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The pseudo-convergent sets and the cuts of an ordered field

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Let F be an ordered field. A pair (A, B) of subsets of F is called a cut of F if $A \cup B = F$ and $a < b$ for any $a \in A$ and $b \in B$. In this paper we define the breadth of a cut of F which, in some sense, gives a measure of the gap between the lower class and the upper class.

The notion of pseudo-convergence with respect to the finest valuation among all compatible valuations in F plays an important role. Namely we can build up intrinsic relations between the cuts and the pseudo-convergent sets of elements of F. The limit of a pseudo-convergent set is by no means unique and the totality of limits can be described by the breadth of the pseudo-convergent set. We can show that the breadth of a pseudo-convergent set coincides with the breadth of the corresponding cut. As an application we give the following theorem: F has no strongly proper cut (see Definition 1.7) if and only if $A_0/M_0 = R$ and (F, v) is maximal as a valued field, where v is the finest valuation and (A_0, M_0) its valuation ring (Theorem 3.7).

§1. The finest valuation and cuts

For an ordered field F, let v be the finest valuation of F. The valuation ring of v is $A_0 := A(F, \boldsymbol{Q}) = \{a \in F;\ |a| < b \text{ for some } b \in \boldsymbol{Q}\}$. The maximal ideal and the value group of v will be denoted by M_0 and G respectively. A pair (A, B) of subsets of F is called a cut of F if $F = A \cup B$ and $A < B$.

DEFINITION 1.1. For a cut (A, B) of F, we put $E(A, B) = \{e \in F;\ b - a > |e| \text{ for any } a \in A \text{ and } b \in B\}$ and we call it the *breadth* of the cut (A, B). If $A = \phi$ or $B = \phi$, then we put $E(A, B) = F$. The breadth $E(A, B)$ is a convex additive subgroup of F.

The breadth of a cut (A, B) is characterized by $E(A, B) = \{e \in F;\ a + |e| \in A \text{ for any } a \in A\}$ or $E(A, B) = \{e \in F;\ b - |e| \in B \text{ for any } b \in B\}$. It is clear that a cut (A, B) is archimedean (for the definition, see [2], Definition 1.1) if and only if the breadth of (A, B) is zero.

DEFINITION 1.2. For a convex subgroup D of F, we put $A_1(D) = F^- \setminus D$, $B_1(D) = F^+ \cup D$, $A_r(D) = F^- \cup D$, $B_r(D) = F^+ \setminus D$, where F^+ (resp. F^-) is the set of positive (resp. negative) elements of F. Clearly $(A_1(D), B_1(D))$ and $(A_r(D), B_r(D))$ are cuts of

F and it is easily shown that $D=E(A_1(D), B_1(D))=E(A_r(D), B_r(D))$.

PROPOSITION 1.3. *For a subset D of F, the following statements are equivalent.*

(1) *D is a convex additive subgroup of F.*

(2) *$(v(F\backslash D), v(\dot{D}))$ is a cut of G (i.e. $v(F\backslash D)\cup v(\dot{D})=G$ and $v(F\backslash D)<v(\dot{D})$).*

(3) *D is the breadth of some cut (A, B) of F.*

Moreover if $A\neq\phi$ and $B\neq\phi$, then $D:=E(A, B)$ is an (integral or fractional) ideal of A_0.

PROOF. (1)⇒(2): Since D is a convex additive subgroup of F, $|a|<|b|$ for any $a\in D$ and $b\in F\backslash D$, and so we have $v(a)\geqq v(b)$ by the compatibility of v. Hence it is sufficient to show that $v(a)\neq v(b)$ for any $a\in D$ and $b\in F\backslash D$. If $v(a)=v(b)$, then $v(b/a)=0$ and so $|b/a|<r$ for some $r\in \boldsymbol{Q}$. This implies that there is a positive integer n such that $|b|<n|a|\in D$, and this contradicts the fact that $b\in F\backslash D$.

(1)⇒(3): A convex subgroup D is the breadth of $(A_1(D), B_1(D))$ and also that of $(A_r(D), B_r(D))$.

The converse assertions (2)⇒(1) and (3)⇒(1) are easily shown and we omit the proofs. Q.E.D.

We borrow from I. Kaplansky [1] the following definitions. Let v be a valuation of a field K and A be its valuation ring. A well-ordered set $\{a_i; i\in I\}$ of elements of K, without a last term, is said to be *pseudo-convergent* if and only if $v(a_j-a_i)<v(a_k-a_j)$ for all $i<j<k$. If $\{a_i\}$ is pseudo-convergent, then $v(a_j-a_i)=v(a_{i+1}-a_i)$ for all $i<j$ ([1], Lemma 2). We denote it by γ_i; $\{\gamma_i\}$ is a monotone increasing set of elements in the value group G. The set of all elements y of K such that $v(y)>\gamma_i$ for all i forms an (integral or fractional) ideal of the valuation ring A; this ideal is called the breadth of $\{a_i\}$ and denoted by $\mathscr{B}(\{a_i\})$. An extension of v, or its valuation ring B, is said to be immediate, if the value group and the residue class field coincide with those of v respectively. The extension of v will be also written by the same symbol v. Let B be an immediate extension of A and L its quotient field. An element x' of L is said to be a limit of the pseudo-convergent set $\{a_i\}$ of elements of K if $v(x'-a_i)=\gamma_i$ for all i.

Let x' be an element in L but not in K; then the set $\mathscr{B}(x'):=\{b\in K; v(b)>v(x'-a)$ for all $a\in K\}$ is called the *breadth* of x'. It is an (integral or fractional) ideal of A. For an element a of K, the breadth $\mathscr{B}(a)$ of a is zero (cf. [4], Definition 3). The definition in this paper is slightly different; namely in [4] the breadth $\mathscr{B}(x')$ is defined for an element x' of B and it is an integral ideal of B.

DEFINITION 1.4. Let D be a convex subgroup of the ordered field F. We say that D is *principal* if the minimal element of $v(\dot{D})$ exists; it is equivalent to the condition that D is a principal fractional ideal of A_0. We say that D is *coprincipal* if the maximal element of $v(F\backslash D)$ exists; it is equivalent to the condition that D is isomorphic to M_0, i.e. $D=aM_0$ for some non-zero element a of F.

For a cut (A, B) and an element c of F, we put $A+c=\{a+c;\ a\in A\}$ and $B+c=\{b+c;\ b\in B\}$. It is clear that $(A+c, B+c)$ is a cut of F and the breadth of $(A+c, B+c)$ coincides with that of (A, B).

DEFINITION 1.5. Let $\{a_i;\ i\in I\}$ be a pseudo-convergent set of elements of F. We put $A_L(\{a_i\}):=\{c\in F;$ there exists $i\in I$ such that $c<a_j$ for any $j>i\}$ and $B_L(\{a_i\}):=F\setminus A_L$. We also put $B_R(\{a_i\}):=\{c\in F;$ there exists $i\in I$ such that $a_j<c$ for any $j>i\}$ and $A_R(\{a_i\}):=F\setminus B_R$.

THEOREM 1.6. *For a pseudo-convergent set $\{a_i\}$ of elements of F, we have* $\mathscr{B}(\{a_i\})=E(A_R(\{a_i\}), B_R(\{a_i\}))=E(A_L(\{a_i\}), B_L(\{a_i\}))$.

PROOF. First we show that $\mathscr{B}(\{a_i\})\subset E(A_L(\{a_i\}), B_L(\{a_i\}))$. Let c be any element of $\mathscr{B}(\{a_i\})$. We must show that $|c|<b-a$ for any $a\in A_L(\{a_i\})$ and $b\in B_L(\{a_i\})$. By the definition of $(A_L(\{a_i\}), B_L(\{a_i\}))$, there exist elements $i, j\in I$ such that $a<a_i<a_j<b$. It follows from the condition $v(c)>v(a_j-a_i)$ that $|c|<a_j-a_i<b-a$.

Next we show that $\mathscr{B}(\{a_i\})\supset E(A_L(\{a_i\}), B_L(\{a_i\})$. Let c be any element of F which is not contained in $\mathscr{B}(\{a_i\})$. Then there exists $j\in I$ such that $v(c)<\gamma_j$ (note that $\{\gamma_i;\ i\in I\}$ has no largest element). Put $d:=|a_{j+1}-a_j|$, $a:=a_j-2d$ and $b:=a_j+2d$. Since $v(b-a)=v(4d)=\gamma_j>v(c)$, we have $b-a<|c|$. Hence it is sufficient to show that $a\in A_L(\{a_i\})$ and $b\in B_L(\{a_i\})$. For any element $k\in I$, $k>j+1$, $v(a_k-a_{j+1})=\gamma_{j+1}>\gamma_j=v(d)$, and so $|a_k-a_{j+1}|<d$. Now it is easily shown that $a<a_{j+1}<b$, $b-a_{j+1}\geqq d$ and $a_{j+1}-a\geqq d$. Hence $a<a_k<b$ for any $k>j+1$. This shows that $a\in A_L(\{a_i\})$ and $b\in B_L(\{a_i\})$.

Similarly we can show $\mathscr{B}(\{a_i\})=E(A_R(\{a_i\}), B_R(\{a_i\})$. Q.E.D.

In [3], Definition 2.1, we gave the definition of a proper cut. In the following definition, we define a strongly proper cut.

DEFINITION 1.7. Let (A, B) be a cut of F and put $D=E(A, B)$. Since D is a convex subgroup of F, F/D has a structure of an ordered group. We put $\bar{A}=\{a+D, a\in A\}\subset F/D$ and $\bar{B}=\{b+D, b\in B\}\subset F/D$. It is easy to see that $\bar{A}\cap\bar{B}=\phi$, and so $(\bar{A}, \bar{B})$ is a cut of F/D. We say that (A, B) is strongly proper if $A\neq\phi$, $B\neq\phi$ and neither max $\bar{A}$ nor min $\bar{B}$ exists.

REMARK 1.8. Let $D(D\neq F)$ be a convex subgroup of F. It is clear that $(A_1(D), B_1(D))$ and $(A_r(D), B_r(D))$ are not strongly proper cuts. Moreover $(A_1(D)+c, B_1(D)+c)$ and $(A_r(D)+c, B_r(D)+c)$ are also not strongly proper cuts for any $c\in F$. Conversely. let (A, B) be any cut with the breadth D, and suppose that (A, B) is not strongly proper. Then we can easily show that there exists an element c of F such that $(A, B)=(A_1(D)+c, B_1(D)+c)$ or $(A, B)=(A_r(D)+c, B_r(D)+c)$.

Let $\{a_i; i\in I\}$ be a pseudo-convergent set of elements of F and put $D=\mathscr{B}(\{a_i\})$. Since $\{\gamma_i\}$ has no maximal element, D is not coprincipal. By [1], Lemma 1, either $v(a_i)<v(a_j)$ for all $i<j$, or $v(a_i)=v(a_j)$ from some point on. We can easily show that zero is a limit of $\{a_i\}$ if and only if $v(a_i)<v(a_j)$ for all $i<j$, and so $v(a_i)=\gamma_i$ for all $i\in I$. Suppose that zero is a limit of $\{a_i\}$. Then we can show that if $\{i\in I; a_i<0\}$ is a cofinal subset of I, then $(A_1(D), B_1(D))=(A_L(\{a_i\}), B_L(\{a_i\}))$ and if $\{i\in I; a_i>0\}$ is a cofinal subset of I, then $(A_r(D), B_r(D))=(A_R(\{a_i\}), B_R(\{a_i\}))$.

Put $b_i:=a_i+c, c\in F$. Then $\{b_i\}$ is a pseudo-convergent set of elements of F and it is clear that $(A_L(\{b_i\}), B_L(\{b_i\}))=(A_L(\{a_i\})+c, B_L(\{a_i\})+c)$ and $(A_R(\{b_i\}), B_R(\{b_i\}))=(A_R(\{a_i\})+c, B_R(\{a_i\})+c)$.

PROPOSITION 1.9. *Let (A, B) be a cut of F such that $A\neq\phi$, $B\neq\phi$. Then the following statements are equivalent.*

(1) *(A, B) is not strongly proper and $D:=E(A, B)$ is not coprincipal.*

(2) *There exists a pseudo-convergent set $\{a_i\}$ which has a limit in F such that $(A, B)=(A_L(\{a_i\}), B_L(\{a_i\}))$.*

(3) *There exists a pseudo-convergent set $\{a_i\}$ which has a limit in F such that $(A, B)=(A_R(\{a_i\}), B_R(\{a_i\}))$.*

Moreover, for a pseudo-convergent set $\{a_i\}$, if $(A_L(\{a_i\}), B_L(\{a_i\}))$ (or $(A_R(\{a_i\}), B_R(\{a_i\}))$) is not strongly proper, then $\{a_i\}$ has a limit in F.

PROOF. (1)⇒(2): By Remark 1.8, we may assume that $(A, B)=(A_1(D), B_1(D))$ or $(A, B)=(A_r(D), B_r(D))$. Since D is not coprincipal, there exists a well-ordered cofinal subset $\{g_i; i\in I\}$ of $v(F\backslash D)$. For any $i\in I$, we take an element a_i such that $v(a_i)=g_i$ and $a_i>0$. Then $\{a_i; i\in I\}$ and $\{-a_i; i\in I\}$ are pseudo-convergent sets of elements of F, and we have $(A_R(\{a_i\}), B_R(\{a_i\}))=(A_r(D), B_r(D))$ and $(A_L(\{-a_i\}), B_L(\{-a_i\}))=(A_1(D), B_1(D))$.

(2)⇒(1): By Theorem 1.6, D is the breadth of $\{a_i\}$, and so it is not coprincipal. Let $x\in F$ be a limit of $\{a_i\}$. We can show that if $\{i\in I; a_i<x\}$ is cofinal in I, then $(A_L(\{a_i\}), B_L(\{a_i\}))=(A_1(D)+x, B_1(D)+x)$ and if $\{i\in I; a_i<x\}$ is not cofinal in I, then $(A_L(\{a_i\}), B_L(\{a_i\}))=(A_r(D)+x, B_r(D)+x)$. Hence $(A_L(\{a_i\}), B_L(\{a_i\}))=(A, B)$ is not strongly proper.

The equivalence of (1) and (3) are proved similarly. The proof of the last statement is similar to the proof of (1)⇒(2) and we omit it. Q.E.D.

PROPOSITION 1.10. *Let $\{a_i; i\in I\}$ be a pseudo-convergent set of elements of F and put $D:=\mathscr{B}(\{a_i\})$. Then the following statements are equivalent.*

(1) *Zero is a limit of $\{a_i\}$.*

(2) *$v(a_i)<v(a_j)$ for all $i<j$.*

(3) *$(A_L(\{a_i\}), B_L(\{a_i\}))=(A_1(D), B_1(D))$ or $(A_r(D), B_r(D))$.*

(4) *$(A_R(\{a_i\}), B_R(\{a_i\}))=(A_1(D), B_1(D))$ or $(A_r(D), B_r(D))$.*

The proof of Proposition 1.10 is a routine exercise and left to the reader.

PROPOSITION 1.11. *Let $\{a_i; i\in I\}$ be a pseudo-convergent set of elements of F. If $\{a_i\}$ has no limit in F, then we have*

$$(A_L(\{a_i\}), B_L(\{a_i\})) = (A_R(\{a_i\}), B_R(\{a_i\})).$$

PROOF. Suppose that there exists an element $x\in A_R\backslash A_L$. Let i be any element of I. Then we have $x\leqq a_j$ for some $j>i$ since $x\in A_R$ and $a_k\leqq x$ for some $k>j$ since $x\in B_L$. Hence $v(x-a_j)\geqq v(a_j-a_k)=g_j>g_i=v(a_j-a_i)$, and so $v(x-a_i)=v(x-a_j+a_j-a_i)=g_i$. This shows that x is a limit of $\{a_i\}$, and we have a contradiction.

Q.E.D

Let v be a valuation of a field K and G be its value group. A q-section of a valuation v is map s: $G\to K$ satisfying

(1) $s(0)=1$

(2) $v(s(g))=g$

(3) $s(g_1+g_2)\equiv s(g_1)\cdot s(g_2) \bmod \dot{K}^2$

for all $g, g_1, g_2\in G$ (cf. [5], §7). Let X be the set of orderings of K which are compatible with v. Then every ordering $P\in X$ induces a character σ_P of the group $G/2G$ defined by $\sigma_P(\bar{g})s(g)\in P$. Let K_v be the residue field of v and Y be the set of orderings of K_v. We denote by $\bar{P}$ the ordering of K_v which is canonically induced by P. Then the map ψ: $X\to\chi(G/2G)\times Y$ defined by $\psi(P)=(\sigma_P, \bar{P})$ is bijective (cf. [5], Theorem 7.9). From this fact, we have immediately the following Proposition 1.12.

PROPOSITION 1.12. *Let L/K be an extension of fields. Let v_K, G_K and s_K (resp. v_L, G_L and s_L) be a valuation, its value group and a q-section of K(resp. L). Suppose that v_K (resp. s_K) is a restriction of v_L (resp. s_L). Let X_K (resp. X_L) be the set of orderings of K(resp. L) which are compatible with v_K (resp. v_L). Then for $P\in X_K$ and $P'\in X_L$, the following statements are equivalent.*

(1) *P' is an extension of P.*

(2) *$\bar{P}'$ is an extension of $\bar{P}$ and $\sigma_P=\sigma_{P'}\circ\rho$* where *$\rho$ is the canonical morphism $G_K/2G_K\to G_L/2G_L$.*

We now go back to the ordered field F. Let B_0 be a maximal immediate extension of A_0 and K be the quotient field of B_0. By Proposition 1.12, there exists a unique ordering of K which is compatible with the valuation of K and is an extension of the ordering of F.

PROPOSITION 1.13. *In the above situation, let x be an element of $K\backslash F$ and let $\{a_i; i\in I\}$ be a pseudo-convergent set of elements of F such that $\{a_i\}$ has no limit in F and x is a limit of $\{a_i\}$ (cf.* [1], Theorem 1). *Put $A:=\{a\in F; a<x\}$ and $B:=\{b\in F; x<b\}$. Then*

we have

$\mathscr{B}(\{a_i\}) = \mathscr{B}(x) = E(A, B)$ *and* $(A, B) = (A_L(\{a_i\}), B_L(\{a_i\})) = (A_R(\{a_i\}), B_R(\{a_i\}))$.

PROOF. First we show that $A \subset A_L(\{a_i\})$. Let a be any element of A. Since a is not a limit of $\{a_i\}$, there exists an element i of I such that $v(x-a) < v(x-a_i)$. Hence $v(x-a) < v(x-a_j)$ for any j > i, and so we have $|x-a_j| < x-a$. Thus $a < a_j$ and this implies $a \in A_L(\{a_i\})$. Similarly we have $B \subset B_R(\{a_i\})$, and by Proposition 1.11, $(A, B) = (A_L(\{a_i\}), B_L(\{a_i\})) = (A_R(\{a_i\}). B_R(\{a_i\}))$. It is known that $\mathscr{B}(\{a_i\}) = \mathscr{B}(x)$ (cf. [4], Remark 1), and this implies that $\mathscr{B}(x) = E(A, B)$ by Proposition 1.6. Q.E.D.

§2. Strongly proper cuts

Let $\{a_i\}$ be a pseudo-convergent set of elements of F which has no limit in F. We put $(A, B) = (A_R(\{a_i\}), B_R(\{a_i\})) = (A_L(\{a_i\}), B_L(\{a_i\}))$. By Proposition 1.9, (A, B) is strongly proper. Conversely, in this section, we show that for any strongly proper cut (A, B) for which $E(A, B)$ is not coprincipal, there exists a pseudo-convergent set $\{a_i\}$ of elements of F such that $\{a_i\}$ has no limit in F and $(A, B) = (A_R(\{a_i\}), B_R(\{a_i\})) = (A_L(\{a_i\}), B_L(\{a_i\}))$. Throughout this section, we fix a cut (A, B) and assume that it is strongly proper and $D := E(A, B)$ is not coprincipal.

LEMMA 2.1. *For any* $g \in v(F \backslash D)$, *there exist* $a \in A$ *and* $b \in B$ *such that* $v(b-a) = g$.

PROOF. Let g' be an element of $v(F \backslash D)$ such that $g < g'$. Let c and d be positive elements of F such that $v(c) = g'$ and $v(d) = g$. Since c is not contained in D, there exist $a' \in A$ and $b \in B$ such that $b - a' < c$, and we have $v(b-a') \geqq g'$. Put $a := a' - d$. Then $v(b-a) = v((b-a') + d) = v(d) = g$. Q.E.D.

LEMMA 2.2. *The following statements hold.*

(1) *For anyelement* x *of* A, *there exists an element* y *of* A *such that* $x < y$ *and* $v(y-x) \in v(F \backslash D)$.

(2) *For anyelement* x *of* B, *there exists an element* y *of* B *such that* $y < x$ *and* $v(y-x) \in v(F \backslash d)$.

(3) *For any* $a \in A$, $b \in B$ *and* $g \in v(F \backslash D)$, *there exist elements* $x \in A$ *and* $y \in B$ *such that* $a < x$, $y < b$ *and* $v(y-x) > g$.

PROOF. First we show (1). If $v(y-x) \in v(\dot{D})$ for any $y \in A$, $x < y$, then $y - x$ is an element of D by Proposition 1.3, and so $\bar{x} = \max \bar{A}$ where $(\bar{A}, \bar{B})$ is a cut of $F \backslash D$. This contradicts the fact that (A, B) is strongly proper. The assertion (2) is proved similarly. Next we show the assertion (3). Let g' be an element of $v(F \backslash D)$ with $g' > \max\{v(b-a), g\}$ (note that D is not coprincipal). By Lemma 2.1, there exist $a' \in A$ and $b' \in B$ such that $v(b'-a') = g'$. We take elements $x \in A$ and $y \in B$ so that $\max\{a, a'\} < x$ and $y < \min\{b, b'\}$. Then we have $v(y-x) > g$. Q.E.D.

Let $\{g_i;\ i \in I\}$ be a well-ordered cofinal subset of $v(F \setminus D)$. Since D is not coprincipal, there is no last element in $\{g_i\}$. For any $i \in I$, we choose elements $a_i \in A$ and $b_i \in B$ such that $v(b_i - a_i) = g_i$.

LEMMA 2.3. *In the above situation, at least one of the following statements holds.*

(1) $\quad v(x - b_i) = g_i$ *for any* $x \in A,\ a_i < x$.

(2) $\quad v(y - a_i) = g_i$ *for any* $y \in B,\ y < b_i$.

PROOF. Suppose to the contrary that $v(x - b_i) > g_i$ for some $x \in A$, $a_i < x$ and $v(y - a_i) > g_i$ for some $y \in B$, $y < b_i$. Then for any $x' \in A$, $x < x'$, and $y' \in B$, $y' < y$, we have $v(y' - a_i) > g_i$ and $v(x' - b_i) > g_i$. These facts imply $v(y' - b_i) = v(y' - a_i + a_i - b_i) = g_i$. Hence $v(x' - y') = v(x' - b_i + b_i - y') = g_i$. This contradicts (3) of Lemma 2.2.

Q.E.D.

DEFINITION 2.4. We put $c_i = b_i$ if $v(x - b_i) = g_i$ for any $x \in A$, $a_i < x$ and $c_i = a_i$ if $v(y - a_i) = g_i$ for any $y \in B$, $y < b_i$. (If $v(x - b_i) = g_i$ and $v(y - a_i) = g_i$ for any $x \in A$ and $y \in B$, then we optionally put $c_i = a_i$ or $c_i = b_i$.)

LEMMA 2.5. *$\{c_i\}$ is a pseudo-convergent set of elements of F, and it has no limit in F.*

PROOF. First we show that $\{c_i\}$ is a pseudo-convergent set. It is sufficient to show that $v(c_j - c_i) = g_i$ for any $i < j$.

Case 1. Suppose $c_i \in A$ and $c_j \in B$. We take $x \in A$ and $y \in B$ so that $\max\{c_i, a_j\} < x, y < \min\{b_i, c_j\}$ and $v(y - x) > g_i$ (cf. Lemma 2.2, (3)). Then $v(c_i - x) = v(c_i - y + y - x) = g_i$. Hence we have $v(c_j - c_i) = v(c_j - x + x - c_i) = g_i$.

Case 2. Suppose $c_i \in A$ and $c_j \in A$. Let y be an element of B such that $y < \min\{b_i, b_j\}$. Then $v(c_j - c_i) = v(c_j - y + y - c_i) = g_i$.

The other cases can be proved similarly; thus $\{c_i\}$ is a pseudo-convergent set.

We now show that $\{c_i\}$ has no limit in F. Let x be any element of A. By Lemma 2.2 (1), there exists an element y of A such that $x < y$ and $v(y - x) \in v(F \setminus D)$. Let i be an element of I with $g_i > v(y - x)$. To show that x is not a limit of $\{c_i\}$, it is sufficient to show that $v(a_i - x) < g_i$. In fact, if $v(a_i - x) < g_i$, then $v(b_i - x) = v(b_i - a_i + a_i - x) < g_i$, and so we have $v(c_i - x) < g_i$. Suppose to the contrary that $v(a_i - x) \geqq g_i$. Then $v(b_i - x) = v(b_i - a_i + a_i - x) \geqq g_i$, and this implies $v(y - x) \geqq v(b_i - x) \geqq g_i$ since $y - x < b_i - x$. Thus we have a contradiction. Hence x is not a limit of $\{c_i\}$. Similarly no element of B is a limit of $\{c_i\}$. Q.E.D.

THEOREM 2.6. *Let (A, B) be a cut of F. Suppose that (A, B) is strongly proper and $D := E(A, B)$ is not coprincipal. Then there exists a pseudo-convergent set $\{c_i\}$ of elements of F, without a limit in F such that*

$$(A, B) = (A_R(\{c_i\}), B_R(\{c_i\}) = (A_L(\{c_i\}), B_L(\{c_i\})).$$

PROOF. Let $\{c_i\}$ be a psuedo-convergent set defined in Definition 2.4. Let a be any element of A. By Lemma 2.2. (1), there exists an element x of A such that $a<x$ and $v(x-a)\in v(F\backslash D)$. For any $i\in I$ with $g_i>v(x-a)$, we have $c_i>a$, and so a is an element of $A_L(\{c_i\})$. Similarly we have $B\subset B_R(\{c_i\})$. By Proposition 1.11, $(A_R(\{c_i\}), B_R(\{c_i\}))=(A_L(\{c_i\}), B_L(\{c_i\}))$, and so $(A, B)=(A_R(\{c_i\}), B_R(\{c_i\}))=(A_L(\{c_i\}), B_L(\{c_i\}))$. Q.E.D.

REMARK 2.7. Let B_0 be a maximal immediate extension of A_0 and K be the quotient field of B_0. We define an equivalence relation $\sim$ in the set $K\backslash F$. For x, $y\in K\backslash F$, we put $x\sim y$ if there is no element of F such that $x<a<y$ or $y<a<x$. We can readily see that $\sim$ is an equivalence relation in the set $K\backslash F$. Put $S=(K\backslash F)/\sim$. Let W be the set of pseudo-convergent sets of elements of F without limits in F. We also define an equivalence reltion $\sim$ in the set W. For $\{a_i\}$, $\{b_j\}\in W$, we put $\{a_i\}\sim\{b_j\}$ if $\{a_i\}$ and $\{b_j\}$ have a common limit. Note that if $\{a_i\}$ and $\{b_j\}$ have a common limit, then $\mathscr{B}(\{a_i\})=\mathscr{B}(\{b_j\})$, and it is easy to see that $\sim$ is an equivalence relation. We put $T=W/\sim$. For $\{a_i\}$, $\{b_i\}\in W$, let x and y be limits of $\{a_i\}$ and $\{b_i\}$ respectively. Then $x\sim y$ if and only if $\{a_i\}\sim\{b_i\}$. Thus we have a canonical bijection φ: $S\to T$ (cf. [1], Theorem 1). Let U be the set of strongly proper cuts of F whose breadths are not coprincipal. For $\{a_i\}\in W$, we put $\psi(\{a_i\})=(A_R(\{a_i\}), B_R(\{a_i\}))=(A_L(\{a_i\}), B_L(\{a_i\}))$. The map ψ: $W\to U$ canonically induces the map (denoted by the same symbol) ψ: $T\to U$. From the arguments in this section and by Proposition 1.13, we can see that ψ: $T\to U$ is bijective.

REMARK 2.8. Let (A, B) be a cut of F. Then (A, B) is filled in K (i.e. there exists an element x of K such that $A<x<B$) if and only if the cut (A, B) is strongly proper and $E(A, B)$ is not coprincipal (cf. Proposition 1.13). Suppose that a cut (A, B) is filled in K and take $x\in K$ so that $A<x<B$. Then we have $\{y\in K;\ A<y<B\}=x+\mathscr{B}(x)$ (here, we regard $\mathscr{B}(x)$ as an ideal of B_0).

REMARK 2.9. The set L of elements of K whose breadths are zero forms a subfield of K (cf. [4], Theorem 5). In [4], L is called the π-completion of F. For any element $x\in L\backslash F$, the equivalence class of x in $K\backslash F$ is $\{x\}$, and the cut of F determined by x is a proper archimedean cut (for the definition, see [2], Definition 1.1 or [3], Definition 2.1). The π-completion L coincides with the completion of F (cf. [3], Definition 2.5).

§3. Applications

THEOREM 3.1. *For an ordered field F, the following statements are equivalent.*
(1) *F has no strongly proper cut (A, B) such that $E(A, B)$ is not coprincipal.*

(2) *(F, v) is a maximal valued field.*

PROOF. (1)⇒(2): Let $\{a_i; i\in I\}$ be any pseudo-convergent set of elements of F. Put $(A, B) := (A_L(\{a_i\}), B_L(\{a_i\}))$. By Proposition 1.6, we have $E(A, B) = \mathscr{B}(\{a_i\})$, and so $D := E(A, B)$ is not coprincipal. Hence (A, B) is not a strongly proper cut, and so $\{a_i\}$ has a limit in F by Proposition 1.9.

The assertion (2)⇒(1) is clear from Theorem 2.6. Q.E.D.

For a cut (A, B) and a positive element c of F, we put $cA = \{ca; a\in A\}$ and $cB = \{cb; b\in B\}$. It is clear that (cA, cB) is also a cut of F. Let D be a convex subgroup of F which is contained in $E(A, B)$. We put $A_D = \{a + D; a\in A\}$ and $B_D = \{b + D; b\in B\}$. (A_D, B_D) is a cut of the ordered group F/D. If $D = E(A, B)$, then $(\bar{A}, \bar{B}) = (A_D, B_D)$ (cf. Definition 1.7).

LEMMA 3.2. *For a cut (A, B) and a positive element c of F, the following statements hold.*

(1) *$E(cA, cB) = cE(A, B)$. Moreover $E(A, B)$ is coprincipal (resp. principal) if and only if $E(cA, cB)$ is coprincipal (resp. principal).*

(2) *(A, B) is strongly proper if and only if so is (cA, cB).*

PROOF. The assertion (1) is easily shown. Suppose (A, B) is not strongly proper. Let x be an element of F such that $\bar{x} = \max \bar{A}$ or $\bar{x} = \min \bar{B}$. (For the definition of $(\bar{A}, \bar{B})$, see Definition 1.7.) Then $\bar{c}\bar{x} = \max \bar{c}\bar{A}$ or $\bar{c}\bar{x} = \min \bar{c}\bar{B}$. Hence (cA, cB) is not strongly proper. Q.E.D.

The following Lemma 3.3 and Lemma 3.4 follow immediately from the definitions.

LEMMA 3.3. *For a cut (A, B) and a convex subgroup D of F which is contained in $E(A, B)$, the following statements are equivalent.*

(1) *$D = E(A, B)$.*

(2) *(A_D, B_D) is archimedean; namely for any positive element α of F/D, there exist $\beta\in A_D$ and $\gamma\in B_D$ such that $\beta - \gamma < \alpha$.*

LEMMA 3.4. *Let (A, B) be a cut of F. Suppose that $E(A, B) = M_0$, $A\cap A_0 \neq \phi$ and $B\cap A_0 \neq \phi$. We put $A' = \{a + M_0; a\in A\cap A_0\}$ and $B' = \{b + M_0; b\in B\cap A_0\}$. Then (A', B') is a cut of A_0/M_0 and (A, B) is strongly proper if and only if (A', B') is proper.*

LEMMA 3.5. *If $A_0/M_0 = \boldsymbol{R}$, then F has no strongly proper cut (A, B) for which $E(A, B)$ is coprincipal.*

PROOF. Let (A, B) be a cut of F for which $D := E(A, B)$ is coprincipal. We must show that (A, B) is not a strongly proper cut. By Lemma 3.2, we may assume that $D = M_0$. We fix a positive element a of $A_0\backslash M_0$. Since a is not an element of D, $b - c < a$ for some $b\in B$ and $c\in A$. We put $d = (b + c)/2$. Then the cut $(A - d, B - d)$ satisfies the condition of Lemma 3.4. By the fact $A_0/M_0 = \boldsymbol{R}$ and Lemma 3.4, $(A-d, B-d)$ is not strongly proper, and so (A, B) is also not strongly proper. Q.E.D.

THEOREM 3.6. *For an ordered field F, the following statements are equivalent.*
(1) $A_0/M_0 = \boldsymbol{R}$.
(2) *F has no strongly proper cut* (A, B) *for which* $E(A, B)$ *is coprincipal.*

PROOF. By Lemma 3.5, it is sufficient to show the assertion (2)⇒(1). Suppose that there exists an element s of $\boldsymbol{R}\backslash A_0/M_0$, Let (C, D) be a cut of A_0/M_0 determined by s. Since A_0/M_0 is a convex subgroup of the additive group F/M_0, we can extend (C, D) to the cut (C', D') of F/M_0. Let (A, B) be the cut of F which is the pullback of (C', D'); namely $(A_{M_0}, B_{M_0}) = (C', D')$. By Lemma 3.3, we have $E(A, B) = M_0$; hence $E(A, B)$ is coprincipal. It is clear that neither $\max C'$ nor $\min D'$ exists, and so (A, B) is strongly proper. This contradicts the assumption of (2). Q.E.D.

The following Theorem 3.7 follows immediately from Theorem 3.1 and Theorem 3.6.

THEOREM 3.7. *For an ordered field F, the following statements are equivalent.*
(1) $A_0/M_0 = \boldsymbol{R}$ *and* (F, v) *is a maximal valued field.*
(2) *F has no strongly proper cut.*

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