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Interior and exterior boundary value problems for the degenerate Monge-Ampere operator

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1. Introduction

This paper deals with interior (exterior) Dirichlet and (Neumann) boundary value problems (b.v.p.) for the real Monge-Ampere (M.A.) equation:

$$\det u_{x_i x_j} = f(|x|)g(|Du|) \qquad \text{in } B_i \text{ (or } B_e), \tag{1}$$

where $B_i = \{x \in \boldsymbol{R}^n; |x| < R\}$, $B_e = \{x \in \boldsymbol{R}^n; |x| > R\}$, $f \geqq 0$, $g(|p|) \geqq 0$.

When we investigate this problem we have in mind the fact that the equation of Gauss curvature of every C^2-smooth surface is given by

$$\det u_{x_i x_j} = K(x)(1 + |Du|^2)^{(n+2)/2} \tag{2}$$

i.e. is of type (1) ($g(t) = (1 + t^2)^{(n+2)/2}$).

Unfortunately the growth of the right-hand side with respect to $|Du|$ leads to the nonexistence results for the Dirichlet b.v.p. even in the case when the Gauss curvature is positive. More precisely, it was shown in [12, 16] that for every $C = \text{const}$ and every $\varepsilon > 0$ there exists C^∞-function φ, $|\varphi| < \varepsilon$ for which the Dirichlet problem for (2) with data $C + \varphi$ on the boundary has no classical convex solution. For this reason only constant boundary data will be considered. This enables us to investigate arbitrary growth of $g(|p|)$. Further on our basic assumption is $g(|p|) \geqq g_0 = \text{const} > 0$ since the more interesting geometric applications satisfy this condition. The degeneration of $g(|p|)$ leads to quite complicated effects like bifurcation of the solutions (see the appendix). We propose complete results for existence, uniqueness and regularity of the classical convex solutions of the M.A. operator with constant data in a ball (B_i, B_e). It is interesting to point out that in this case each classical solution turns out to be a radially symmetric one.

2. Statement of the main results

Because of the lack of space we shall formulate and prove only interior Dirichlet (D_i) and exterior Neumann (N_e) problems for equation (1). By the same methods we can prove similar results for (D_e) and (N_i). Further on the short notations

$$u_{ij} = u_{x_i x_j}, \qquad u_i = u_{x_i}$$

will be used and the summation convention $a^{ij}u_{ij} = \sum_{i,j=1}^{n} a^{ij}u_{ij}$ is understood.

Thus we will study the following b.v.p.:

$$(D_i) \qquad \begin{cases} \det u_{ij} = f(|x|)g(|Du|) \text{ in } B_i\,, \\ u = C \text{ on } \partial B_i\,; \end{cases}$$

$$(N_e) \qquad \begin{cases} \det u_{ij} = f(|x|)g(|Du|) \text{ in } B_e\,, \\ \dfrac{\partial u}{\partial \nu} = C \text{ on } \partial B_e\,, \\ \lim\limits_{|x|\to\infty} \left(\dfrac{d}{d|x|} u|_{S(|x|)}\right) = C_1\,, \quad \lim\limits_{|x|\to\infty} (u - C_1|x|)|_{S(|x|)} = C_2\,, \end{cases}$$

where ν is the unit inner normal to ∂B_e, $S(|x|)$ is the sphere $\{y \in \boldsymbol{R}^n, |y| = |x|\}$ and C, C_1, C_2 are constants.

The first result of our paper is

PROPOSITION 1. *Consider b.v.p. (D_i) $((N_e))$ and suppose that $f(|x|) \geqq 0$, $g(|p|) \geqq g_0 > 0$, $f \in C(\overline{B_i})$ $(f \in C(\overline{B_e}))$, $g \in C^1(\boldsymbol{R}^n)$. Then every $C^2(\overline{B_i})$ $(C^2(\overline{B_e}))$ convex solution of (D_i) $((N_e))$ is radially symmetric.*

Now we can formulate the following existence and uniqueness results.

THEOREM 2. *Suppose $f(|x|) \in C^{n-1,\alpha}(\overline{B_i})$, $n \geqq 2$, $0 < \alpha \leqq 1$, $f \geqq 0$, $g \in C^1(\boldsymbol{R}^n)$, $g(|p|) \geqq g_0 = const > 0$. Then the problem (D_i) has a unique convex solution $u \in C^{2,\alpha/n}(\overline{B_i})$ iff inequality (3) holds, i.e.*

$$\int_0^\infty (t^{n-1}/g(t))\,dt > \int_0^R t^{n-1}f(t)\,dt\,. \tag{3}$$

Moreover, if $f \in C^{n,\alpha}(\overline{B_i})$ then $u \in C^{2,(\alpha+1)/n}(\overline{B_i})$.

THEOREM 3. *Suppose $f(|x|) \in C^{n-1}(\overline{B_e})$, $n \geqq 2$, $f \geqq 0$, $g(|p|) \in C^1(\boldsymbol{R}^n)$, $g(|p|) \geqq g_0 = const > 0$. Then the problem (N_e) has a unique convex solution $u \in C^2(\overline{B_e})$ iff*

$$\begin{gathered} C_1 \geqq C \geqq 0\,, \quad \int_R^\infty t^{n-1}f(t)\,dt = \int_C^{C_1} (t^{n-1}/g(t))\,dt\,, \\ \int_R^\infty \left(\int_t^\infty s^{n-1}f(s)\,ds\right)dt < \infty\,, \end{gathered} \tag{4}$$

and for $C = 0$, $f(t) = (t - R)^{n-1}f_1(t)$, where $0 \leqq f_1 \in C(\overline{B_e})$.

Moreover, if $C = 0$ and f has a $C^{n-1,\alpha}(C^{n,\alpha})$ smooth zero extension in $\boldsymbol{R}^n$ then $u \in C^{2,\alpha/n}(\overline{B_e})$ $(C^{2,(\alpha+1)/n}(\overline{B_e}))$.

If $C > 0$ *and* $f \in C^k(\overline{B_e})(C^\infty(\overline{B_e}))$, $g \in C^k(\boldsymbol{R}^n)(C^\infty(\boldsymbol{R}^n))$ *then* $u \in C^{k+2}(\overline{B_e})$ $(C^\infty(\overline{B_e}))$.

The result in Theorem 2 is the best possible one as the following examples show.

EXAMPLE 1. Consider the problem

$$\begin{cases} \det u_{ij} = f_p(|x|) & \text{in } B_i = \{|x| < 2\}\,, \\ u = C & \text{on } \partial B_i\,, \end{cases}$$

where $f_p(t) = 0$ for $0 \leqq t \leqq 1$ and $f_p(t) = (t-1)^p$ for $1 \leqq t \leqq 2$. If $p = n - 1 + \alpha$, $0 < \alpha \leqq 1$, then $f_p \in C^{n-1,\alpha}(\overline{B_i})$. According to the formula for the unique convex solution u of the problem (D_i) proposed in the proof of Theorem 2 we have that $u \in C^{2,\alpha/n}(\overline{B_i}) \backslash C^{2,\alpha/n+\varepsilon}(\overline{B_i})$ for every $\varepsilon > 0$. If $p = n + \alpha$, then $f_p \in C^{n,\alpha}(\overline{B_i})$ but the solution $u \in C^{2,(\alpha+1)/n}(\overline{B_i}) \backslash C^{2,(\alpha+1)/n+\varepsilon}(\overline{B_i})$.

The next example shows that the further regularity of f and g does not imply further regularity of the solution.

EXAMPLE 2. Consider the problem

$$\begin{cases} \det u_{ij} = |x|^2 & \text{in } B_i\,, \\ u = C & \text{on } \partial B_i\,. \end{cases}$$

The right-hand side is infinitely smooth but the solution

$$u(x) = C + (n/(2n+2)) \cdot (n/(n+2))^{1/n} (|x|^{2+2/n} - R^{2+2/n})$$

is of the class $C^{2,2/n}(\overline{B_i}) \backslash C^{2,2/n+\varepsilon}(\overline{B_i})$ for every $\varepsilon > 0$.

Let us now give some sufficient conditions for further regularity of the solutions.

PROPOSITION 4. *Suppose* $0 \leqq f(|x|) \in C^\infty(\overline{B_i})$, $g(|p|) \in C^\infty(\boldsymbol{R}^n)$, $g(|p|) \geqq g_0 =$ *const* > 0. *Then the solution of* (D_i) *belong to* $C^\infty(\overline{B_i})$ *if*

$$|x| \left(\int_0^{|x|} t^{n-1} f(t)\, dt \right)^{1/n} \in C^\infty(N_I)\,, \tag{5}$$

where N_I *is a neighborhood* (*ngbh*) *of the set* $I = \{x \in \overline{B_i};\ \int_0^{|x|} t^{n-1} f(t)\, dt = 0\}$.

REMARK 1. The solution u of (D_i) belongs to $C^\infty(\overline{B_i} \backslash I)$ if $f \in C^\infty(\overline{B_i})$, $g \in C^\infty(\boldsymbol{R}^n)$. Condition (5) guarantees the infinite smoothness of u in a ngbh of I. For wide classes of equations, for example when $f(|x|) = |x|^{2m} f_1(|x|)$, $f_1(0) > 0$, $g \in C^\infty$, condition (5) is also necessary for C^∞ regularity (since (5) is equivalent to the condition m/n is an integer).

PROPOSITION 5. *Suppose* $0 \leqq f(|x|) \in C^\infty(\overline{B_e})$, $g(|p|) \in C^\infty(\boldsymbol{R}^n)$, $g(|p|) \geqq g_0 =$ *const* > 0. *Then the solution of* (N_e) *with* $C = 0$ *is* $C^\infty(\overline{B_e})$ *if*

$$\left(\int_R^{|x|} t^{n-1} f(t)\, dt\right)^{1/n} \in C^\infty(N_E)\,, \tag{6}$$

where N_E *is a ngbh of the set*

$$E = \left\{x \in \overline{B_e};\ \int_R^{|x|} t^{n-1} f(t)\, dt = 0\right\}.$$

COROLLARY. *The Dirichlet problem* (D_i) *for equation* (2) *has a unique classical solution iff*

$$\frac{1}{n} > \int_0^R t^{n-1} f(t)\, dt\,.$$

The uniformly elliptic M.A. operator (i.e. when the right-hand side is positive) in strictly convex bounded domains has been studied by [1, 4, 6, 9] and others. As for the degenerate case existence of generalized (C^0, $C^{1,1/n}$ or $C^{1,1}$) solutions was proved (see [2, 14, 15]).

In [11] existence of radially symmetric $C^k(\overline{B_i})$ solutions of the problem (D_i) under the stronger conditions $g \equiv 1$, f has a zero of finite order at the origin was proved. The equation (2) of surfaces having prescribed Gauss curvature is not contained in the class of operators considered in [11]. The precise regularity results (Theorem 2, Prop. 4) can not be obtained by the methods developped in [11].

The above observations stimulated our investigations of the degenerate M.A. operator in a ball with constant data.

We hope that our results will be useful in further investigations of the classical solvability.

Very little is known about the solvability of b.v.p. for M.A. operator in unbounded domains even in the uniformly elliptic case. In this direction we were influenced by the papers of Kusano and Usami [7] and Kusano, Naito and Swanson [8] where radially symmetric solutions for nonlinear Laplace operators were obtained. More precisely, we use the ideas of the above mentioned authors in order to state the exterior Neumann problem (N_e) and to obtain results for the uniqueness of the convex classical solutions.

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3. Proofs of the main results

We shall prove at first the radial symmetry of the classical convex solutions of (D_i) and (N_e).

PROOF OF PROPOSITION 1. We shall give a detailed proof of the Dirichlet problem (D_i) and we shall only point out the differences occuring for the problem (N_e).

Suppose $u, v \in C^2(\overline{B_i})$ are convex solutions of b.v.p. (D_i). Let $\varepsilon > 0$ be a positive constant and let us consider the function

$$w = u + \varepsilon(e^{a|x|^2/2} - e^{aR^2/2}).$$

If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the Hessian matrix $\{u_{x_ix_j}(x)\}$, by rotation of the coordinate system we obtain the inequalities

$$\begin{aligned}
\det w_{x_ix_j} &= \det(\operatorname{diag}\lambda_i + (\varepsilon a Id + \varepsilon a^2 x_i x_j)e^{a|x|^2/2}\\
&= (\lambda_1 + \varepsilon a e^{a|x|^2/2})\dots(\lambda_n + \varepsilon a e^{a|x|^2/2})\\
&\quad + \sum(\lambda_1 + \varepsilon a e^{a|x|^2/2})\dots(\lambda_{i-1} + \varepsilon a e^{a|x|^2/2})\varepsilon a^2 x_i^2 e^{a|x|^2/2}\\
&\quad \times (\lambda_{i+1} + \varepsilon a e^{a|x|^2/2})\dots(\lambda_n + \varepsilon a e^{a|x|^2/2})\\
&> \lambda_1\dots\lambda_n + n\varepsilon a(\lambda_1\dots\lambda_n)^{(n-1)/n}\cdot e^{a|x|^2/2}\\
&\quad + \varepsilon a^2 e^{a|x|^2/2}\sum_{i=1}^{n}\lambda_1\dots\lambda_{i-1}x_i^2\lambda_{i+1}\dots\lambda_n.
\end{aligned}$$

In the above inequalities we use the well-known inequality for the geometric and arithmetic means. Since $\lambda_1\lambda_2\dots\lambda_n = f(|x|)g(|Du|)$ it follows that

$$\begin{aligned}
&\det w_{ij}/g(|Dw|) - \det v_{ij}/g(|Dv|) > \varepsilon a[f^{(n-1)/n}\cdot(K_1 - K_2|x|f^{1/n})\\
&\quad + a\Sigma\lambda_1\cdots\lambda_{i-1}x_i^2\lambda_{i+1}\dots\lambda_n]\cdot e^{a|x|^2/2}/g(|Dw|),
\end{aligned}$$

where the constants K_1, K_2 do not depend on ε and a if the inequality $\varepsilon a e^{aR^2/2} \leqq 1$ holds. Consequently for the linear operator $L = A^{ij}\partial^2/\partial x_i\partial x_j$ ($A^{ij} = \int_0^1 B^{ij}(t)\,dt$, B^{ij} are the cofactors of the matrix $(tw + (1-t)v)_{x_ix_j}$) we obtain the inequality

$$L(w - v) > \varepsilon a[f^{(n-1)/n}\cdot(K_1 - K_2|x|f^{1/n}) + a\Sigma\lambda_1\dots\lambda_{i-1}x_i^2\lambda_{i+1}\dots\lambda_n] \tag{7}$$

at the point $y \in B_i$ where $w - v$ attains its maximum.

In the set $\overline{B_i}\backslash B_0$, $B_0 = \{x \in \overline{B_i};\, K_1 - K_2|x|f^{1/n} > 0\}$ the matrix $\{u_{ij}\}$ is strictly positive since $f(|x|) \geq K_3 > 0$ so that $\lambda_i \geq K_4 = \text{const} > 0$, $i = 1, 2, \dots, n$ with constant K_4 independent of ε and a. Moreover, $|x| \geq K_5 = \text{const} > 0$ in $\overline{B_i}\backslash B_0$ and we have the estimate: $L(w - v) > 0$ for $y \in B_i\backslash B_0$ when a is a sufficiently large constant independent of ε. In the set $\overline{B_0}$ it follows trivially that the right-hand side of inequality (7) is nonnegative, i.e., $L(w - v) > 0$ for $y \in B_0$. Since $w - v \leq 0$ on ∂B_i from the maximum principle we have $w - v \leq 0$, i.e. $u - v \leq \varepsilon e^{aR^2}$ for $0 < \varepsilon < (e^{-aR^2/2})/a$. Letting $\varepsilon \to 0$ we obtain $u \leq v$ in $\overline{B_i}$. In the same way the inequality $v \leq u$ holds in $\overline{B_i}$, i.e., $u \equiv v$ in $\overline{B_i}$.

In order to prove the radial symmetry of the convex solutions of (N_e) we will show that $u \leq v$ in $\bar{B}_e$. Suppose that

$$\sup_{B_e}(u - v) = (u - v)(z) = b > 0 \quad (|z| < \infty \text{ from the boundary data}).$$

We consider the auxiliary function $w = u + \varepsilon e^{a|x|^2/2}$ in the annulus $H = \{x \in \boldsymbol{R}^n; R < |x| < R_0\}$, where $R_0 > R$ is such that $z \in H$ and $\sup|u - v| < b/2$ on $\{|x| = R_0\}$. Since $\dfrac{\partial(w - v)}{\partial v} = u_v - v_v + \varepsilon a R e^{aR^2/2} > 0$ on $\{|x| = R\}$, where v is the unit inner normal to ∂B_e, it follows that $w - v$ does not attain its maximum on ∂B_e. Moreover, $(w - v)(z) > b$ and $\sup|w - v| < b$ on $\{|x| = R_0\}$ if $0 < \varepsilon < (b/2)\cdot e^{-aR_0^2/2}$ so that $w - v$ attains its maximum at the interior point $z_0 \in H$. In the same way as above we obtain the inequality $L(w - v)(z_0) > 0$ if $0 < a$ is sufficiently large (a is independent of ε) and $\varepsilon < (e^{-aR_0^2/2})/aR_0$. This fact contradicts our assumption, i.e., $u \leq v$ in $\bar{B}_e$. In the same way the inequality $v \leq u$ in $\bar{B}_e$ holds, i.e., $u \equiv v$ in B_e. The observation that the b.v.p. (D_i) and (N_e) are invariant under the action of the orthogonal group $SO(n)$ completes the proof of Proposition 1.

PROOF OF THEOREM 2. *Necessity.* There are no difficulties to check that every $C^2(\overline{B_i})$ convex solution which according to Proposition 1 is radially symmetric satisfies the ordinary differential equation:

$$v''v'^{n-1} = r^{n-1}f(r)g(v') \qquad \text{in } [0, R], \tag{8}$$
$$v'(0) = 0, \qquad v(R) = C.$$

So the identity

$$\int_0^{v'(r)} (t^{n-1}/g(t))\,dt = \int_0^r t^{n-1}f(t)\,dt$$

($v'(r) \geqq 0$, $v''(r) \geqq 0$ from the convexity of the solution $u(x) = V(|x|)$, $|x| = r$) holds for $r \in [0, R]$ and

$$\int_0^R t^{n-1}f(t)\,dt < \int_0^\infty (t^{n-1}/g(t))\,dt.$$

Sufficiency. Let us introduce the functions

$$F_i(r) = \left(\int_0^r t^{n-1}f(t)\,dt\right)^{1/n} \quad \text{and} \quad G_i(y) = \left(\int_0^y (t^{n-1}/g(t))\,dt\right)^{1/n}.$$

The differentiable function G_i is strictly monotonically increasing in $(-\infty, \infty)$ when n is odd and in $[0, \infty)$ when n is even, respectively; i.e. $G_i' > 0$, so that the inverse function G_i^{-1} is well defined and differentiable in $(G_i(-\infty), G_i(\infty))$ when

n is odd and $[0, G_i(\infty))$ when n is even, respectively. We will only check that G_i is differentiable at the origin. From L'Hospital's rule we have

$$G_i'(0) = \lim_{y\to 0} y^{n-1} \Bigg/ \left(ng(y) \left(\int_0^y \frac{t^{n-1}\, dt}{g(t)} \right)^{(n-1)/n} \right)$$
$$= (1/ng(0)) \left(\lim_{y\to 0} y^n \Bigg/ \left(\int_0^y \frac{t^{n-1}\, dt}{g(t)} \right)^{(n-1)/n} \right) = (ng(0))^{-1/n} > 0\,.$$

Therefore from (3) the function $G_i^{-1}(F_i(r))$ is well defined and continuous for $r \in [0, R]$. We will prove that

$$v = C - \int_{|x|}^R G_i^{-1}(F_i(t))\, dt$$

is a convex solution of (D_i). An easy calculation shows that $v(r)$ belongs to the class $C^1([0, R])$ and $v'(0) = 0$, $v(R) = C$. Since F_i is differentiable outside the set $I = \{x \in \overline{B}_i\colon \int_0^{|x|} t^{n-1} f(t)\, dt = 0\}$ we have $v(|x|) \in C^2(\overline{B}_i \setminus I)$.

In order to obtain the C^2 smoothness of $v(|x|)$ we shall check it in a ngbh of the set I. Let $r_0 = \inf\{r \in [0, R]; f(r) = 0\}$ so that $F_i(r) \neq 0$ for $r > r_0$. Hence it follows that $F_i \in C^2((r_0, R])$, $G_i^{-1} \in C^2$ and $v \in C^3((r_0, R])$. In the interval $[0, r_0)$ (for $r_0 > 0$) we have $v \equiv \text{const.}$, i.e., the function v belongs to the class $C^3([0, R] \setminus \{r_0\})$.

(i) We will first consider the case $r_0 = 0$ and will show that $v \in C^3$, i.e., $G_i \in C^2$ and $F_i \in C^2$ if $f(0) > 0$. In fact

$$F_i(r) = r\left[\frac{f(0)}{n} + \int_0^1 s^{n-1}(f(sr) - f(0))\, ds\right]^{1/n} = r f_1^{1/n}(r) \in C^2$$

in a sufficiently small ngbh of the origin since

$$F_i'(r) = f_1^{1/n}(r) + \frac{r\displaystyle\int_0^1 s^n f'(sr)\, ds}{f_1^{(n-1)/n}(r)}\,, \quad F_i'(0) = f_1^{1/n}(0) = \left(\frac{f(0)}{n}\right)^{1/n} > 0 \quad \text{and}$$

$$\lim_{r\to 0} \frac{F_i'(r) - F_i'(0)}{r} = \lim_{r\to 0} \frac{f_1^{1/n}(r) - f_1^{1/n}(0)}{r} + \lim_{r\to 0} \frac{\displaystyle\int_0^r s^n f'(sr)\, ds}{f_1^{(n-1)/n}(r)}$$
$$= \frac{f'(0)}{n(n+1)\left(\dfrac{f(0)}{n}\right)^{(n-1)/n}} + \frac{f'(0)}{(n+1)\left(\dfrac{f(0)}{n}\right)^{(n-1)/n}}\,.$$

In the same way we check that $G_i \in C^2$ as $g(0) > 0$.

Suppose $f(0) = 0$. Then for $F_i(r)$ the estimate

$$0 \leqq F_i(r) \leqq K_6 \left| \int_0^r t^n \, dt \right|^{1/n} \leqq K_7 r^{(n+1)/n}$$

holds. Since $v'(0) = 0$ it follows that v is twice differentiable at the origin and $v''(0) = 0$.

In order to show that v'' is Hölder continuous at 0 with exponent α/n, $0 < \alpha \leqq 1$ it is enough to prove the estimate

$$|F_i'(r)| \leqq K_8 r^{1/n} \quad \text{since} \quad v''(r) = \frac{F_i'(r)}{G_i'(G_i^{-1}(F_i(r)))}.$$

Let us note that the direct application of l'Hospital's rule does not lead to any results when F_i has zero of sufficiently large order (including ∞). That is why we will adopt a different approach. For this purpose the following inequality will be proved

$$r^{n-1/(n-1)} f^{n/(n-1)}(r) \leqq 2K_9 \int_0^r t^{n-1} f(t) \, dt \tag{9}$$

for sufficiently small positive r. Let

$$h_0(r) = r^{n-1/(n-1)} f^{n/(n-1)}(r) - 2K_9 \int_0^r t^{n-1} f(t) \, dt$$

and K_9 be sufficiently large so that the inequality $(n - 1/(n-1))^{n-1} f(r) \leqq K_9^{n-1} r$ holds in a ngbh N_0 of the origin (we remind that $f(0) = 0$). Then $h_0'(r) \leqq 0$ for $r \in N_0$ provided the inequality

$$\frac{n}{n-1} r^{1-1/(n-1)} f'(r) - K_9 f^{(n-2)/(n-1)}(r) \leqq 0 \tag{10}$$

is fulfilled. From $h_0(0) = 0$ and $h_0'(r) \leqq 0$ we immediately derive (9). It is clear that (10) holds at the points $r \in N_0$ for which $f'(r) \leqq 0$ or when $n = 2$. Suppose that $n > 2$. Then at the points $r \in N_1 \subset N_0$ for which $f'(r) > 0$, inequality (10) is equivalent to the inequality

$$h_1(r) = r(f'(r))^{(n-1)/(n-2)} - 2K_{10} f(r) \leqq 0.$$

Now $h_1(0) = 0$ and $h_1'(r) \leqq 0$ in N_1 if

$$\frac{n-1}{n-2} r f''(r) (f'(r))^{1/(n-2)} - K_{10} f'(r) \leqq 0. \tag{11}$$

When $n = 3$ or $f''(r) \leqq 0$ the above inequality follows trivially. Let us suppose that $n > 3$. Then for $r \in N_2 \subset N_1$, $f''(r) > 0$, inequality (11) is equivalent to the following one:

$$h_2(r) = r^{(n-2)/(n-3)} (f''(r))^{(n-2)/(n-3)} - 2K_{11} f'(r) \leqq 0.$$

By induction for the functions

$$h_m(r) = r^{(n-2)/(n-m-1)}(f^{(m)}(r))^{(n-m)/(n-m-1)} - 2K_{m+9}f^{(m-1)}(r)$$

we will prove, when $n > m$, the inequality $h_m(r) \leqq 0$ at the points $r \in N_{m-1} = \{r \in N_{m-2}; f^{(m-1)}(r) > 0\}$. For this purpose it is enough to prove that $h'_m(r) \leqq 0$, i.e.,

$$r^{(n-2)/(n-m-1)}f^{(m+1)}(r) - K_{m+9}(f^{(m)}(r))^{(n-m-2)/(n-m-1)} \leqq 0 \tag{12}$$

when $r \in N_m = \{r \in N_{m-1}; f^{(m)}(r) > 0\}$. But for $m = n-2$ inequality (12) is of the following type:

$$r^{n-2}f^{(n-1)}(r) \leqq K_{n+7}$$

which trivially follows from the smoothness of f.

(ii) Let us now suppose that $r_0 > 0$. Since $f \in C^{n-1,\alpha}[0, R]$ and $f \equiv 0$ in $[0, r_0]$ we have $f(r_0) = f'(r_0) = \cdots = f^{(n-1)}(r_0) = 0$ and $|f^{(n-1)}(r)| \leqq C_3|r - r_0|^\alpha$ in a ngbh N_0 of r_0. Therefore

$$\begin{aligned} 0 \leqq F_i(r) &= \left(\int_{r_0}^r t^{n-1}f(t)\,dt\right)^{1/n} \\ &= \left(\int_{r_0}^r t^{n-1}\frac{(t-r_0)^{n-1}}{(n-2)!}\int_0^1 (1-s)^{n-2}f^{(n-1)}(r_0 + s(t-r_0))\,ds\,dt\right)^{1/n} \\ &\leqq C_4\left(\int_{r_0}^r (t-r_0)^{n-1+\alpha}\,dt\right)^{1/n} \leqq C_5|r - r_0|^{(n+\alpha)/n}. \end{aligned}$$

Thus $0 \leqq \dfrac{v'(r)}{|r-r_0|} = \dfrac{G_i^{-1}(F_i(r))}{|r-r_0|} \leqq C_6|r-r_0|^{\alpha/n}$ and $v''(r_0) = 0$.

Now we will show that v'' is Hölder continuous at r_0 with exponent α. For this purpose we will prove the estimate:

$$g_0(r) = f^{n/(n-1)}(r)(r-r_0)^{-\alpha/(n-1)} - 2K'_0\int_{r_0}^r t^{n-1}f(t)\,dt \leqq 0 \tag{13}$$

for $r > r_0$, $r \in N_0$ where N_0 is a ngbh of r_0. Repeating the same procedure as in (i) we obtain (13); the only difference being the using of the auxiliary functions

$$g_m(r) = (f^{(m)}(r))^{(n-m)/(n-m-1)} - 2K'_m(r-r_0)^{\alpha/(n-m-1)}f^{(m-1)}(r)$$

instead of $h_m(r)$.

The smoothness of the function $u(x) = v(|x|)$ follows trivially from the smoothness of $v(r) \in C^{2,\alpha/n}([0, R])$ as well as $v'(0) = 0$. More precisely, $u_{ij}(x) =$

$\left(v'' - \frac{v'}{|x|}\right)\frac{x_i x_j}{|x|^2} + \frac{v'}{|x|}\delta^{ij}$ so that $u_{ij}(0) = \delta^{ij}v''(0)$. Since $u_{ij}(x) - u_{ij}(0) = \delta^{ij}\int_0^1 (v''(s|x|) - v''(0))\,ds + \frac{x_i x_j}{|x|^2}\int_0^1 (v''(|x|) - v''(s|x|))\,ds$ we immediately obtain the Hölder continuity of the second derivatives u_{ij}.

The proof that $u(x)$ is a convex solution of (D_i) is the same as in [11] and we omit it.

SKETCH OF THE PROOF OF THEOREM 3. *Necessity.* Suppose $u(x)$ is a classical convex solution of (N_e). Then from Proposition 1 $u(x) = v(|x|)$ is a convex radially symmetric solution i.e. $v' \geqq 0$, $v'' \geqq 0$ because of the positiveness of the Hessian matrix $\{v_{ij}\}$. Consequently $0 \leqq C = v'(R) \leqq v'(\infty) = C_1$. The case $C = C_1$ is trivial as then $v' \equiv C$ i.e. $v(|x|) = C|x| + \text{const}$ and $f(|x|) \equiv 0$.

As in the proof of Theorem 2 from (8) we obtain the identity

$$\int_R^\infty t^{n-1}f(t)\,dt = \int_C^{C_1} (t^{n-1}/g(t))\,dt\,.$$

Moreover, from (8) it follows that

$$f(r) = \frac{v''(r)(r-R)^{n-1}(\int_0^1 v''(\theta r + (1-\theta)R)\,d\theta)^{n-1}}{r^{n-1}g(v'(r))} = (r-R)^{n-1}f_1(r)\,,$$

$0 \leqq f_1 \in C(\overline{B_e})$ if $C = 0$. Let us now introduce the functions

$$F_e(r) = \left(\int_R^r t^{n-1}f(t)\,dt\right)^{1/n}, \qquad G_e(y) = \left(\int_C^y (t^{n-1}/g(t))\,dt\right)^{1/n}.$$

Then

$$v(r) = K + \int_R^r G_e^{-1}(F_e(t))\,dt \quad \text{and} \quad G_e^{-1}(F_e(\infty)) = C_1 = v'(\infty)\,.$$

So

$$v(r) = K + \int_R^r (G_e^{-1}(F_e(t)) - G_e^{-1}(F(\infty))\,dt + C_1(r-R)$$

and $\lim_{r\to\infty}(v(r) - C_1 r)$ exists if and only if $\int_R^\infty (G_e^{-1}(F_e(\infty)) - G_e^{-1}(F_e(t))\,dt < \infty$. Since $0 < a \leqq (G_e^{-1})' \leqq b$; $a, b = \text{const}$, then

$$\begin{aligned}\int_R^r (G_e^{-1}(F_e(\infty)) - G_e^{-1}(F_e(t)))\,dt &\geqq a\int_R^r (F_e(\infty) - F_e(t))\,dt \\ &\geqq a_1 \int_R^r \int_t^\infty s^{n-1}f(s)\,ds\,dt \geqq 0\,,\end{aligned}$$

$a_1 > 0$, using the elementary inequality

$$\frac{A-B}{nA^{(n-1)/n}} \leqq A^{1/n} - B^{1/n} \leqq \frac{A-B}{A^{(n-1)/n}} \quad \text{for} \quad A > B > 0\,.$$

Sufficiency. Consider the case $C = 0$ at first. From condition (4) the function

$$v(|x|) = \int_R^{|x|} G_e^{-1}(F_e(t))\,dt + C_2 + C_1 R + \int_R^{\infty} G_e^{-1}(F_e(\infty)) - G_e^{-1}(F_e(t))\,dt$$

is well defined and differentiable for $|x| \geqq R$ and satisfies the boundary conditions. As it was shown in Theorem 2, $G_e^{-1} \in C^1([0, G_e(C_1)])$. To verify that $F_e \in C^1([R, \infty))$ we use l'Hospital's rule for $r = R$ and the representation $f = (r-R)^{n-1} f_1(r)$, $0 \leqq f_1 \in C(\overline{B_e})$. The smoothness of F_e for $r > R$ is due to the fact that $f \in C^{n-1}(\overline{B_e})$ and to l'Hospital's rule (see Theorem 2). Thus it follows that $v \in C^2([R, \infty))$ and then $u \in C^2(\overline{B_e})$.

To complete the proof of Theorem 3 when $C > 0$ we introduce the functions $G_e^* = G_e^n$, $F_e^* = F_e^n$ and note that $(G_e^*)' > 0$ in $[C, C_1]$. Repeating the same procedure as in the proof of necessity we conclude that the function

$$v(r) = \text{const} + \int_R^r (G_e^*)^{-1}(F_e^*(t))\,dt$$

satisfies the b.v.p. (N_e). Obviously $F_e^* \in C^{k+1}$, $(G_e^*)^{-1} \in C^{k+1}$ so that $v \in C^{k+2}(\overline{B_e})$.

The $C^{2,\alpha/n}$ smoothness of the solution u can be proved in the same way as it was done in Theorem 2.

Proof of Proposition 4. At first we shall prove the smoothness of the solution at the origin. From the representation

$$\mathbf{G}_i(t) = t\left[\frac{1}{ng(0)} - \int_0^1 s^{n-1}\frac{g(st) - g(0)}{g(0)g(st)}\,ds\right]^{1/n}$$

in a sufficiently small ngbh $N_0 = [0, \delta)$ of the origin it follows that $G_i \in C^\infty(N_0)$ and that G_i can be extended as an odd C^∞ function in $(-\delta, \delta)$. From (5) we have $F_i \in C^\infty(N_0)$ and can be extended as an odd function in $(-\delta, \delta)$. Thus $v'(r) = G_i^{-1}(F_i(r))$ can be extended as an odd C^∞ function in $(-\delta, \delta)$, and consequently

$$v(|x|) = \int_0^{|x|} G_i^{-1}(F_i(t))\,dt - \int_0^R G_i^{-1}(F_i(t))\,dt + C$$

belongs to $C^\infty(|x| < \delta)$.

It is easy to check the smoothness of u for $|x| > 0$ as $|x| \in C^\infty(|x| > 0)$.

The proof of Proposition 5 is similar and we omit it.

Appendix

We shall state now some open problems concerning the (D_i) and (N_e) b.v.p.. At first we shall note that a new effect arises when the function g vanishes at 0. It concerns the so-called bifurcation of the solutions.

EXAMPLE 3. Consider the (D_i) b.v.p. and assume that $g(|Du|) = |Du|^k g_1(|Du|)$, $g_1 > 0$, $f(r) \not\equiv 0$. We claim that:

(i) if $k \leqq n-1$, $f \in C^{n-k-1}(\overline{B}_i)$, $g_1 \in C^1$, $g_1 \geqq g_0 = \text{const} > 0$ and $\displaystyle\int_0^\infty \frac{t^{n-k-1}}{g_1(t)}\,dt > \int_0^R t^{n-1} f(t)\,dt$, then there exist at least two convex solutions of (D_i), namely $u_1 = \text{const}$, $u_2 \in C^2(\overline{B}_i)$, $u_1 \not\equiv u_2$;

(ii) if $k \geqq n$, $g \in C^1$, $f \in C^1$ then $u \equiv \text{const}$ is the unique convex radially symmetric solution of the problem (D_i).

The elementary proof similar to the proof of Theorem 2 is left to the reader.

PROBLEM 1. Investigate the (D_i) and (N_e) b.v.p. with a right-hand side f depending on $|x| = r$, u and $|Du|$ and find conditions (necessary, sufficient) such that the corresponding problems possess unique classical convex solutions. Find out conditions when non-uniqueness arises.

PROBLEM 2. Let us consider an arbitrary strictly convex and bounded domain Ω in $\boldsymbol{R}^n$. It is not clear for which right-hand sides $f(x, u, Du) \geqq 0$ and for which data $u|_{\partial\Omega}$, $\left(\left.\dfrac{\partial u}{\partial \nu}\right|_{\partial\Omega}\right)$ the b.v.p. $(D_i)((N_e))$ has a unique convex classical solution. It is worth while studying these problems even in a ball but in the non-radially symmetric case.

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