or in* $\tilde{K}$.

(f) *Center* $(K_1) =$ *Center* $(K_2) = \{1\}$.

(g) *Let* α *be an element of* $\mathrm{Aut}(K)$ *or* $\mathrm{Aut}(\tilde{K})$. *Then,* $\alpha | K_i = id$ *implies* $\alpha = id$ *for* $i = 1$ *and* 2.

PROPOSITION 3.2 (Bannai [1]). *There is a quadruple* $(K, \tilde{K}, K_1, K_2)$ *satisfying the conditions in Theorem* 3.1. *In fact, we can take* $K = \mathrm{Aut}(M_{12})$, $\tilde{K} = \mathrm{Aut}(M_{22})$, $K_1 \cong \mathrm{Aut}(A_6)$ *and* $K_2 \cong \mathrm{Aut}(PSL(2, \mathbf{Z}_{11}))$, *where* M_n *denotes the Mathieu group of degree* n *and* A_k *is the alternating group on* k *elements. This quadruple satisfies also the following conditions* (b′) *and* (d′) *stronger than the conditions* (b) *and* (d):

(b′) $\mathrm{Out}(K) = \mathrm{Out}(\tilde{K}) = \{id\}$.

(d′) *Some element of* K_1 *(or* K_2*) is not conjugate to any element of a group isomorphic to* K_2 *(or* K_1 *resp.) in any ambient group.*

PROOF OF PROPOSITION 3.2. We use the Atlas [2]. According to the Atlas (pp. 31–33) $K = \mathrm{Aut}(M_{12})$ is a non-trivial split extension of M_{12} by $\mathbf{Z}_2$. We take one of the maximal subgroup $\cong M_{10}{:}2 \cong A_6 \cdot 2^2$ (in the notation of the Atlas) of M_{12} $(\subset K)$ as K_1. K_1 is isomorphic to $\mathrm{Aut}(A_6)$ by these isomorphisms. Let K_2 be a maximal subgroup $\cong L_2(11){:}2$ (in the notation of the Atlas) of K such that $K_2 \cap M_{12}$ is isomorphic to $PSL(2, \mathbf{Z}_{11})$ $(= L_2(11)$ in the notation of the Atlas) and K_2 is a non-trivial split extension of $K_2 \cap M_{12}$ by $\mathbf{Z}_2$. Since $\mathrm{Out}(PSL(2, \mathbf{Z}_{11})) \cong \mathbf{Z}_2$ and the extension is non-trivial, the isomorphism class of K_2 is uniquely determined. Similarly by the Atlas (p. 39) $\tilde{K} = \mathrm{Aut}(M_{22})$ is a non-trivial split extension of M_{22} by $\mathbf{Z}_2$. $\tilde{K}_1$ is a maximal subgroup $\cong M_{10}{:}2 \cong A_6 \cdot 2^2$ of $\tilde{K}$ and $\tilde{K}_2$ is a maximal subgroup $\cong L_2(11){:}2$ of $\tilde{K}$. We know that $\tilde{K}_1 \cong K_1$ and $\tilde{K}_2 \cong K_2$ by these data. We note here that $|K| = 2^7 3^3 5{\cdot}11$, $|\tilde{K}| = 2^8 3^2 5{\cdot}7{\cdot}11$, $|K_1| = 2^5 3^2 5$ and $|K_2| = 2^3 3{\cdot}5{\cdot}11$. This shows the condition (a). We prove now the other conditions. (b′) Since $\mathrm{Aut}(M_{12})$ is a semi-direct product of M_{12} with $\mathbf{Z}_2$ and $\mathrm{Aut}(\mathbf{Z}_2) = \{id\}$, we see that $\mathrm{Aut}(\mathrm{Aut}(M_{12})) = \mathrm{Aut}(M_{12})$. Hence, $\mathrm{Out}(\mathrm{Aut}(M_{12})) = \{id\}$. The same is true for $\mathrm{Aut}(M_{22})$. It is known also that $\mathrm{Aut}(\mathrm{Aut}(G)) = \mathrm{Aut}(G)$ for any non-abelian simple group G. (c) By the construction K_2 is a maximal subgroup of K and hence $N(K_2, K)$ is K_2 or K. But the latter means that K_2 is a normal subgroup of K. Since M_{12} is the unique normal subgroup of K, we get $N(K_2, K) = K_2$. The same argument implies that $N(\tilde{K}_i, \tilde{K}) = \tilde{K}_i$ for $i = 1$ and 2. By the Atlas (p. 33) M_{12} has another maximal subgroup isomorphic to K_1 which is not conjugate to K_1 in M_{12} but conjugate to K_1 in K. Hence, $N(K_1, K)$ is contained in M_{12}. So, $N(K_1, K)$ is K_1, because K_1 is maximal but not normal in M_{12}. (d′) An element of order 11 of K_2 cannot be conjugate to any element of a group isomorphic to K_1, because 11 does not divide $|K_1|$. By the Atlas (p. 5 and p. 7) K_1 has an element of order 8 and K_2 does not. So, such an element cannot be conjugate to any element of a group isomorphic to K_2. (e) Since $\tilde{K}$ has an element of order 7 and 7 does not divide $|K_1|$ and $|K_2|$, such an element is not contained in the union of the conjugates of $\tilde{K}_1$ and $\tilde{K}_2$. By the Atlas (p. 33) $K - M_{12}$ has an element of order 6. If such an element is conjugate to an element g of $K_1 \cup K_2$, g must be contained in $K_2 - K_2 \cap M_{12}$ because M_{12} is normal in K. But there is no element of order 6 in $K_2 - K_2 \cap M_{12} \cong \mathrm{Aut}\,(L_2(11)) - L_2(11)$ by the Atlas (p. 7). (f) Since $K_i \supset A_5$ and $\mathit{Center}\,(A_5) = \{1\}$, $\mathit{Center}\,(K_i) = \{1\}$ for $i = 1$ and 2. (g) This comes from (b′), (c) and (f). Every automorphism of K is an inner-automorphism and $N(K_1, K) = K_1$. So, $\alpha(K_1) = K_1$ means that α is the conjugation by an element g of K_1. But $\alpha|K_1 = id$ and $\mathit{Center}\,(K_1) = \{1\}$ imply $g = 1$. So, $\alpha = id$. The argument applies to the other cases, too. The proof is outlined by Bannai [1] but to the description of the present proof only the author is responsible. q.e.d.

We start now a proof of Theorem 3.1.

We take a maximal tree T by applying the Zorn lemma. We take a maximal tree T' of a barycentric subdivision Γ' of Γ so that T' contains the barycentric subdivision of T. We consider that $\pi = \pi_1(\mathscr{G}, \Gamma', T')$ hereafter.

Take an $\alpha \in \mathrm{Aut}\,(\pi)$. Suppose $|K| \leqq |\tilde{K}|$. (Otherwise argue about $V(\Gamma)$ before $V_E(\Gamma')$.) For $e \in V_E(\Gamma')$ we have an $e' \in V(\Gamma')$ and a $g \in \pi$ such that $\alpha(\mathscr{G}_e) \subset g\mathscr{G}_{e'}g^{-1}$ by (1) of Lemma 2.3. Since $\tilde{K} \cong \mathscr{G}_e \subset \alpha^{-1}(g\mathscr{G}_{e'}g^{-1})$, we see that $e' \in V_E(\Gamma')$. Since $\alpha^{-1}(g\mathscr{G}_{e'}g^{-1}) = \alpha^{-1}(g)\alpha^{-1}(\mathscr{G}_{e'})(\alpha^{-1}(g))^{-1}$, the correspondence $e \to e'$ gives a bijection of $V_E(\Gamma')$ by (2) of Lemma 2.3 and the condition (e). For $v \in V(\Gamma) \subset V(\Gamma')$ we have an $e' \in V_E(\Gamma')$ or a $v' \in V(\Gamma)$ and a $g \in \pi$ such that $\alpha(\mathscr{G}_v) \subset g\mathscr{G}_{e'}g^{-1}$ or $\alpha(\mathscr{G}_v) \subset g\mathscr{G}_{v'}g^{-1}$ respectively. But the former case cannot be possible by (2) of Lemma 2.3, the condition (e) and the fact that $\alpha^{-1}(g\mathscr{G}_{e'}g^{-1})$ is a conjugate of $\mathscr{G}_e$ with $e = \alpha^{-1}(e')$. So, by the same reasoning as before the correspondence $v \to v'$ gives also a bijection of $V(\Gamma)$. To prove that α induces an automorphism of Γ we have to show that $\tau(\alpha(e)) = \alpha(\tau(e))$ for $e \in V_E(\Gamma')$.

Fix an edge $e \in E_+(\Gamma)$ (or $\in \bar{E}_+(\Gamma)$). By composing some conjugation we may assume that $\alpha(\mathscr{G}_e) = \mathscr{G}_{\alpha(e)}$ and α coincides with the identity with respect to fixed identifications of $\mathscr{G}_e$ and $\mathscr{G}_{\alpha(e)}$ with $\tilde{K}$. We know that $\alpha(\mathscr{G}_{\tau(e)}) = g\mathscr{G}_{\alpha(\tau(e))}g^{-1}$ for some $g \in \pi$.

Claim 1. *Suppose that* $\mathscr{G}_{\alpha(e)} \cap g\mathscr{G}_{\alpha(\tau(e))}g^{-1} \supset K_2$ (*or* K_1 *resp.*) *in* $\mathscr{G}_{\alpha(e)} = \tilde{K}$. *Then,* $\alpha(\tau(e)) = \tau(\alpha(e))$.

PROOF OF CLAIM 1. Let a be an element of K_2 (or K_1 resp.) $\subset \mathscr{G}_{\alpha(e)} = \tilde{K}$ which is not conjugate to any element of K_1 (or K_2 resp.) in K and $\tilde{K}$. Such an element exists by the condition (d). By the assumption $g^{-1}ag \in \mathscr{G}_{\alpha(\tau(e))}$. So, both a and $g^{-1}ag$ are cyclically reduced elements. We consider a subgroup $\pi(\Gamma_1)$ of π containing a and g which is the fundamental group of a graph of groups over a finite connected subgraph Γ_1 of Γ'. By retaking T' we may assume that $\mathscr{G}_{e'} = K_1$ (or K_2 resp.) for $e' \in E(\Gamma') - E(T')$. Let $k \geqq 1$ and assume that if $k \geqq 2$ then $\Gamma_{k'}$ $(1 \leqq k' < k)$ are already defined. If Γ_k contains a cycle, we take an edge e_k on the cycle and put $\Gamma_{k+1} = \Gamma_k - e_k$. Otherwise we take an edge e_k and put $\Gamma_{k+1} =$ the connected component of $\Gamma_k - e_k$ containing $\alpha(e)$ and $\Gamma'_{k+1} =$ the other component. In both cases we may take e_k so that $\mathscr{G}_{e_k} = K_1$ (or K_2 resp.). Then, $\pi(\Gamma_k)$ is either an HNN extension $\langle G = \pi(\Gamma_{k+1}),$ $t;\ tat^{-1} = \varphi(a)$ for $a \in H\rangle$ or a free product with amalgamation $\langle G = \pi(\Gamma_{k+1}),$ $G' = \pi(\Gamma'_{k+1});\ a = \varphi(a)$ for $a \in H\rangle$, where H is an edge group $\mathscr{G}_{e_k}$ embedded in $\mathscr{G}_{\tau(e_k)}$ and φ is an into-isomorphism induced from $\mathscr{G}_{e_k} = \mathscr{G}_{\bar{e}_k}$. We may assume that both a and $g_k^{-1}ag_k$ are cyclically reduced with $g_1 = g$. Now we have two cases: either (i) a is conjugate to an element of H in $\pi(\Gamma_k)$ or (ii) otherwise. (If we assume the condition (d′), we choose $a \in K_2$ (or K_1 resp.) not conjugate to any element of a group isomorphic to K_1 (or K_2 resp.) and we see from the beginning that the case (i) does not occur by the choice of e_k with $H = \mathscr{G}_{e_k} \cong K_1$ (or K_2 resp.).) In case (i) we see that a is conjugate to an element of H (or possibly $\varphi(H)$ in the case of HNN extension) by an element g_{k+1} of G by (2) of Lemma 2.2. In case (ii) $g_k^{-1}ag_k$ is contained in G and $g_k \in G$ by (3) of Lemma 2.2. Define $g_{k+1} = g_k$ in the case (ii). So, in both cases we can argue further on $\pi(\Gamma_{k+1})$ unless Γ_{k+1} contains no edges with group K_1 (or K_2 resp.). Since Γ_1 is a finite graph, we see that at some n the subgraph Γ_n consists of one vertex $v \in V(\Gamma)$ and a finite number of vertices $v_i \in V_E(\Gamma')$ and edges $e_i \in E(\Gamma')$ with $\mathscr{G}_{e_i} = K_2$ (or K_1 resp.) connecting v and v_i. If every step passes through the case (ii), then $g_k = g$ for $1 \leqq k \leqq n$ and g is contained in $\pi(\Gamma_n)$. This implies easily that $\alpha(e)$ is one of v_i and v is $\alpha(\tau(e))$, that is, $\tau(\alpha(e)) = \alpha(\tau(e))$. If the last step passes through the case (i), then we see that a is conjugate to an element of K_1 (or K_2 resp.) in K or $\tilde{K}$ by (2) of Lemma 2.2. This contradicts the choice of a. In the remaining cases we can change the order of the steps so that the last step passes through the case (i). q.e.d.

So, we get a map ρ: Aut$(\pi) \to$ Aut(Γ), which is immediately seen to be a homomorphism of groups with respect to the compositions. To show that ρ is surjective we consider for a while $\pi = \pi_1(\mathscr{G}, \Gamma', v_0)$ with $v_0 \in V(\Gamma)$. Take a $\beta \in$ Aut(Γ). Choose a path γ from v_0 to $\beta(v_0)$ in T'. An element

$$a_0 e_1 a_1 e_2 \dots e_n a_n \quad (v_n = v_0,\ v_i = \iota(e_{i+1}) = \tau(e_i) \text{ and } a_i \in \mathscr{G}_{v_i})$$

of π is transformed to the element

$$\gamma a_0 \beta(e_1) a_1 \beta(e_2) \dots \beta(e_n) a_n \bar{\gamma}\,, \quad \text{where } a_i \text{ is considered now in } \mathcal{G}_{\beta(v_i)}\,.$$

Then, this induces an automorphism $\tilde{\beta}$ of $\pi = \pi_1(\mathcal{G}, \Gamma', v_0)$. It is not difficult to see that $\rho(\tilde{\beta}) = \beta$.

Now ρ induces a surjective homomorphism $\bar{\rho}\colon \mathrm{Out}(\pi) \to \mathrm{Aut}(\Gamma)$. Take $\alpha \in \mathrm{Aut}(\pi)$ with $\rho(\alpha) = id$. We will show that α is conjugate to the identity. Abbreviate $v_0 = 0$. Then, $\alpha(\mathcal{G}_0) = g\mathcal{G}_0 g^{-1}$ for some $g \in \pi$. Putting $\alpha_1 = g^{-1}\alpha g$ we have $\alpha_1(\mathcal{G}_0) = \mathcal{G}_0$. By the condition (b) we have an $h \in \mathcal{G}_0$ with $h\alpha_1 h^{-1} | \mathcal{G}_0 = id$. So, we define a new α by $h\alpha_1 h^{-1}$ and will show that $\alpha = id$.

Recall $\alpha \in \mathrm{Aut}(\pi)$ with $\pi = \pi_1(\mathcal{G}, \Gamma', T')$, $\rho(\alpha) = id$, $\alpha(\mathcal{G}_0) = \mathcal{G}_0$ and $\alpha | \mathcal{G}_0 = id$. If $e \in E_+(\Gamma)$ and $\iota(e) = 0$, then $\alpha | \mathcal{G}_{e_1} = id$ and $\mathcal{G}_{e_1} = K_1 \subset \tilde{K} = \mathcal{G}_e$. So, $\mathcal{G}_{e_1} \subset \alpha(\mathcal{G}_e) = g\mathcal{G}_e g^{-1}$ for some $g \in \pi$. In particular, $g\mathcal{G}_e g^{-1} \cap \mathcal{G}_{\iota(e)} \supset K_1$ in $\mathcal{G}_{\iota(e)}$.

Claim 2. (1) *Suppose that $\mathcal{G}_e \cap g\mathcal{G}_{\tau(e)} g^{-1} \supset K_2$ (or K_1) in $\mathcal{G}_e = \tilde{K}$ for an $e \in E_+(\Gamma)$ (or $\bar{E}_+(\Gamma)$ resp.) and a $g \in \pi$. Then, $g \in \mathcal{G}_{\tau(e)}$.*

(2) *Suppose that $g\mathcal{G}_e g^{-1} \cap \mathcal{G}_{\iota(e)} \supset K_1$ (or K_2) in $\mathcal{G}_{\iota(e)} = K$ for an $e \in E_+(\Gamma)$ (or $\bar{E}_+(\Gamma)$ resp.) and a $g \in \pi$. Then, $g \in \mathcal{G}_e$.*

Proof of Claim 2. (1) Let H be the edge group $\mathcal{G}_e \cap \mathcal{G}_{\tau(e)}$ isomorphic to K_2 (or K_1 resp.). Put $X = g^{-1}Hg$. Then, X is a subset of $\mathcal{G}_{\tau(e)}$ and $gXg^{-1} = H$. Note that H is the edge group on the edge $e' \in E(\Gamma')$ if and only if $\mathcal{G}_{e'} \cong K_2$ (or K_1 resp.) and e' is contiguous to $\tau(e)$. We consider a subgroup $\pi(\Gamma_1)$ of π containing H and g which is over a finite connected subgraph Γ_1 of Γ'. Using the condition (d) and Lemma 2.2 as in the proof of Claim 1 we see that g is contained in $\pi(T_{\tau(e),e})$, where $T_{\tau(e),e}$ is the tree consisting of the edges e' contiguous to $\tau(e)$ with $\mathcal{G}_{e'} \cong K_2$ (or K_1 resp.). (If we assume the condition (d′), it suffices to apply (3) of Lemma 2.2 to the edge groups isomorphic to K_1 (or K_2 resp.) repeatedly, because an $a \in X$ is not conjugate to any element of a group isomorphic to K_1 (or K_2 resp.).) Define a tree T_k by $V(T_k) = \{v, v_1, \dots, v_k\}$, $E(T_k) = \{e_1, \dots, e_k\}$, $\iota(e_i) = v$ and $\tau(e_i) = v_i$. We can identify $T_{\tau(e),e} = T_n$ for some n with $v = \tau(e)$ and consider $H \subset \mathcal{G}_v$. Applying (2) of Lemma 2.2 to $\pi(T_n) = \langle \pi(T_{n-1}), \mathcal{G}_{v_n};\ a = \varphi(a) \text{ for } a \in H \rangle$ with $gXg^{-1} = H$, we see that g is a product of elements of $N(H, \pi(T_{n-1}))$, elements of $N(H, \mathcal{G}_{v_n})$ and an element g_1 of $\pi(T_{n-1})$ with $g_1 X g_1^{-1} = H$. We may assume $g_k \in \pi(T_{n-k})$, with $g_k X g_k^{-1} = H$, are already defined. Then, by the same argument g_k is a product of elements of $N(H, \pi(T_{n-k-1}))$, elements of $N(H, \mathcal{G}_{v_{n-k}})$ and an element g_{k+1} of $\pi(T_{n-k-1})$ with $g_{k+1} X g_{k+1}^{-1} = H$. If we consider $X = H$ for a while, an element of $N(H, \pi(T_{n-k}))$ turns out a product of elements of $N(H, \pi(T_{n-k-1}))$ and elements of $N(H, \mathcal{G}_{v_{n-k}})$. Consequently, g is a product of elements of $N(H, \mathcal{G}_{v_n})$, $\dots$, elements of $N(H, \mathcal{G}_{v_1})$, elements of $N(H, \pi(T_0) = \mathcal{G}_v)$ and an element g_n of $\mathcal{G}_v$. Since $N(H, \mathcal{G}_{v_i}) = N(H, \mathcal{G}_v) = H$ by the condition (c), g is a product of elements of H and $g_n \in \mathscr{G}_{\tau(e)}$, that is, $g \in \mathscr{G}_{\tau(e)}$. This completes the proof of (1). The proof of (2) is almost the same and omitted. q.e.d.

By (2) of Claims 2 we see that $g \in \mathscr{G}_e$ and hence $g\mathscr{G}_e g^{-1} = \mathscr{G}_e$, that is, $\alpha(\mathscr{G}_e) = \mathscr{G}_e$. Since $\alpha|\mathscr{G}_{e_1} = id$, the condition (g) for $\tilde{K}$ implies $\alpha|\mathscr{G}_e = id$. Now $\mathscr{G}_e \supset \mathscr{G}_{e_2} = K_2 \subset \alpha(\mathscr{G}_{\tau(e)}) = g\mathscr{G}_{\tau(e)}g^{-1}$ for another $g \in \pi$. By (1) of Claim 2 we have $g \in \mathscr{G}_{\tau(e)}$ and hence $\alpha(\mathscr{G}_{\tau(e)}) = \mathscr{G}_{\tau(e)}$. Since $\alpha|\mathscr{G}_{e_2} = id$, the condition (g) for K implies $\alpha|\mathscr{G}_{\tau(e)} = id$. Let now $e \in E_+(\Gamma)$ and $\tau(e) = 0$. Then, applying Claim 2 for $\bar{e}$ in the same way we see that $\alpha(\mathscr{G}_e) = \mathscr{G}_e$ and $\alpha|\mathscr{G}_e = id$ and then $\alpha(\mathscr{G}_{\iota(e)}) = \mathscr{G}_{\iota(e)}$ and $\alpha|\mathscr{G}_{\iota(e)} = id$. Repeating this process from $v \in V(\Gamma)$ along the maximal tree T we see that $\alpha(\mathscr{G}_{ve}) = \mathscr{G}_{ve}$ and $\alpha|\mathscr{G}_{ve} = id$ for any $ve \in V(T') = V(\Gamma')$. What we have not yet proved is that $\alpha(t) = t$ for $t = t_e$ with $e \in E(\Gamma') - E(T')$. Put $\alpha_t = \alpha(t)$. Then, $\alpha_t a \alpha_t^{-1} = tat^{-1}$, that is, $(t^{-1}\alpha_t)a(t^{-1}\alpha_t)^{-1} = a$ for any $a \in \mathscr{G}_e$. In particular, $(t^{-1}\alpha_t)H(t^{-1}\alpha_t)^{-1} = H$ with $H = \mathscr{G}_e$.

CLAIM 3. *Let $g \in \pi$ and $H = \mathscr{G}_e = \mathscr{G}_{\tau(e)} \cap \mathscr{G}_{\iota(e)}$ for an $e \in E(\Gamma')$. Then, $gHg^{-1} = H$ implies $g \in H$.*

PROOF OF CLAIM 3. By the assumption we have $g\mathscr{G}_{\tau(e)}g^{-1} \cap \mathscr{G}_{\iota(e)} \supset H$ ($= K_1$ or K_2). So, by applying (1) of Claim 2 in case $\iota(e) \in E(\Gamma)$ and $\tau(e) \in V(\Gamma)$ and (2) of Claim 2 in case $\tau(e) \in E(\Gamma)$ and $\iota(e) \in V(\Gamma)$, we see that $g \in \mathscr{G}_{\tau(e)}$. Since we have no non-trivial normalizer of H in $\mathscr{G}_{\tau(e)}$ by the condition (c), we see that $g \in H$. q.e.d.

By Claim 3 we have $t^{-1}\alpha_t \in H$. But *Center* $(H) = \{1\}$ by the condition (f). This implies $t^{-1}\alpha_t = 1$, that is, $\alpha_t = t$. This completes the proof of Theorem 3.1 and consequently the main theorem stated in the introduction. q.e.d.

4. Remarks related to the homotopy theory

We know the following lemma. So, we get the corollary mentioned in the introduction.

LEMMA 4.1. (1) $\mathscr{E}_0(K(\pi, 1)) = \mathrm{Aut}\,(\pi)$, *and* (2) $\mathscr{E}(K(\pi, 1)) = \mathrm{Out}\,(\pi)$.

COROLLARY 4.2. *For any group G there is an Eilenberg-MacLane space $K(\pi, 1)$ such that G is isomorphic to $\mathscr{E}(K(\pi, 1))$.*

PROOF OF LEMMA 4.1. (1) The based homotopy class of a map f: $K(\pi, 1) \to K(\pi, 1)$ induces a homomorphism $\pi_1(f)$: $\pi = \pi_1(K(\pi, 1)) \to \pi$. This defines a canonical homomorphism π_1: $\mathscr{E}_0(K(\pi, 1)) \to \mathrm{Aut}\,(\pi)$. If $\alpha \in \mathrm{Aut}\,(\pi)$ is given, there is a map f_1 on the 2-skeleton of $K(\pi, 1)$ to $K(\pi, 1)$ so that $\alpha = \pi_1(f_1)$. Then, there is no obstruction to extend to f: $K(\pi, 1) \to K(\pi, 1)$ with $\alpha = \pi_1(f)$. Suppose $\pi_1(f) = \pi_1(f')$. Then, f and f' are homotopic on the

1-skeleton and the homotopy extends on the whole $K(\pi, 1)$ without obstruction. This finishes a proof of (1). (2) is a direct corollary of (1), since both the groups are the orbit spaces by the adjoint action of $\pi_1(K(\pi, 1))$. q.e.d.

Note that de Groot's example X is given by replacing each edge of a non-directed graph Γ of Theorem 2.1 with a rigid space. Here, a rigid space is defined by the property that any continuous map of the rigid space to itself is either the identity or a contraction to a point. So, it is easy to see that $\mathrm{Aut}(\Gamma) = \mathrm{Homeo}(X) = \mathscr{E}(X)$.

5. Case of finite groups

Due to Kojima [5] or Kawauchi [4] every finite group G is realized by the full isometry group $\mathrm{Iso}(M)$ of some closed hyperbolic 3-manifold M. Hence, any finite group G is represented by $\mathrm{Out}(\pi_1(M))$, since Mostow's theorem [8] says that the natural homomorphism $\mathrm{Iso}(M) \to \mathrm{Out}(\pi_1(M))$ is an isomorphism for any closed hyperbolic n-manifold M if $n \geqq 3$. Note that M is a $K(\pi_1(M), 1)$ and $\mathrm{Out}(\pi_1(M)) \cong \mathscr{E}(M)$. Note also that $\mathrm{Out}(\pi_1(M)) \cong N(\pi_1(M), PSL(2, \mathbf{C}))/\pi_1(M)$ by Mostow's theorem.

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