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Fibred Sasakian spaces with vanishing contact Bochner curvature tensor

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Introduction

There have been many attempts to clarify geometric meanings of Bochner curvature since S. Bochner [3] introduced it as a Kaehlerian analogue of conformal curvature in 1949. S. Tachibana [12] gave the expression of Bochner curvature tensor in real form, M. Matsumoto and S. Tanno [10] proved that a Kaehlerian space with vanishing Bochner curvature tensor and of constant scalar curvature is a complex space form or a locally product of two complex space forms of constant holomorphic sectional curvature c (≥ 0) and $-c$. Y. Kubo [8], I. Hasegawa and T. Nakane [5] obtained necessary conditions for a Kaehler manifold with vanishing Bochner curvature tensor to be a complex space form.

On the other hand, M. Matsumoto and G. Chūman [9] defined the contact Bochner (briefly, C-Bochner) curvature tensor in a Sasakian space and studied its properties. A Sasakian space form is a space with vanishing C-Bochner curvature tensor.

In this paper, we discuss properties of fibred Sasakian spaces with vanishing C-Bochner curvature tensor and construct an example of Sasakian space with vanishing C-Bochner curvature tensor which is not a Sasakian space form. As to notations and terminologies, we refer to the previous papers [7, 13].

Throughout this paper, the ranges of indices are as follows:

$$A, B, C, D, E = 1, 2, \dots, m\,,$$
$$h, i, j, k, l = 1, 2, \dots, m\,,$$
$$a, b, c, d, e = 1, 2, \dots, n\,,$$
$$\alpha, \beta, \gamma, \delta, \varepsilon = n + 1, \dots, n + p = m\,.$$

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§ 1. Preliminaries

Let $\{\tilde{M}, M, \tilde{g}, \pi\}$ be a fibred Riemannian space, that is, $\{\tilde{M}, \tilde{g}\}$ is an m-dimensional total space with projectable Riemannian metric $\tilde{g}$, M an

n-dimensional base space, and $\pi: \tilde{M} \to M$ a projection with maximal rank n. The fibre passing through a point $\tilde{P}$ in $\tilde{M}$ is denoted by $\overline{M}$, and it is p-dimensional, $n + p = m$.

We take coordinate neighborhoods $(\tilde{U}, z^h)$ in $\tilde{M}$ and (U, x^a) in M such that $\pi(\tilde{U}) = U$, then the projection π is expressed by equations

$$x^a = x^a(z^h)\,, \tag{1.1}$$

with Jacobian $(\partial x^a/\partial z^i)$ of maximum rank n. Take a fibre $\overline{M}$ such that $\overline{M} \cap \tilde{U} \neq \varnothing$. Then there are local coordinates y^α in $\overline{M} \cap \tilde{U}$ and (x^a, y^α) form a coordinate system in $\tilde{U}$.

If we put

$$E_i^{\ a} = \frac{\partial x^a}{\partial z^i} \qquad \text{and} \qquad C^h_{\ \alpha} = \frac{\partial z^h}{\partial y^\alpha}\,, \tag{1.2}$$

then $E_i^{\ a}$ are components of a local covector field E^a in $\tilde{U}$ for each fixed index a, and $C^h_{\ \alpha}$ are those of a vector field C_α for each fixed index α. The vector fields C_α form a natural frame tangent to $\overline{M}$ and

$$E_i^{\ a} C^i_{\ \beta} = 0\,. \tag{1.3}$$

The induced metric tensor $\bar{g}$ in each fibre $\overline{M}$ is given by

$$\bar{g}_{\gamma\beta} = \tilde{g}(C_\gamma, C_\beta)\,. \tag{1.4}$$

If we put

$$g_{cb} = \tilde{g}(E_c, E_b)\,, \tag{1.5}$$

then g_{cb} are components of the metric tensor g with respect to (x^a) in the base space M. We put

$$E^h_{\ a} = \tilde{g}^{hi} g_{ab} E_i^{\ b} \qquad \text{and} \qquad C_i^{\ \alpha} = \tilde{g}_{ih} \bar{g}^{\alpha\beta} C^h_{\ \beta}\,.$$

We write the frame (E_B) for (E_b, C_β) in all, if necessary. Let $h_{\gamma\beta}{}^a$ be components of the second fundamental tensor with respect to the normal vector E_a and $L = (L_{cb}{}^\alpha)$ the normal connection of each fibre $\overline{M}$. Then we have

$$h_{\gamma\beta}{}^a = h_{\beta\gamma}{}^a \qquad \text{and} \qquad L_{cb}{}^\alpha + L_{bc}{}^\alpha = 0\,. \tag{1.6}$$

Denoting by $\tilde{\nabla}$ the Riemannian connection of the total space $\tilde{M}$, we have the following equations [6, 7, 13]:

$$\begin{aligned}
\tilde{\nabla}_j E^h_{\ b} &= \Gamma^a_{cb} E_j^{\ c} E^h_{\ a} - L_{cb}{}^\alpha E_j^{\ c} C^h_{\ \alpha} + L_b{}^a{}_\gamma C_j^{\ \gamma} E^h_{\ a} - h_\gamma{}^\alpha{}_b C_j^{\ \gamma} C^h_{\ \alpha}\,,\\
\tilde{\nabla}_j C^h_{\ \beta} &= L_c{}^a{}_\beta E_j^{\ c} E^h_{\ a} - (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j^{\ c} C^h_{\ \alpha} + h_{\gamma\beta}{}^a C_j^{\ \gamma} E^h_{\ a} + \bar{\Gamma}^\alpha_{\gamma\beta} C_j^{\ \gamma} C^h_{\ \alpha}\,,\\
\tilde{\nabla}_j E_i^{\ a} &= -\Gamma^a_{cb} E_j^{\ c} E_i^{\ b} - L_c{}^a{}_\beta (E_j^{\ c} C_i^{\ \beta} + C_j^{\ \beta} E_i^{\ c}) - h_{\gamma\beta}{}^a C_j^{\ \gamma} C_i^{\ \beta}\,,\\
\tilde{\nabla}_j C_i^{\ \alpha} &= L_{cb}{}^\alpha E_j^{\ c} E_i^{\ b} + (h_\beta{}^\alpha{}_c - P_{c\beta}{}^\alpha) E_j^{\ c} C_i^{\ \beta} + h_\gamma{}^\alpha{}_b C_j^{\ \gamma} E_i^{\ b} - \bar{\Gamma}^\alpha_{\gamma\beta} C_j^{\ \gamma} C_i^{\ \beta}\,,
\end{aligned} \tag{1.7}$$

where Γ_{cb}^{a} and $\bar{\Gamma}_{\gamma\beta}^{\alpha}$ are connection coefficients of the projection $\nabla = p\tilde{\nabla}$ and $\bar{\nabla}$ of the induced metric $\bar{g}$ in $\bar{M}$.

The curvature tensor of $\tilde{M}$ is defined by

$$\tilde{K}(\tilde{X}, \tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in $\tilde{M}$. We put

$$\tilde{K}(E_D, E_C)E_B = \tilde{K}_{DCB}{}^{a}E_a + \tilde{K}_{DCB}{}^{\alpha}C_\alpha \,, \tag{1.8}$$

then $\tilde{K}_{DCB}{}^{A}$ are components of the curvature tensor with respect to the frame (E_B). Denoting by $\tilde{K}_{kji}{}^{h}$ components of the curvature tensor of $\tilde{M}$ in $(\tilde{U}, z^h)$, we have the relations

$$\tilde{K}_{DCB}{}^{A} = \tilde{K}_{kji}{}^{h}E^{k}{}_{D}E^{j}{}_{C}E^{i}{}_{B}E_{h}{}^{A} \,. \tag{1.9}$$

The structure equations of $\tilde{M}$ are written as follows:

$$\tilde{K}_{dcb}{}^{a} = K_{dcb}{}^{a} - L_{d}{}^{a}{}_{\varepsilon}L_{cb}{}^{\varepsilon} + L_{c}{}^{a}{}_{\varepsilon}L_{db}{}^{\varepsilon} + 2L_{dc}{}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon} \,, \tag{1.10}$$

$$\tilde{K}_{dcb}{}^{\alpha} = -{}^{*}\nabla_{d}L_{cb}{}^{\alpha} + {}^{*}\nabla_{c}L_{db}{}^{\alpha} - 2L_{dc}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} \,, \tag{1.11}$$

$$\begin{aligned}\tilde{K}_{dc\beta}{}^{\alpha} = {}&{}^{*}\nabla_{c}h_{\beta}{}^{\alpha}{}_{d} - {}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{c} + 2{}^{**}\nabla_{\beta}L_{dc}{}^{\alpha} + L_{de}{}^{\alpha}L_{c}{}^{e}{}_{\beta} \\ &- L_{ce}{}^{\alpha}L_{d}{}^{e}{}_{\beta} - h_{\varepsilon}{}^{\alpha}{}_{d}h_{\beta}{}^{\varepsilon}{}_{c} + h_{\varepsilon}{}^{\alpha}{}_{c}h_{\beta}{}^{\varepsilon}{}_{d} \,,\end{aligned} \tag{1.12}$$

$$\tilde{K}_{d\gamma b}{}^{a} = {}^{*}\nabla_{d}L_{b}{}^{a}{}_{\gamma} - L_{d}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{b} + L_{db}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} - L_{b}{}^{a}{}_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{d} \,, \tag{1.13}$$

$$\tilde{K}_{d\gamma b}{}^{\alpha} = -{}^{*}\nabla_{d}h_{\gamma}{}^{\alpha}{}_{b} + {}^{**}\nabla_{\gamma}L_{db}{}^{\alpha} + L_{d}{}^{e}{}_{\gamma}L_{eb}{}^{\alpha} + h_{\gamma}{}^{\varepsilon}{}_{d}h_{\varepsilon}{}^{\alpha}{}_{b} \,, \tag{1.14}$$

$$\tilde{K}_{\delta\gamma b}{}^{a} = L_{\delta\gamma b}{}^{a} + h_{\delta}{}^{\varepsilon}{}_{b}h_{\gamma\varepsilon}{}^{a} - h_{\gamma}{}^{\varepsilon}{}_{b}h_{\delta\varepsilon}{}^{a} \,, \tag{1.15}$$

$$\tilde{K}_{\delta\gamma\beta}{}^{a} = {}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} - {}^{**}\nabla_{\gamma}h_{\delta\beta}{}^{a} \,, \tag{1.16}$$

$$\tilde{K}_{\delta\gamma\beta}{}^{\alpha} = \bar{K}_{\delta\gamma\beta}{}^{\alpha} + h_{\delta\beta}{}^{e}h_{\gamma}{}^{\alpha}{}_{e} - h_{\gamma\beta}{}^{e}h_{\delta}{}^{\alpha}{}_{e} \,, \tag{1.17}$$

where we have put

$$K_{dcb}{}^{a} = \partial_{d}\Gamma_{cb}^{a} - \partial_{c}\Gamma_{db}^{a} + \Gamma_{de}^{a}\Gamma_{cb}^{e} - \Gamma_{ce}^{a}\Gamma_{db}^{e} \,, \tag{1.18}$$

$${}^{*}\nabla_{d}L_{cb}{}^{\alpha} = \partial_{d}L_{cb}{}^{\alpha} - \Gamma_{dc}^{e}L_{eb}{}^{\alpha} - \Gamma_{db}^{e}L_{ce}{}^{\alpha} + Q_{d\varepsilon}{}^{\alpha}L_{cb}{}^{\varepsilon} \,, \tag{1.19}$$

$${}^{*}\nabla_{d}L_{c}{}^{a}{}_{\beta} = \partial_{d}L_{c}{}^{a}{}_{\beta} + \Gamma_{de}^{a}L_{c}{}^{e}{}_{\beta} - \Gamma_{dc}^{e}L_{e}{}^{a}{}_{\beta} - Q_{d\beta}{}^{\varepsilon}L_{c}{}^{a}{}_{\varepsilon} \,, \tag{1.20}$$

$${}^{*}\nabla_{d}h_{\gamma\beta}{}^{a} = \partial_{d}h_{\gamma\beta}{}^{a} + \Gamma_{de}^{a}h_{\gamma\beta}{}^{e} - Q_{d\gamma}{}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - Q_{d\beta}{}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} \,, \tag{1.21}$$

$${}^{*}\nabla_{d}h_{\beta}{}^{\alpha}{}_{b} = \partial_{d}h_{\beta}{}^{\alpha}{}_{b} - \Gamma_{db}^{e}h_{\beta}{}^{\alpha}{}_{e} + Q_{d\varepsilon}{}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - Q_{d\beta}{}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} \,, \tag{1.22}$$

$${}^{**}\nabla_{\delta}L_{cb}{}^{\alpha} = \partial_{\delta}L_{cb}{}^{\alpha} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}L_{cb}{}^{\varepsilon} - L_{c}{}^{e}{}_{\delta}L_{eb}{}^{\alpha} - L_{b}{}^{e}{}_{\delta}L_{ce}{}^{\alpha} \,, \tag{1.23}$$

$${}^{**}\nabla_{\delta}L_{b}{}^{a}{}_{\beta} = \partial_{\delta}L_{b}{}^{a}{}_{\beta} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}L_{b}{}^{a}{}_{\varepsilon} + L_{e}{}^{a}{}_{\delta}L_{b}{}^{e}{}_{\beta} - L_{b}{}^{e}{}_{\delta}L_{e}{}^{a}{}_{\beta} \,, \tag{1.24}$$

$${}^{**}\nabla_{\delta}h_{\gamma\beta}{}^{a} = \partial_{\delta}h_{\gamma\beta}{}^{a} - \bar{\Gamma}_{\delta\gamma}^{\varepsilon}h_{\varepsilon\beta}{}^{a} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\gamma\varepsilon}{}^{a} + L_{e}{}^{a}{}_{\delta}h_{\gamma\beta}{}^{e} \,, \tag{1.25}$$

$${}^{**}\nabla_{\delta}h_{\beta}{}^{\alpha}{}_{b} = \partial_{\delta}h_{\beta}{}^{\alpha}{}_{b} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}h_{\beta}{}^{\varepsilon}{}_{b} - \bar{\Gamma}_{\delta\beta}^{\varepsilon}h_{\varepsilon}{}^{\alpha}{}_{b} - L_{b}{}^{e}{}_{\delta}h_{\beta}{}^{\alpha}{}_{e} \,, \tag{1.26}$$

$$L_{\delta\gamma b}{}^{a} = \partial_{\delta}L_{b}{}^{a}{}_{\gamma} - \partial_{\gamma}L_{b}{}^{a}{}_{\delta} + L_{e}{}^{a}{}_{\delta}L_{b}{}^{e}{}_{\gamma} - L_{e}{}^{a}{}_{\gamma}L_{b}{}^{e}{}_{\delta} \,, \tag{1.27}$$

$$\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \partial_{\delta}\bar{\Gamma}_{\gamma\beta}^{\alpha} - \partial_{\gamma}\bar{\Gamma}_{\delta\beta}^{\alpha} + \bar{\Gamma}_{\delta\varepsilon}^{\alpha}\bar{\Gamma}_{\gamma\beta}^{\varepsilon} - \bar{\Gamma}_{\gamma\varepsilon}^{\alpha}\bar{\Gamma}_{\delta\beta}^{\varepsilon} \,. \tag{1.28}$$

We denote by $\tilde{K}_{CB}$, K_{cb} and $\bar{K}_{\gamma\beta}$ components of the Ricci tensors of $\{\tilde{M}, \tilde{g}\}$, the base space $\{M, g\}$ and each fibre $\{\bar{M}, \bar{g}\}$ respectively. Then we have the relations

$$(1.29)\qquad \tilde{K}_{cb} = K_{cb} - 2L_{ce}{}^{\varepsilon}L_{b}{}^{e}{}_{\varepsilon} - h_{\alpha}{}^{\varepsilon}{}_{c}h_{\varepsilon}{}^{\alpha}{}_{b} + (1/2)({}^{*}\nabla_{c}h_{\varepsilon}{}^{\varepsilon}{}_{b} + {}^{*}\nabla_{b}h_{\varepsilon}{}^{\varepsilon}{}_{c}),$$

$$(1.30)\qquad \tilde{K}_{\gamma b} = {}^{**}\nabla_{\gamma}h_{\varepsilon}{}^{\varepsilon}{}_{b} - {}^{**}\nabla_{\varepsilon}h_{\gamma}{}^{\varepsilon}{}_{b} + {}^{*}\nabla_{e}L_{b}{}^{e}{}_{\gamma} - 2h_{\gamma}{}^{\varepsilon}{}_{e}L_{b}{}^{e}{}_{\varepsilon},$$

$$(1.31)\qquad \tilde{K}_{\gamma\beta} = \bar{K}_{\gamma\beta} - h_{\gamma\beta}{}^{e}h_{\varepsilon}{}^{\varepsilon}{}_{e} + {}^{*}\nabla_{e}h_{\gamma\beta}{}^{e} - L_{a}{}^{e}{}_{\gamma}L_{e}{}^{a}{}_{\beta}.$$

Denoting by $\tilde{K}$, K and $\bar{K}$ the scalar curvatures of $\tilde{M}$, M and each fibre $\bar{M}$ respectively, we obtain the relation

$$(1.32)\qquad \tilde{K} = K^{L} + \bar{K} - L_{cb\varepsilon}L^{cb\varepsilon} - h_{\gamma\beta e}h^{\gamma\beta e} - h_{\gamma}{}^{\gamma}{}_{e}h_{\beta}{}^{\beta e} + 2{}^{*}\nabla_{e}h_{\varepsilon}{}^{\varepsilon e},$$

where K^{L} is the horizontal lift of K.

§2. Complex space form and Sasakian space form

We recall properties of a complex space form and a Sasakian space form in connection with Bochner curvature tensor and C-Bochner curvature tensor, for the sake of the future.

We consider an n-dimensional Kaehlerian space M and denote the complex structure by J. The tensor H_{cb} defined by

$$(2.1)\qquad H_{cb} = J_{c}{}^{e}K_{eb},$$

is skew-symmetric in the indices. The Bochner curvature tensor on M is defined by

$$(2.2)\qquad \begin{aligned} B_{dcb}{}^{a} = K_{dcb}{}^{a} &+ \frac{1}{n+4}(K_{db}\delta_{c}^{a} - K_{cb}\delta_{d}^{a} + g_{db}K_{c}{}^{a} - g_{cb}K_{d}{}^{a} \\ &+ H_{db}J_{c}{}^{a} - H_{cb}J_{d}{}^{a} + J_{db}H_{c}{}^{a} - J_{cb}H_{d}{}^{a} + 2H_{dc}J_{b}{}^{a} \\ &+ 2J_{dc}H_{b}{}^{a}) + \frac{K}{(n+2)(n+4)}(g_{db}\delta_{c}^{a} - g_{cb}\delta_{d}^{a} + J_{db}J_{c}{}^{a} \\ &- J_{cb}J_{d}{}^{a} + 2J_{dc}J_{b}{}^{a}),\end{aligned}$$

[3, 8, 10, 12].

A Kaehlerian space M is called a *complex space form* if the curvature tensor is of the form

$$(2.3)\qquad K_{dcb}{}^{a} = (c/4)(\delta_{d}^{a}g_{cb} - \delta_{c}^{a}g_{db} + J_{d}{}^{a}J_{cb} - J_{c}{}^{a}J_{db} - 2J_{dc}J_{b}{}^{a}).$$

The constant holomorphic sectional curvature c of M is equal to $4K/n(n+2)$.

The following proposition is well known [12].

PROPOSITION 2.1. *A Kaehlerian space M is a complex space form if and only if M is an Einstein space and the Bochner curvature tensor $B_{dcb}{}^a$ vanishes.*

Next we consider a p-dimensional Sasakian manifold $\overline{M}$ and denote the contact metric structure by $(\bar\phi_\beta{}^\alpha, \bar\xi^\alpha, \bar\eta_\beta, \bar g_{\beta\alpha})$. They satisfy the relations

$$\begin{aligned}&\bar\phi^2 = -I + \bar\eta \otimes \bar\xi\,, \quad \bar\eta \otimes \bar\phi = 0\,, \quad \bar\phi(\bar\xi) = 0\,, \quad \bar\eta(\bar\xi) = 1\,,\\ &\bar\nabla \bar\eta = \bar\phi\,, \quad (\bar\nabla_{\bar X}\bar\phi)\bar Y = \bar g(\bar X, \bar Y)\bar\xi - \bar\eta(\bar Y)\bar X\end{aligned} \tag{2.4}$$

[1], where $\bar\nabla$ is the Riemannian connection on $\overline{M}$ and $\bar X$, $\bar Y$ are arbitrary vector fields. The tensor $\bar H_{\beta\alpha}$ defined by $\bar H_{\beta\alpha} = \bar\phi_\beta{}^\gamma \bar K_{\gamma\alpha}$ is skew-symmetric in α and β.

The C-Bochner curvature on $\overline{M}$ is defined by

$$\begin{aligned}B_{\delta\gamma\beta}{}^\alpha = \bar K_{\delta\gamma\beta}{}^\alpha + \frac{1}{p+3}\{&\bar K_{\delta\beta}\delta_\gamma^\alpha - \bar K_{\gamma\beta}\delta_\delta^\alpha + \bar g_{\delta\beta}\bar K_\gamma{}^\alpha - \bar g_{\gamma\beta}\bar K_\delta{}^\alpha\\ &+ \bar H_{\delta\beta}\bar\phi_\gamma{}^\alpha + \bar H_{\gamma\beta}\bar\phi_\delta{}^\alpha - \bar\phi_{\delta\beta}\bar H_\gamma{}^\alpha + \bar\phi_{\gamma\beta}\bar H_\delta{}^\alpha + 2\bar H_{\delta\gamma}\bar\phi_\beta{}^\alpha + 2\bar\phi_{\delta\gamma}\bar H_\beta{}^\alpha\\ &- \bar K_{\delta\beta}\bar\eta_\gamma\bar\xi^\alpha + \bar K_{\gamma\beta}\bar\eta_\delta\bar\xi^\alpha - \bar\eta_\delta\bar\eta_\beta\bar K_\gamma{}^\alpha + \bar\eta_\gamma\bar\eta_\beta\bar K_\delta{}^\alpha\\ &- (\bar k + p - 1)(\bar\phi_{\delta\beta}\bar\phi_\gamma{}^\alpha - \bar\phi_{\gamma\beta}\bar\phi_\delta{}^\alpha + 2\bar\phi_{\delta\gamma}\bar\phi_\beta{}^\alpha)\\ &- (\bar k - 4)(\bar g_{\delta\beta}\delta_\gamma^\alpha - \bar g_{\gamma\beta}\delta_\delta^\alpha)\\ &+ \bar k(\bar g_{\delta\beta}\bar\eta_\gamma\bar\xi^\alpha + \bar\eta_\delta\bar\eta_\beta\delta_\gamma^\alpha - \bar g_{\gamma\beta}\bar\eta_\delta\bar\xi^\alpha - \bar\eta_\gamma\bar\eta_\beta\delta_\delta^\alpha)\}\,,\end{aligned} \tag{2.5}$$

where $\bar k = (\bar K + p - 1)/(p + 1)$. It can be constructed from the Bochner curvature tensor in a Kaehlerian space by the fibering of Boothby-Wang (see [9]).

If the Ricci curvature $\bar K_{\beta\alpha}$ on $\overline{M}$ is of the form

$$\bar K_{\beta\alpha} = a\bar g_{\beta\alpha} + b\bar\eta_\beta\bar\eta_\alpha\,, \tag{2.6}$$

with constants a and b, we call $\overline{M}$ an *η-Einstein space.* Since we have the equation

$$\bar K_{\beta\alpha}\bar\xi^\alpha = (p-1)\bar\eta_\beta$$

in a Sasakian space, the constants a and b satisfy the relation

$$a + b = p - 1\,. \tag{2.7}$$

A Sasakian space $\overline{M}$ is called a *Sasakian space form* if the curvature tensor is of the form

$$\begin{aligned}\bar K_{\delta\gamma\beta}{}^\alpha = \frac{\bar c + 3}{4}(\delta_\delta^\alpha\bar g_{\gamma\beta} - \delta_\gamma^\alpha\bar g_{\delta\beta}) - \frac{\bar c - 1}{4}(&\delta_\delta^\alpha\bar\eta_\gamma\bar\eta_\beta - \delta_\gamma^\alpha\bar\eta_\delta\bar\eta_\beta\\ &+ \bar g_{\gamma\beta}\bar\eta_\delta\bar\xi^\alpha - \bar g_{\delta\beta}\bar\eta_\gamma\bar\xi^\alpha + \bar\phi_{\delta\beta}\bar\phi_\gamma{}^\alpha - \bar\phi_{\gamma\beta}\bar\phi_\delta{}^\alpha + 2\bar\phi_{\delta\gamma}\bar\phi_\beta{}^\alpha)\,.\end{aligned} \tag{2.8}$$

Contracting this equation in α and β, we see that the Sasakian space form is an η-Einstein space with constants

$$a = \{\bar{c}(p+1) + 3p - 5\}/4 \qquad \text{and} \qquad b = (p+1)(1-\bar{c})/4\,.$$

The constant $\bar{c}$ is conversely given by $\bar{c} = (4a - 3p + 5)/(p+1)$ by means of (2.8) and it is known that the C-Bochner curvature tensor of the Sasakian space form vanishes identically.

Conversely we assume that $\overline{M}$ is an η-Einstein space and the C-Bochner curvature tensor vanishes. Then we get

$$\begin{aligned} \bar{K} &= (p-1)(a+1)\,, \\ \bar{k} &= (p-1)(a+2)/(p+1) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \bar{K}_{\delta\gamma\beta}{}^{\alpha} = \frac{1}{p+3}\{&(2a - \bar{k} + 4)(\delta_\delta^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma^\alpha \bar{g}_{\delta\beta}) - (2a - \bar{k} - p + 1)(\delta_\delta^\alpha \bar{\eta}_\gamma \bar{\eta}_\beta \\ &- \delta_\gamma^\alpha \bar{\eta}_\delta \bar{\eta}_\beta + \bar{g}_{\gamma\beta}\bar{\eta}_\delta \bar{\xi}^\alpha - \bar{g}_{\delta\beta}\bar{\eta}_\gamma \bar{\xi}^\alpha + \bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha + \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha)\} \end{aligned} \tag{2.10}$$

by use of $b = p - a - 1$. Therefore $\overline{M}$ becomes a Sasakian space form of constant $\bar{\phi}$-holomorphic sectional curvature $\bar{c} = (4a - 3p + 5)/(p+1)$. Thus the following result is valid.

PROPOSITION 2.2. *A Sasakian space $\overline{M}$ is a Sasakian space form if and only if $\overline{M}$ is η-Einstein and has the vanishing C-Bochner curvature tensor.*

§3. Fibred Sasakian space with vanishing contact Bochner curvature tensor

We consider a fibred Riemannian space $\tilde{M}$ such that the base space M is almost Hermitian and each fibre $\overline{M}$ is almost contact metric, and denote the lift of the almost Hermitian structure of M to the total space $\tilde{M}$ by the same characters (J, g) and the almost contact metric structure of each fibre $\overline{M}$ by $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. The present author [7] has introduced an almost contact metric structure $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on the total space $\tilde{M}$ by putting

$$\begin{aligned} \tilde{\phi} &= J_b{}^a E^b \otimes E_a + \bar{\phi}_\beta{}^\alpha C^\beta \otimes C_\alpha\,, \\ \tilde{\xi} &= \bar{\xi}^\alpha C_\alpha, \qquad \tilde{\eta} = \bar{\eta}_\alpha C^\alpha \qquad \text{and} \\ \tilde{g} &= g_{ba} E^b \otimes E^a + \bar{g}_{\beta\alpha} C^\beta \otimes C^\alpha\,. \end{aligned} \tag{3.1}$$

The structure is said to be *induced* on $\tilde{M}$. Conversely, it is known [13] that a fibred almost contact metric space with $\tilde{\phi}$-invariant fibres tangent to $\tilde{\xi}$ defines an almost Hermitian structure in the base space and an almost contact metric structure in each fibre.

If the horizontal mapping covering any curve in M is an isometry (resp. conformal mapping) of fibres, then $\tilde{M}$ is called a *fibred Riemannian space with isometric* (resp. *conformal*) *fibres*. A necessary and sufficient condition for $\tilde{M}$ to have isometric (resp. conformal) fibres is $h_{\gamma\beta}{}^a = 0$ (resp. $h_{\gamma\beta}{}^a = \bar{g}_{\gamma\beta}A^a$, where $A = A^aE_a$ is the mean curvature vector of each fibre $\bar{M}$ in $\tilde{M}$), see [6, 13].

We recall the following propositions for the later use.

PROPOSITION 3.1 ([7]). *The induced almost contact metric structure* $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ *on* $\tilde{M}$ *is Sasakian if and only if*

(1) *the base space M is Kaehlerian.*

(2) *each fibre $\bar{M}$ is Sasakian,*

(3) $L_{cb}{}^\gamma = J_{cb}\bar{\xi}^\gamma$,

(4) $h_\gamma{}^\lambda{}_b\bar{\phi}_\lambda{}^\mu - h_\gamma{}^\mu{}_aJ_b{}^a = 0$ *and*

(5) $*\nabla_c\bar{\phi}_\alpha{}^\gamma = 0$,

where we have put

$$*\nabla_c\bar{\phi}_\alpha{}^\gamma = \partial_c\bar{\phi}_\alpha{}^\gamma + (P_{c\beta}{}^\gamma - h_\beta{}^\gamma{}_c)\bar{\phi}_\alpha{}^\beta - (P_{c\alpha}{}^\beta - h_\alpha{}^\beta{}_c)\bar{\phi}_\beta{}^\gamma\,.$$

PROPOSITION 3.2([7]). *If a fibred Sasakian space $\tilde{M}$ with induced structure has conformal fibres, then $\tilde{M}$ has isometric and totally geodesic fibres.*

Now we assume that a fibred Sasakian space $\tilde{M}$ has conformal fibres and the C-Bochner curvature tensor on $\tilde{M}$ vanishes. If we put $\tilde{H}_{ji} = \tilde{\phi}_j{}^k\tilde{K}_{ki}$, then the tensor $\tilde{H}_{ji}$ satisfies the equations

$$\tilde{H}_{ij} + \tilde{H}_{ji} = 0\,, \tag{3.2}$$

$$\tilde{H}_{ji}\tilde{\xi}^j = 0\,, \tag{3.3}$$

$$\tilde{H}_{ki}\tilde{\phi}_j{}^k = -\tilde{K}_{ji} + (m-1)\tilde{\eta}_j\tilde{\eta}_i\,, \tag{3.4}$$

$$\tilde{H}_{ij}\tilde{\phi}^{ij} = \tilde{K} - m + 1\,, \tag{3.5}$$

and, by means of the equation in $\tilde{M}$ similar to (2.5), the curvature tensor $\tilde{K}_{kji}{}^h$ of $\tilde{M}$ is given by the expression

$$\begin{aligned}\tilde{K}_{kji}{}^h = -\frac{1}{m+3}\{&(\tilde{K}_{ki}\delta_j^h - \tilde{K}_{ji}\delta_k^h + \tilde{g}_{ki}\tilde{K}_j{}^h - \tilde{g}_{ji}\tilde{K}_k{}^h + \tilde{H}_{ki}\tilde{\phi}_j{}^h \\ &- \tilde{H}_{ji}\tilde{\phi}_k{}^h + \tilde{\phi}_{ki}\tilde{H}_j{}^h - \tilde{\phi}_{ji}\tilde{H}_k{}^h + 2\tilde{H}_{kj}\tilde{\phi}_i{}^h + 2\tilde{\phi}_{kj}\tilde{H}_i{}^h \\ &- \tilde{K}_{ki}\tilde{\eta}_j\tilde{\xi}^h + \tilde{K}_{ji}\tilde{\eta}_k\tilde{\xi}^h - \tilde{K}_j{}^h\tilde{\eta}_k\tilde{\eta}_i + \tilde{K}_k{}^h\tilde{\eta}_j\tilde{\eta}_i) \\ &- (\tilde{k} + m - 1)(\tilde{\phi}_{ki}\tilde{\phi}_j{}^h - \tilde{\phi}_{ji}\tilde{\phi}_k{}^h + 2\tilde{\phi}_{kj}\tilde{\phi}_i{}^h) \\ &- (\tilde{k} - 4)(\tilde{g}_{ki}\delta_j^h - \tilde{g}_{ji}\delta_k^h) \\ &+ \tilde{k}(\tilde{g}_{ki}\tilde{\eta}_j\tilde{\xi}^h + \tilde{\eta}_k\tilde{\eta}_i\delta_j^h - \tilde{g}_{ji}\tilde{\eta}_k\tilde{\xi}^h - \tilde{\eta}_j\tilde{\eta}_i\delta_k^h)\}\,,\end{aligned} \tag{3.6}$$

where $\tilde{k} = (\tilde{K} + m - 1)/(m + 1)$

By the equations (1.29) ~ (1.32), Propositions 3.1 and 3.2, we have

$$\tilde{K}_{ji}E^j{}_cE^i{}_b = K_{cb} - 2g_{cb}\,, \tag{3.7}$$

$$\tilde{K}_{ji}E^j{}_cC^i{}_\beta = 0\,, \tag{3.8}$$

$$\tilde{K}_{ji}C^j{}_\gamma C^i{}_\beta = \bar{K}_{\gamma\beta} + n\bar{\eta}_\gamma\bar{\eta}_\beta\,, \tag{3.9}$$

$$\tilde{H}_{ji}E^j{}_cE^i{}_b = H_{cb} - 2J_{cb}\,, \tag{3.10}$$

$$\tilde{H}_{ji}E^j{}_cC^i{}_\alpha = 0\,, \tag{3.11}$$

$$\tilde{H}_{ji}C^j{}_\gamma C^i{}_\beta = \bar{H}_{\gamma\beta} \tag{3.12}$$

and

$$\tilde{K} = K^L + \bar{K} - n\,. \tag{3.13}$$

Referring the expression (3.6) to the frame $(E_A) = (E_a, C_\alpha)$, we obtain the equations

$$\begin{aligned} K_{dcb}{}^a = &-\frac{1}{m+3}(K_{db}\delta_c^a - K_{cb}\delta_d^a + K_c{}^a g_{db} - K_d{}^a g_{cb} + H_{db}J_c^a \\ &- H_{cb}J_d^a + H_c^a J_{db} - H_d{}^a J_{cb} + 2H_{dc}J_b{}^a + 2H_b{}^a J_{dc}) \\ &+ \frac{K+\bar{K}+p-1}{(m+1)(m+3)}(J_{db}J_c^a - J_{cb}J_d^a + 2J_{dc}J_b{}^a \\ &+ g_{db}\delta_c^a - g_{cb}\delta_d^a)\,, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &K_{db}\delta_\gamma^\alpha - (k-2)(g_{db}\delta_\gamma^\alpha + J_{db}\bar{\phi}_\gamma{}^\alpha) + \bar{K}_\gamma{}^\alpha g_{db} + (k+n-m-1)g_{db}\bar{\eta}_\gamma\bar{\xi}^\alpha \\ &\quad + K_{eb}J_d^e\bar{\phi}_\gamma{}^\alpha + \bar{K}_\beta{}^\alpha\bar{\phi}_\gamma{}^\beta J_{db} - K_{db}\bar{\eta}_\gamma\bar{\xi}^\alpha = 0 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \bar{K}_{\delta\gamma\beta}{}^\alpha = &-\frac{1}{m+3}\{(\bar{K}_{\delta\beta}\delta_\gamma^\alpha - \bar{K}_{\gamma\beta}\delta_\delta^\alpha + \bar{K}_\gamma{}^\alpha\bar{g}_{\delta\beta} - \bar{K}_\delta{}^\alpha\bar{g}_{\gamma\beta} + \bar{H}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha \\ &- \bar{H}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + \bar{H}_\gamma{}^\alpha\bar{\phi}_{\delta\beta} - \bar{H}_\delta{}^\alpha\bar{\phi}_{\gamma\beta} + 2\bar{H}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha + 2\bar{H}_\beta{}^\alpha\bar{\phi}_{\delta\gamma} \\ &- \bar{K}_{\delta\beta}\bar{\eta}_\gamma\bar{\xi}^\alpha + \bar{K}_{\gamma\beta}\bar{\eta}_\delta\bar{\xi}^\alpha - \bar{K}_\gamma{}^\alpha\bar{\eta}_\delta\bar{\eta}_\beta + \bar{K}_\delta{}^\alpha\bar{\eta}_\beta\bar{\eta}_\gamma) \\ &+ n(\bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha - \bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha + \bar{\eta}_\gamma\bar{\xi}^\alpha\bar{g}_{\delta\beta} - \bar{\eta}_\delta\bar{\xi}^\alpha\bar{g}_{\gamma\beta}) \\ &- (\tilde{k}+m-1)(\bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha - \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha) \\ &- (\tilde{k}-4)(\bar{g}_{\delta\beta}\delta_\gamma^\alpha - \bar{g}_{\gamma\beta}\delta_\delta^\alpha) \\ &+ \tilde{k}(\bar{g}_{\delta\beta}\bar{\eta}_\gamma\bar{\xi}^\alpha + \bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha - \bar{g}_{\gamma\beta}\bar{\eta}_\delta\bar{\xi}^\alpha - \bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha)\} \end{aligned} \tag{3.16}$$

by means of the equations (1.9), (1.10), (1.14), (1.17) and (3.9) ~ (3.13). Moreover, contracting g^{db} and the indices γ and α in (3.15), we obtain

$$(p-1)(p+1)K + n(n+2)(\bar{K} + p - 1) = 0\,. \tag{3.17}$$

By use of this equation and (3.14), the curvature tensor of M is given by

$$(3.18)\quad \begin{aligned} K_{dcb}{}^{a} = & -\frac{1}{n+p+3}(K_{db}\delta_c^a - K_{cb}\delta_d^a + K_c{}^a g_{db} - K_d{}^a g_{cb} + H_{db}J_c{}^a \\ & - H_{cb}J_d{}^a + H_c{}^a J_{db} - H_d{}^a J_{cb} + 2H_{dc}J_b{}^a + 2H_b{}^a J_{dc}) \\ & + \frac{K(n-p+1)}{n(n+2)(n+p+3)}(J_{db}J_c{}^a - J_{cb}J_d{}^a + 2J_{dc}J_b{}^a + g_{db}\delta_c^a - g_{cb}\delta_d^a)\,. \end{aligned}$$

Hence, comparing this expression in the case of $p = 1$ with (2.2), we can state that

PROPOSITION 3.3. *If a fibred Sasakian space $\tilde{M}$ has 1-dimensional fibres and the C-Bochner curvature tensor of $\tilde{M}$ vanishes, then so does the Bochner curvature tensor of the base space M.*

In the case of $p \neq 1$, by the contraction in the indices a and d of (3.18), we get

$$(3.19)\quad K_{cb} = (K/n)g_{cb}\,,$$

and the base space M is an Einstein space provided $n > 2$. Substituting (3.19) into (3.18) and noting $H_{cb} = (K/n)J_{cb}$, we get

$$(3.20)\quad K_{dcb}{}^{a} = \frac{K}{n(n+2)}(g_{cb}\delta_d^a - g_{db}\delta_c^a + J_{cb}J_d{}^a - J_{db}J_c{}^a - 2J_{dc}J_b{}^a)\,.$$

Hence we can state

LEMMA 3.4. *Let $\tilde{M}$ be a fibred Sasakian space with conformal fibres of dimension $p \neq 1$. If the C-Bochner curvature tensor of $\tilde{M}$ vanishes, then the base space M is a complex space form provided $n > 2$.*

On the other hand, from the equation (3.17), we get

$$(3.21)\quad K = -n(n+2)\left\{\frac{\bar{K}}{(p-1)(p+1)} + \frac{1}{p+1}\right\}.$$

Substituting this into (3.16), we see that the curvature tensor of the fibre $\bar{M}$ has the expression

$$\begin{aligned} \bar{K}_{\delta\gamma\beta}{}^{\alpha} = & -\frac{1}{n+p+3}\Big[(\bar{K}_{\delta\beta}\delta_\gamma^\alpha - \bar{K}_{\gamma\beta}\delta_\delta^\alpha + \bar{K}_\gamma{}^\alpha \bar{g}_{\delta\beta} - \bar{K}_\delta{}^\alpha \bar{g}_{\gamma\beta} + \bar{H}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha \\ & - \bar{H}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + \bar{H}_\gamma{}^\alpha\bar{\phi}_{\delta\beta} - \bar{H}_\delta{}^\alpha\bar{\phi}_{\gamma\beta} + 2\bar{H}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha + 2\bar{H}_\beta{}^\alpha\bar{\phi}_{\delta\gamma} \\ & - \bar{K}_{\delta\beta}\bar{\eta}_\gamma\bar{\xi}^\alpha + \bar{K}_{\gamma\beta}\bar{\eta}_\delta\bar{\xi}^\alpha - \bar{K}_\gamma{}^\alpha\bar{\eta}_\delta\bar{\eta}_\beta + \bar{K}_\delta{}^\alpha\bar{\eta}_\beta\bar{\eta}_\gamma) \\ & + n(\bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha - \bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha + \bar{\eta}_\gamma\bar{\xi}^\alpha\bar{g}_{\delta\beta} - \bar{\eta}_\delta\bar{\xi}^\alpha\bar{g}_{\gamma\beta}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{n+p+1}\left\{\left(1-\frac{n(n+2)}{(p+1)(p-1)}\right)\bar{K}-\frac{n(n+2)}{p+1}+(n+p)^2+p\right\} \\
&\times(\bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha-\bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha+2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha) \\
&-\frac{1}{n+p+1}\left\{\left(1-\frac{n(n+2)}{(p+1)(p-1)}\right)\bar{K}-\frac{n(n+2)}{p+1}-4n-3p-5\right\} \\
&\times(\bar{g}_{\delta\beta}\delta_\gamma^\alpha-\bar{g}_{\gamma\beta}\delta_\delta^\alpha) \\
&+\frac{1}{n+p+1}\left\{\left(1-\frac{n(n+2)}{(p+1)(p-1)}\right)\bar{K}-\frac{n(n+2)}{p+1}+p-1\right\} \\
&\left.\times(\bar{g}_{\delta\beta}\bar{\eta}_\gamma\bar{\xi}^\alpha+\bar{\eta}_\delta\bar{\eta}_\beta\delta_\gamma^\alpha-\bar{g}_{\gamma\beta}\bar{\eta}_\delta\bar{\xi}^\alpha-\bar{\eta}_\gamma\bar{\eta}_\beta\delta_\delta^\alpha)\right].
\end{aligned}
\tag{3.22}
$$

Then the contraction with respect to α and δ gives

$$
\begin{aligned}
n\bar{K}_{\gamma\beta}=&-\frac{n}{p-1}(\bar{K}\bar{\eta}_\gamma\bar{\eta}_\beta-\bar{K}\bar{g}_{\gamma\beta}) \\
&+\left\{np+n-p+1+\frac{(p+1)(p-1)-n(n+2)}{n+p+1}\right\}\bar{\eta}_\gamma\bar{\eta}_\beta \\
&+\left\{p-2n-1+\frac{n(n+2)-(p+1)(p-1)}{n+p+1}\right\}\bar{g}_{\gamma\beta}\,.
\end{aligned}
\tag{3.23}
$$

Differentiating covariantly this equation on $\bar{M}$, noting $\bar{\nabla}_\beta\bar{K}_\alpha{}^\beta=(1/2)(\bar{\nabla}_\alpha K)$ and using (2.4), we have

$$
\frac{1}{2}\bar{\nabla}_\beta\bar{K}=\frac{1}{p-1}\{\bar{\nabla}_\beta\bar{K}-(\bar{\nabla}_\gamma\bar{K})\bar{\xi}^\gamma\bar{\eta}_\beta\}\,. \tag{3.24}
$$

Transvecting $\bar{\xi}^\beta$, we see $\bar{\xi}^\beta\bar{\nabla}_\beta\bar{K}=0$ and furthermore

$$
\bar{\nabla}_\beta\bar{K}=0 \tag{3.25}
$$

provided $p>3$, that is, $\bar{K}$ is constant on each fibre $\bar{M}$. Therefore it follows from (3.23) that the Ricci tensor $\bar{K}_{\beta\alpha}$ of $\bar{M}$ has the form

$$
\bar{K}_{\beta\alpha}=a\bar{g}_{\beta\alpha}+b\bar{\eta}_\beta\bar{\eta}_\alpha\,, \tag{3.26}
$$

where the constant coefficients a and b are put by

$$
\begin{aligned}
a&=\frac{1}{n}\left\{p-2n-1+\frac{n(n+2)-(p+1)(p-1)}{n+p+1}+\frac{n}{p-1}\bar{K}\right\}, \\
b&=\frac{1}{n}\left\{n+np-p+1+\frac{(p+1)(p-1)-n(n+2)}{n+p+1}-\frac{n}{p-1}\bar{K}\right\}
\end{aligned}
$$

and satisfy

$$a + b = p - 1 .$$

Substituting (3.26) into (3.16) and taking account of $\bar{H}_{\beta\alpha} = a\bar{\phi}_{\beta\alpha}$, we obtain the equation

$$\begin{aligned}\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \frac{1}{n+p+3}\{&2a(\delta_\delta^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma^\alpha \bar{g}_{\delta\beta})\\ &- (p-a-1)(\delta_\gamma^\alpha \bar{\eta}_\delta \bar{\eta}_\beta - \delta_\delta^\alpha \bar{\eta}_\gamma \bar{\eta}_\beta + \bar{g}_{\delta\beta}\bar{\eta}_\gamma \bar{\xi}^\alpha - \bar{g}_{\gamma\beta}\bar{\eta}_\delta \bar{\xi}^\alpha)\\ &+ a(\bar{g}_{\delta\beta}\bar{\eta}_\gamma \bar{\xi}^\alpha - \bar{g}_{\gamma\beta}\bar{\eta}_\delta \bar{\xi}^\alpha + \delta_\gamma^\alpha \bar{\eta}_\delta \bar{\eta}_\beta - \delta_\delta^\alpha \bar{\eta}_\beta \bar{\eta}_\gamma)\\ &- 2a(\bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha - \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha)\\ &- n(\bar{\eta}_\delta \bar{\eta}_\beta \delta_\gamma^\alpha - \bar{\eta}_\gamma \bar{\eta}_\beta \delta_\delta^\alpha + \bar{\eta}_\gamma \bar{\xi}^\alpha \bar{g}_{\delta\beta} - \bar{\eta}_\delta \bar{\xi}^\alpha \bar{g}_{\gamma\beta})\\ &+ (\tilde{k} + n + p - 1)(\bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha - \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha)\\ &+ (\tilde{k} - 4)(\bar{g}_{\delta\beta}\delta_\gamma^\alpha - \bar{g}_{\gamma\beta}\delta_\delta^\alpha)\\ &- \tilde{k}(\bar{g}_{\delta\beta}\bar{\eta}_\gamma \bar{\xi}^\alpha + \bar{\eta}_\delta \bar{\eta}_\beta \delta_\gamma^\alpha - \bar{g}_{\gamma\beta}\bar{\eta}_\delta \bar{\xi}^\alpha - \bar{\eta}_\gamma \bar{\eta}_\beta \delta_\delta^\alpha)\} ,\end{aligned}$$

that is,

$$\begin{aligned}\bar{K}_{\delta\gamma\beta}{}^{\alpha} = \frac{1}{p+1}\{&(a+2)(\delta_\delta^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma^\alpha \bar{g}_{\delta\beta})\\ &+ (p-a-1)(\delta_\delta^\alpha \bar{\eta}_\gamma \bar{\eta}_\beta - \delta_\gamma^\alpha \bar{\eta}_\delta \bar{\eta}_\beta - \bar{g}_{\delta\beta}\bar{\eta}_\gamma \bar{\xi}^\alpha + \bar{g}_{\gamma\beta}\bar{\eta}_\delta \bar{\xi}^\alpha\\ &+ \bar{\phi}_{\delta\beta}\bar{\phi}_\gamma{}^\alpha - \bar{\phi}_{\gamma\beta}\bar{\phi}_\delta{}^\alpha + 2\bar{\phi}_{\delta\gamma}\bar{\phi}_\beta{}^\alpha)\}\end{aligned} \tag{3.27}$$

by use of

$$\tilde{k} = -\frac{(a+2)(n-p+1)}{p+1}. \tag{3.28}$$

Thus we obtain

LEMMA 3.5. *Let $\tilde{M}$ be a fibred Sasakian space with conformal fibres of dimension $p > 3$. If the C-Bochner curvature of $\tilde{M}$ vanishes, then the fibre $\bar{M}$ is a Sasakian space form of constant $\bar{\phi}$-holomorphic sectional curvature $\bar{c} = (4a - 3p + 5)/(p + 1)$.*

Combining Lemmas 3.4 and 3.5, we have established

THEOREM 3.6. *Let $\tilde{M}$ be a fibred Sasakian space with base space M of dimension $n > 2$ and conformal fibres of dimension $p > 3$. If the C-Bochner curvature of $\tilde{M}$ vanishes, then the base space M is a complex space form and each fibre $\bar{M}$ is a Sasakian space form.*

§ 4. Examples

As we have shown in [7], a Sasakian space form $E^m(-3)$ is a fibred space having a Euclidean base space E^n of even dimension and a Sasakian space form $E^p(-3)$ as fibre. It is a trivial example.

Next, we shall give a fibred Sasakian space with vanishing C-Bochner curvature tensor, which is not a Sasakian space form.

Let $C^{n/2}$ be a complex space of complex dimension $n/2$ and denote complex coordinates by x^s, $s = 1, 2, \ldots, n/2$, and their conjugates by $\bar{x}^s$. If we consider the real valued function

$$F = (2/c) \log S\,, \qquad S = 1 + (c/2) \textstyle\sum_s x^s \bar{x}^s$$

with real constant c, then the metric tensor

$$g_{st*} = \frac{\partial^2 F}{\partial x^s \partial \bar{x}^t} = \frac{\delta_{st}}{S} - \frac{c\bar{x}^s x^t}{2S^2}$$

defines a Fubini-Study metric of constant holomorphic sectional curvature c [2]. If we put

$$\omega_s = -i\frac{\partial F}{\partial x^s} = -\frac{i\bar{x}^s}{S}\,, \qquad \omega_{s*} = i\frac{\partial F}{\partial \bar{x}^s} = \frac{ix^s}{S}\,, \tag{4.1}$$

then the fundamental 2-form $J = 2ig_{st*}\, dx^s \wedge d\bar{x}^t$ is given by

$$J_{ab} = (1/2)(\partial_a \omega_b - \partial_b \omega_a)\,. \tag{4.2}$$

If $c > 0$, then the 1-form $\omega = \omega_a\, dx^a$ is locally defined in the complex space form M. If $c < 0$, the 1-form ω is globally defined in the open domain

$$\{x^s | \textstyle\sum_s x^s \bar{x}^s < -2/c\}$$

in $C^{n/2}$, which is the underlying space of the complex space form M. If $c = 0$, the 1-form ω is globally defined in the complex Euclidean space $M = C^{n/2}$ by putting $S = 1$ in (4.1). The equation (4.2) may be valid in real coordinates.

Let $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{g})$ be a p-dimensional Sasakian space form with constant $\bar{\phi}$-holomorphic sectional curvature $-c-3$. We take the product space $M \times \bar{M}$ as the underlying space of $\tilde{M}$, and put

$$\tilde{g}_{ji} = \begin{pmatrix} g_{ba} + \omega_b \omega_a & \omega_b \bar{\eta}_\alpha \\ \bar{\eta}_\beta \omega_a & \bar{g}_{\beta\alpha} \end{pmatrix},$$

$$\tilde{\phi}_i{}^h = \begin{pmatrix} J_b{}^a & 0 \\ -J_b{}^d \omega_d \bar{\xi}^\alpha & \bar{\phi}_\beta{}^\alpha \end{pmatrix} \quad \text{and} \tag{4.3}$$

$$\tilde{\xi}^h = \begin{pmatrix} 0 \\ \bar{\xi}^\alpha \end{pmatrix}$$

with respect to the coordinate system $z^h = (x^a, y^\alpha)$. Then we have

$$\tilde{\eta}_i = \tilde{g}_{ih}\tilde{\xi}^h = (\omega_b, \bar{\eta}_\beta)$$

and verify that $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an almost contact metric structure on $\tilde{M}$. The covariant components of the metric $\tilde{g}$ are equal to

$$\tilde{g}^{ih} = \begin{pmatrix} g^{ba} & -\omega^b\bar{\xi}^\alpha \\ -\bar{\xi}^\beta\omega^a & \bar{g}^{\beta\alpha} + (\omega_d\omega^d)\bar{\xi}^\beta\bar{\xi}^\alpha \end{pmatrix},$$

where $\omega^b = \omega_a g^{ba}$.

The vector fields $E^A = (E^a, C^\alpha)$ and $E_A = (E_b, C_\beta)$ are given by

$$\text{(4.4)}\qquad \begin{aligned} &E_i{}^a = (\delta_b^a, 0)\,, && C_i^\alpha = (\bar{\xi}^\alpha\omega_b, \delta_\beta^\alpha) \\ &E^h{}_b = \begin{pmatrix} \delta_b^a \\ -\omega_b\bar{\xi}^\alpha \end{pmatrix}, && C^h{}_\beta = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \end{pmatrix} \end{aligned}$$

and E_A form a frame field in $\tilde{M}$ and we have the relations

$$\text{(4.5)}\qquad \tilde{g}(E_c, E_b) = g_{cb} \quad\text{and}\quad \tilde{g}(C_\beta, C_\alpha) = \bar{g}_{\beta\alpha}\,.$$

Therefore the space $\tilde{M}$ has an induced almost contact fibred structure.

By straightforward computations on account of properties of the Kaehlerian structure in the base space M and the Sasakian structure in the fibre $\overline{M}$, the connection $\tilde{\nabla}$ of $\tilde{g}$ in the total space $\tilde{M}$ has the following coefficients with respect to the coordinate system $z^h = (x^a, y^\alpha)$:

$$\text{(4.6)}\qquad \begin{aligned} &\tilde{\Gamma}_{cb}^a = \Gamma_{cb}^a + J_c{}^a\omega_b + J_b{}^a\omega_c\,, \\ &\tilde{\Gamma}_{cb}^\alpha = \tfrac{1}{2}(\nabla_b\omega_c + \nabla_c\omega_b)\bar{\xi}^\alpha + (J_{ac}\omega_b + J_{ab}\omega_c)\omega^a\bar{\xi}^\alpha\,, \\ &\tilde{\Gamma}_{c\beta}^a = J_c{}^a\bar{\eta}_\beta\,, \\ &\tilde{\Gamma}_{c\beta}^\alpha = -\omega^e J_{ce}\bar{\eta}_\beta\bar{\xi}^\alpha - \omega_c\bar{\phi}_\beta{}^\alpha\,, \\ &\tilde{\Gamma}_{\gamma\beta}^a = 0\,, \\ &\tilde{\Gamma}_{\gamma\beta}^\alpha = \bar{\Gamma}_{\gamma\beta}^\alpha\,, \end{aligned}$$

where Γ_{cb}^a and $\bar{\Gamma}_{\gamma\beta}^\alpha$ are connection coefficients of ∇ in M and $\bar{\nabla}$ in $\overline{M}$ respectively. Then it follows from the equations (1.7) that the second fundamental tensor $h = (h_{\gamma\beta}{}^a)$ with respect to E_a is equal to

$$\text{(4.7)}\qquad h_{\beta\alpha}{}^a = \tilde{\Gamma}_{\beta\alpha}^a = 0$$

and the normal connection $L = (L_{cb}{}^\alpha)$ of each fibre $\overline{M}$ in $\tilde{M}$ is

$$\text{(4.8)}\qquad L_{cb}{}^\alpha = J_{cb}\bar{\xi}^\alpha\,.$$

Therefore each fibre is totally geodesic. According to (4.6), we can see that

$$\tilde{\nabla}_c\tilde{\phi}_{\beta\alpha} = \partial_c\tilde{\phi}_{\beta\alpha} - \tilde{\Gamma}^d_{c\beta}\tilde{\phi}_{d\alpha} - \tilde{\Gamma}^\gamma_{c\beta}\tilde{\phi}_{\gamma\alpha} - \tilde{\Gamma}^d_{c\alpha}\tilde{\phi}_{\beta d} - \tilde{\Gamma}^\gamma_{c\alpha}\tilde{\phi}_{\beta\gamma}$$

are equal to zero. From this fact and (4.4), we have

$$*\nabla_c\bar{\phi}_{\beta\alpha} = (\tilde{\nabla}_j\tilde{\phi}_{ih})E^j{}_cC^i{}_\beta C^h{}_\alpha = 0 .$$

Hence, by means of Proposition 3.1, $\tilde{M}$ is a fibred Sasakian space with the base space M and the fibre $\bar{M}$.

Put $q = n/2$ and $r = (p-1)/2$ for short, and take a $\tilde{\phi}$-basis $\{e_1, \ldots, e_m\}$ at every point of $\tilde{M}$ such that $e_1, \ldots, e_q$, $e_{q+1} = \tilde{\phi}e_1, \ldots, e_n = \tilde{\phi}e_q$ are horizontal vectors and $e_{n+1}, \ldots, e_{n+r}, e_{n+r+1} = \tilde{\phi}e_{n+1}, \ldots, e_{n+p-1} = \tilde{\phi}e_{n+r}$, $e_m = \tilde{\xi}$ are vertical vectors. We denote by $H(X, Y)$ the sectional curvature with respect to the plane spanned by X and Y. By means of (1.10) ~ (1.17) and (4.7) ~ (4.8), we obtain

$$\begin{aligned}
&H(e_s, \tilde{\phi}e_s) = c - 3 && \text{for} \quad 1 \leq s \leq q\,,\\
&H(e_s, e_t) = \frac{c}{4} && \text{for} \quad 1 \leq s\,, t \leq q\,, s \neq t\,,\\
&H(e_\alpha, \tilde{\phi}e_\alpha) = -c - 3 && \text{for} \quad n+1 \leq \alpha \leq n+r\,,\\
&H(e_\alpha, e_\beta) = -\frac{c}{4} && \text{for} \quad n+1 \leq \alpha\,, \beta \leq n+r\,, \alpha \neq \beta \quad \text{and}\\
&H(e_\alpha, e_s) = 0\,,
\end{aligned}$$

and see that the relation

$$8H(e_\lambda, e_\mu) - 6 = H(e_\lambda, \tilde{\phi}e_\lambda) + H(e_\mu, \tilde{\phi}e_\mu) \qquad (\lambda \neq \mu) \tag{4.9}$$

is satisfied for λ, $\mu = 1, \ldots, q, n+1, \ldots, n+r$. That the equation (4.9) is satisfied for a $\tilde{\phi}$-basis is an equivalent condition to the vanishing C-Bochner curvature tensor in a Sasakian space of dimension $m \geq 5$ [cf. 4, 11]. Hence $\tilde{M}$ is a Sasakian space with vanishing C-Bochner curvature tensor but not of constant $\tilde{\phi}$-holomorphic sectional curvature because $H(e_s, \tilde{\phi}e_s) \neq H(e_\alpha, \tilde{\phi}e_\alpha)$. This is an example we seek for.

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