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Radial entire solutions of the linear equation $\Delta u + \lambda p(|x|)u = 0$

Dedicated to Professor Tosihusa Kimura on his 60th birthday

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In this paper we are concerned with radial entire solutions of the linear elliptic differential equation

$$(1_\lambda) \qquad \Delta u + \lambda p(|x|)u = 0, \qquad x \in \boldsymbol{R}^N,$$

where Δ is the N-dimensional Laplacian, $|x|$ denotes the Euclidean length of $x \in \boldsymbol{R}^N$, and λ is a positive parameter. We always assume that $N \geqq 3$ and p satisfies

(2) $p \in C[0, \infty)$, $p(t) \geqq 0$ on $[0, \infty)$, and $p(t) \not\equiv 0$ on $[T, \infty)$ for every $T \geqq 0$.

The theorem below requires the further conditions

$$(3) \qquad \int_0^\infty tp(t)\, dt < \infty$$

and

$$(4) \qquad \int_0^\infty t^{N-1}p(t)\, dt < \infty .$$

The primary motivation for this paper comes from the observation that very little is known about the asymptotic property of radial entire solutions even for simple linear equations of the form (1_λ), whereas there are many results concerning the existence and asymptotic property of positive entire solutions of the *nonlinear* equation

$$(5) \qquad \Delta u + K(x)|u|^{\gamma-1}u = 0, \qquad x \in \boldsymbol{R}^N, \qquad \gamma \neq 1 .$$

For some recent literature on equation (5) the reader is referred to the papers [1–3, 5–8] and the references cited therein.

Now let us consider the linear equation (1_λ). Assume that (3) is satisfied, and put

$$(6) \qquad P = \int_0^\infty tp(t)\, dt .$$

In [5, Remark 2.2], Kawano has shown that if the parameter $\lambda > 0$ is small enough so that $\lambda P < N - 2$, then equation (1_λ) has a radial entire solution $u(t)$, $t = |x|$, such that $u(t) > 0$ for $t \geqq 0$ and $\lim u(t)$ as $t \to \infty$ exists and is positive. It is natural to consider the case of $\lambda P \geqq N - 2$. However, as far as the author is aware, there is no result with respect to this case. The objective of this paper is to determine what happens to the numbers of zeros and the asymptotic behavior as $|x| \to +\infty$ of radial entire solutions of (1_λ) as the parameter λ grows to $+\infty$.

The next theorem can be shown.

THEOREM (I) *Let* (3) *be satisfied. There exist* λ_0 *and* λ_1 *with* $0 < \lambda_0 \leqq \lambda_1 < +\infty$ *such that*

(i) *if* $\lambda \in (0, \lambda_0)$, *then every nontrivial radial entire solution* $u(t)$ *of* (1_λ) *has no zero in* $[0, \infty)$ *and has the asymptotic behavior that*

(7) $\lim_{t\to\infty} u(t)$ *exists and is a non-zero finite value*;

(ii) *if* $\lambda \in [\lambda_0, \lambda_1]$, *then every nontrivial radial entire solution* $u(t)$ *of* (1_λ) *has no zero in* $[0, \infty)$ *and has the asymptotic behavior that*

(8) $\lim_{t\to\infty} t^{N-2}u(t)$ *exists and is a non-zero finite value*;

(iii) *if* $\lambda \in (\lambda_1, +\infty)$, *then every nontrivial radial entire solution of* (1_λ) *has at least one zero in* $[0, \infty)$.

(II) *Let* (4) *be satisfied. Then, in addition to* λ_1 *in* (I), *there exist* λ_k $(k = 2, 3, \ldots)$ *with* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots$ *and* $\lim_{k\to\infty} \lambda_k = +\infty$ *such that if* $\lambda \in (\lambda_k, \lambda_{k+1}]$ $(k = 1, 2, \ldots)$, *then every nontrivial radial entire solution* $u(t)$ *of* (1_λ) *has exactly* k *zeros in* $[0, \infty)$.

A nontrivial radial entire function $u(t)$, where $t = |x|$, is a solution of (1_λ) if and only if $u(t)$ is a solution of the equation

$$(9_\lambda) \qquad u'' + \frac{N-1}{t}u' + \lambda p(t)u = 0\,, \qquad t > 0\,,$$

satisfying $u(0) = c$ and $u'(0) = 0$ for some real number $c \neq 0$. Then, since $\tilde{u}(t) = u(t)/u(0)$ is also a solution of (9_λ) satisfying $\tilde{u}(0) = 1$, $\tilde{u}'(0) = 0$, there is no loss of generality in assuming that u satisfies

$$(10) \qquad u(0) = 1\,, \qquad u'(0) = 0\,.$$

We denote by $u_\lambda(t)$ the solution of the initial value problem (9_λ)–(10). It can be easily verified that, for every $\lambda > 0$, $u_\lambda(t)$ is uniquely defined on $[0, \infty)$ and satisfies

$$(11) \qquad u_\lambda(t) = 1 - \frac{\lambda}{N-2}\int_0^t \left[1 - \left(\frac{s}{t}\right)^{N-2}\right] sp(s)u_\lambda(s)\,ds\,, \qquad t \geqq 0\,.$$

In discussing the properties of $u_\lambda(t)$, the results for solutions of equations of the type

$$(12) \qquad v'' + \frac{N-1}{t}v' + q(t)v = 0\,, \qquad t \geqq t_0\,,$$

will be effectively used.

LEMMA 1. *Let $t_0 > 0$. Suppose that $q \in C[t_0, \infty)$ and*

$$(13) \qquad \int_{t_0}^\infty t|q(t)|\,dt < \infty\,.$$

Then equation (12) *has a fundamental system of solutions $\{v_1(t), v_2(t)\}$ such that*

$$\lim_{t\to\infty} v_1(t) = 1 \quad \textit{and} \quad \lim_{t\to\infty} t^{N-2}v_2(t) = 1\,.$$

PROOF. Set $w(t) = tv([t/(N-2)]^{1/(N-2)})$. We find that $w(t)$ is a solution of the equation

$$(14) \qquad w'' + Q(t)w = 0\,, \qquad t \geqq t_1\,,$$

where $Q(t) = t^{-4}[t/(N-2)]^{2(N-1)/(N-2)}q([t/(N-2)]^{1/(N-2)})$, $t_1 = (N-2)t_0^{N-2}$ and that condition (13) is rewritten as

$$(15) \qquad \int_{t_1}^\infty t|Q(t)|\,dt < \infty\,.$$

It is well known that if $Q \in C[t_1, \infty)$ satisfies (15), then (14) has a fundamental system of solutions $\{w_1(t), w_2(t)\}$ such that

$$\lim_{t\to\infty} \frac{w_1(t)}{t} = 1 \quad \text{and} \quad \lim_{t\to\infty} w_2(t) = 1$$

(see, for example, Hartman [4, Corollary 9.1, p. 380]). Then $v_1(t) = (N-2)^{-1}t^{-N+2}w_1((N-2)t^{N-2})$ and $v_2(t) = t^{-N+2}w_2((N-2)t^{N-2})$ give the desired linearly independent solutions of (12). The proof is complete.

Lemma 1 implies in particular that, under condition (13), each nontrivial solution $v(t)$ of (12) has a finite number of zeros in $[t_0, \infty)$ and satisfies one of the next two asymptotic conditions:

$$\lim_{t\to\infty} v(t) \quad \text{exists and is a non-zero finite value}\,,$$

or

$$\lim_{t\to\infty} t^{N-2}v(t) \quad \text{exists and is a non-zero finite value}.$$

LEMMA 2. *Let* $t_0 > 0$. *Suppose that* $q \in C[t_0, \infty)$ *and* $q(t) \geqq 0$ *for* $t \geqq t_0$. *If there is a solution* $v(t)$ *of* (12) *having no zero in* $[t_0, \infty)$, *then*

$$(t^{N-2} - t_0^{N-2}) \int_t^\infty s^{-N+3} q(s)\, ds \leqq N-2, \qquad t \geqq t_0. \tag{16}$$

PROOF. We may suppose that $v(t) > 0$ for $t \geqq t_0$. An easy calculation shows that v satisfies

$$(t^{-N+3}(t^{N-2}v(t))')' + tq(t)v(t) = 0, \qquad t \geqq t_0, \tag{17}$$

and that

$$(t^{N-2}v(t))' \geqq 0, \qquad t \geqq t_0. \tag{18}$$

Then, integrating (17) once, we find that

$$t^{-N+3}(t^{N-2}v(t))' \geqq \int_t^\infty sq(s)v(s)\, ds, \qquad t \geqq t_0. \tag{19}$$

From (19) together with (18) it follows that, for $t \geqq t_0$,

$$\begin{aligned} t^{N-2}v(t) &\geqq t_0^{N-2}v(t_0) + \int_{t_0}^t s^{N-3}\left(\int_s^\infty \sigma q(\sigma)v(\sigma)\, d\sigma\right) ds \\ &\geqq \frac{1}{N-2}(t^{N-2} - t_0^{N-2}) \int_t^\infty \sigma q(\sigma)v(\sigma)\, d\sigma \\ &\geqq \frac{1}{N-2}(t^{N-2} - t_0^{N-2})t^{N-2}v(t) \int_t^\infty \sigma^{-N+3} q(\sigma)\, d\sigma, \end{aligned}$$

yielding (16). The proof of Lemma 2 is complete.

We now return to studying the properties of $u_\lambda(t)$, the solution of the problem (9_λ)–(10). By Lemma 1 we can conclude that, if (3) is satisfied, then, for each $\lambda > 0$, $u_\lambda(t)$ has a finite number of zeros in $[0, \infty)$ and satisfies either

$$\lim_{t\to\infty} u_\lambda(t) \quad \text{exists and is a non-zero finite value}, \tag{20}$$

or

$$\lim_{t\to\infty} t^{N-2}u_\lambda(t) \quad \text{exists and is a non-zero finite value}. \tag{21}$$

LEMMA 3. *Let* $0 < \mu < \lambda$. *If* $u_\lambda(t) > 0$ *on* $[0, t_\lambda)$, *where* $0 < t_\lambda \leqq \infty$, *then* $u_\mu(t) \geqq u_\lambda(t)$ (>0) *on* $[0, t_\lambda)$.

PROOF. First we claim that $u_\mu(t) > 0$ on $[0, t_\lambda)$. Assume to the contrary that $u_\mu(t)$ has a zero in $[0, t_\lambda)$. Then there is a $t_\mu \in (0, t_\lambda)$ such that $u_\mu(t_\mu) = 0$ and $u_\mu(t) > 0$ on $[0, t_\mu)$. Clearly, $u'_\mu(t_\mu) < 0$. Define the function $W(t)$ by

$$W(t) = t^{N-1}[u'_\mu(t)u_\lambda(t) - u_\mu(t)u'_\lambda(t)]\,, \qquad t \geqq 0\,. \tag{22}$$

The derivative of W is given by

$$W'(t) = (\lambda - \mu)t^{N-1}p(t)u_\mu(t)u_\lambda(t)\,, \qquad t \geqq 0\,.$$

By the assumptions, $W'(t) \geqq 0$ on $[0, t_\mu]$; that is, $W(t)$ is nondecreasing on $[0, t_\mu]$. However, $W(0) = 0$ and $W(t_\mu) = t_\mu^{N-1}u'_\mu(t_\mu)u_\lambda(t_\mu) < 0$, contradicting the nondecreasing property of W. Thus $u_\mu(t) > 0$ on $[0, t_\lambda)$.

Reconsider the function $W(t)$ defined by (22). We have $W'(t) \geqq 0$ on $[0, t_\lambda)$ and $W(0) = 0$, and so $W(t) \geqq 0$ on $[0, t_\lambda)$. Then,

$$\left(\frac{u_\mu(t)}{u_\lambda(t)}\right)' = \frac{W(t)}{t^{N-1}[u_\lambda(t)]^2} \geqq 0\,, \qquad 0 \leqq t < t_\lambda\,,$$

and consequently

$$\frac{u_\mu(t)}{u_\lambda(t)} \geqq \frac{u_\mu(0)}{u_\lambda(0)} = 1\,, \qquad 0 \leqq t < t_\lambda\,.$$

This proves Lemma 3.

As a corollary of Lemma 3 we have the following lemma.

LEMMA 4. *Let* $0 < \mu < \lambda$. (I) *If* $u_\lambda(t) > 0$ *for* $t \geqq 0$, *then* $u_\mu(t) \geqq u_\lambda(t)$ *for* $t \geqq 0$; *in particular* $u_\mu(t) > 0$ *for* $t \geqq 0$.

(II) *If* $u_\mu(t)$ *has a zero in* $[0, \infty)$, *then* $u_\lambda(t)$ *has a zero in* $[0, \infty)$. *Let* t_λ^1 *and* t_μ^1 *be the first zeros of* $u_\lambda(t)$ *and* $u_\mu(t)$, *respectively. Then* $t_\lambda^1 \leqq t_\mu^1$.

LEMMA 5. *Suppose that* (3) *holds. Then, for each* $\lambda > 0$,

$$|u_\lambda(t)| \leqq \exp(\lambda P/(N-2))\,, \qquad t \geqq 0\,, \tag{23}$$

where P *is given by* (6).

PROOF. Use of (11) gives

$$|u_\lambda(t)| \leqq 1 + \frac{\lambda}{N-2}\int_0^t sp(s)|u_\lambda(s)|\,ds\,, \qquad t \geqq 0\,,$$

and hence by Gronwall's inequality we have

$$|u_\lambda(t)| \leqq \exp\left(\frac{\lambda}{N-2}\int_0^t sp(s)\,ds\right), \qquad t \geqq 0\,.$$

Then the assertion (23) is clear.

LEMMA 6. *Suppose that* (3) *holds. Then, for* $\lambda > 0$ *and* $\mu > 0$,

$$(24) \qquad |u_\lambda(t) - u_\mu(t)| \leqq |\lambda - \mu| \frac{P}{N-2} \exp\left(\frac{\lambda + \mu}{N-2} P\right), \qquad t \geqq 0\,,$$

where P *is defined by* (6).

PROOF. In view of (11) and (23) we can estimate as follows:

$$\begin{aligned} |u_\lambda(t) - u_\mu(t)| &\leqq \frac{|\lambda - \mu|}{N-2}\int_0^t sp(s)|u_\lambda(s)|\,ds + \frac{\mu}{N-2}\int_0^t sp(s)|u_\lambda(s) - u_\mu(s)|\,ds \\ &\leqq \frac{|\lambda - \mu|}{N-2} P \exp\left(\frac{\lambda P}{N-2}\right) + \frac{\mu}{N-2}\int_0^t sp(s)|u_\lambda(s) - u_\mu(s)|\,ds \end{aligned}$$

for $t \geqq 0$. An application of Gronwall's inequality gives (24).

LEMMA 7. *Suppose that* (3) *holds. There is* $\lambda' > 0$ *such that if* $\lambda \in (0, \lambda']$, *then* $u_\lambda(t) > 0$ *for* $t \geqq 0$ *and* $u_\lambda(t)$ *satisfies* (20).

This lemma is a consequence of Kawano's result [5, Remark 2.2].

PROOF OF THEOREM. (I) Define the subsets Λ_0 and Λ_1^+ of $\boldsymbol{R}_+ = (0, +\infty)$ by

$$\Lambda_0 = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) > 0 \text{ for } t \geqq 0\}\,, \quad \text{and}$$

$$\Lambda_1^+ = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ has at least one zero in } [0, \infty)\}\,.$$

It is clear that $\boldsymbol{R}_+ = \Lambda_0 \cup \Lambda_1^+$, $\Lambda_0 \cap \Lambda_1^+ = \phi$. By Lemma 7, Λ_0 is non-empty. It can be shown that Λ_1^+ is also non-empty. To see this, assume the contrary. Then $\boldsymbol{R}_+ = \Lambda_0$, i.e., $u_\lambda(t) > 0$ on $[0, \infty)$ for every $\lambda > 0$. Applying Lemma 2 to the case of $q(t) = \lambda p(t)$, we find that

$$(25) \qquad \lambda(t^{N-2} - t_0^{N-2})\int_t^\infty s^{-N+3}p(s)\,ds \leqq N-2\,, \qquad t \geqq t_0\,,$$

for all $\lambda > 0$, where $t_0 > 0$ is an arbitrarily fixed number. In (25), fix t and let $\lambda \to +\infty$. Then we are led to a contradiction. Thus Λ_1^+ is non-empty. Besides, Λ_1^+ is an open subset of $\boldsymbol{R}_+$ because of the continuous dependence of

$u_\lambda(t)$ on λ. From (I) of Lemma 4 we see that if $0 < \mu < \lambda$ and $\lambda \in \Lambda_0$, then $\mu \in \Lambda_0$. Therefore we can conclude that Λ_0 and Λ_1^+ are of the forms

$$\Lambda_0 = (0, \lambda_1] \quad \text{and} \quad \Lambda_1^+ = (\lambda_1, +\infty)$$

for some $\lambda_1 > 0$.

Consider the subsets Λ_0^n and Λ_0^p of $\boldsymbol{R}_+$ defined by

$$\Lambda_0^n = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ is positive on } [0, \infty) \text{ and satisfies (20)}\}\,, \quad \text{and}$$

$$\Lambda_0^p = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ is positive on } [0, \infty) \text{ and satisfies (21)}\}\,.$$

We have $\Lambda_0 = \Lambda_0^n \cup \Lambda_0^p$ and $\Lambda_0^n \cap \Lambda_0^p = \phi$. Lemma 7 means that Λ_0^n is non-empty. By (I) of Lemma 4 we see that if $0 < \mu < \lambda$ and $\lambda \in \Lambda_0^n$, then $\mu \in \Lambda_0^n$. It follows from Lemma 6 that Λ_0^n is an open subset of $\boldsymbol{R}_+$. Therefore Λ_0^n is of the form $\Lambda_0^n = (0, \lambda_0)$ for some λ_0, $0 < \lambda_0 \leqq \lambda_1$. Then $\Lambda_0^p = [\lambda_0, \lambda_1]$. The set Λ_0^p may consist of a single point. The proof of part (I) is complete.

(II) We have shown in (I) that, for every $\lambda > \lambda_1$, $u_\lambda(t)$ has a zero in $[0, \infty)$. Let t_λ^1 be the first zero of $u_\lambda(t)$. By (II) of Lemma 4, t_λ^1 is non-increasing for $\lambda > \lambda_1$. Since $u_{\lambda_1}(t)$ has no zero in $[0, \infty)$, the continuous dependence of u_λ on the parameter λ implies that $t_\lambda^1 \to +\infty$ as $\lambda \to \lambda_1 + 0$. Define the subsets Λ_1 and Λ_2^+ of $\boldsymbol{R}_+ = (0, +\infty)$ by

$$\Lambda_1 = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ has exactly one zero in } [0, \infty)\}, \quad \text{and}$$

$$\Lambda_2^+ = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ has at least two zeros in } [0, \infty)\}.$$

Obviously, $\Lambda_1^+ = \Lambda_1 \cup \Lambda_2^+$, $\Lambda_1 \cap \Lambda_2^+ = \phi$, where $\Lambda_1^+ = (\lambda_1, +\infty)$ consists of all $\lambda > 0$ such that $u_\lambda(t)$ has at least one zero in $[0, \infty)$. It will be shown that Λ_1 is non-empty. In fact, if this is not true, then, for every $\lambda > \lambda_1$, $u_\lambda(t)$ has at least two zeros in $[0, \infty)$. Let t_λ^2 be the second zero of $u_\lambda(t)$. It is clear that $0 < t_\lambda^1 < t_\lambda^2$ for $\lambda > \lambda_1$, and so $\lim t_\lambda^2 = +\infty$ as $\lambda \to \lambda_1 + 0$. Since t_λ^2 is the second zero of $u_\lambda(t)$, we have $u_\lambda'(t_\lambda^2) > 0$ $(\lambda > \lambda_1)$. Noticing that $u_\lambda'(t)$ is given by

$$u_\lambda'(t) = -\frac{\lambda}{t^{N-1}} \int_0^t s^{N-1} p(s) u_\lambda(s)\, ds\,, \qquad t \geqq 0\,, \tag{26}$$

we see that

$$\int_0^{t_\lambda^2} s^{N-1} p(s) u_\lambda(s)\, ds < 0 \qquad \text{for } \lambda > \lambda_1\,. \tag{27}$$

To take the limit as $\lambda \to \lambda_1 + 0$ in (27), first note that $u_\lambda(t) \to u_{\lambda_1}(t)$ as $\lambda \to \lambda_1 + 0$ at each point $t \in [0, \infty)$. Lemma 5 and condition (4) show that, for $\lambda \in (\lambda_1, \lambda_1 + 1]$, $t^{N-1} p(t) u_\lambda(t)$ is uniformly bounded on $[0, \infty)$ by an $L^1(0, \infty)$-function. Then, by the dominated convergence theorem, it is seen that

$$\int_0^\infty s^{N-1}p(s)u_{\lambda_1}(s)\,ds = \lim_{\lambda\to\lambda_1+0}\int_0^{t_\lambda^2} s^{N-1}p(s)u_\lambda(s)\,ds \leqq 0\,.$$

However this is a contradiction since $u_{\lambda_1}(t)$ is positive throughout $[0, \infty)$. Thus we conclude that Λ_1 is non-empty.

Assume next that Λ_2^+ is empty. Then $u_\lambda(t)$ has exactly one zero in $[0, \infty)$ for each $\lambda > \lambda_1$. (The zero of $u_\lambda(t)$ is denoted by t_λ^1.) The nonincreasing property of t_λ^1 means that, for all $\lambda \geqq \lambda_1 + 1$, $u_\lambda(t) < 0$ on $[t_{\lambda_1+1}^1 + 1, \infty)$. An application of Lemma 2 to the case of $q(t) = \lambda p(t)$ and $t_0 = t_{\lambda_1+1}^1 + 1$ shows that

$$\lambda[t^{N-2} - (t_{\lambda_1+1}^1 + 1)^{N-2}]\int_t^\infty s^{-N+3}p(s)\,ds \leqq N-2\,, \qquad t \geqq t_{\lambda_1+1}^1 + 1,$$

which leads to a contradiction in the limit $\lambda \to +\infty$. Thus Λ_2^+ is non-empty.

According to (II) of Lemma 4 and the well-known Sturm-type theorem (see, e.g., Swanson [9, Theorem 1.6]) we see that, if $\lambda_1 < \mu < \lambda$ and $\lambda \in \Lambda_1$, then $\mu \in \Lambda_1$. Furthermore it is clear that Λ_2^+ is an open subset of $\boldsymbol{R}_+$. By these facts, it is seen that Λ_1 and Λ_2^+ are of the forms

$$\Lambda_1 = (\lambda_1, \lambda_2] \quad \text{and} \quad \Lambda_2^+ = (\lambda_2, +\infty)$$

for some λ_2 $(>\lambda_1)$.

We proceed with the same arguments. For every $\lambda > \lambda_2$, $u_\lambda(t)$ has at least two zeros in $[0, \infty)$. Denote the second zero of $u_\lambda(t)$ by t_λ^2. It can be shown that $t_\lambda^2 \to +\infty$ as $\lambda \to \lambda_2 + 0$. Define the sets Λ_2 and Λ_3^+ by

$$\Lambda_2 = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ has exactly two zeros in } [0, \infty)\}\,,$$

$$\Lambda_3^+ = \{\lambda \in \boldsymbol{R}_+ : u_\lambda(t) \text{ has at least three zeros in } [0, \infty)\}.$$

Clearly, $\Lambda_2^+ = \Lambda_2 \cup \Lambda_3^+$, $\Lambda_2 \cap \Lambda_3^+ = \phi$, and as in the above, Λ_3^+ is a non-empty open subset of $\boldsymbol{R}_+$. We can show that Λ_2 is non-empty. To show this, assume the contrary. Then, for $\lambda > \lambda_2$, $u_\lambda(t)$ has at least three zeros in $[0, \infty)$. Arguing as in the above, we see that

$$\int_0^\infty s^{N-1}p(s)u_{\lambda_2}(s)\,ds = \lim_{\lambda\to\lambda_2+0}\int_0^{t_\lambda^3} s^{N-1}p(s)u_\lambda(s)\,ds \geqq 0\,, \tag{28}$$

where t_λ^3 is the third zero of $u_\lambda(t)$. If $u_{\lambda_2}(t)$ has a non-zero finite limit as $t \to +\infty$, then it follows from Lemma 6 that there exists $\delta > 0$ and $T > 0$ such that $u_\lambda(t)$ has no zero in $[T, \infty)$ for $\lambda \in (\lambda_2, \lambda_2 + \delta]$. But this contradicts the fact that $u_\lambda(t) = 0$ at $t = t_\lambda^2$ and $\lim t_\lambda^2 = +\infty$ as $\lambda \to \lambda_2 + 0$. Hence the limit $\lim t^{N-2}u_{\lambda_2}(t)$ as $t \to +\infty$ exists and is negative. Since $\lim u_{\lambda_2}(t) = 0$ as $t \to +\infty$, $u_{\lambda_2}(t_{\lambda_2}^1) = 0$ and $u_{\lambda_2}(t) < 0$ on $(t_{\lambda_2}^1, \infty)$, it is possible to take a number $t^* \in (t_{\lambda_2}^1, \infty)$ such that $u'_{\lambda_2}(t^*) = 0$. Then equality (26) with $\lambda = \lambda_2$ implies

$$\int_0^{t^*} s^{N-1}p(s)u_{\lambda_2}(s)\,ds = 0\,. \tag{29}$$

From (28) and (29) it follows that

$$\int_{t^*}^{\infty} s^{N-1}p(s)u_{\lambda_2}(s)\,ds \geqq 0\,,$$

which is a contradiction because $u_{\lambda_2}(t) < 0$ for $t \geqq t^*$. Thus Λ_2 is non-empty. From (II) of Lemma 4 and the above-mentioned Sturm-type theorem we see that, if $\lambda_2 < \mu < \lambda$ and $\lambda \in \Lambda_2$, then $\mu \in \Lambda_2$. Therefore, Λ_2 and Λ_3^+ are of the forms

$$\Lambda_2 = (\lambda_2, \lambda_3] \quad \text{and} \quad \Lambda_3^+ = (\lambda_3, +\infty)$$

for some $\lambda_3 (> \lambda_2)$.

To complete the proof of part (II), continue the same arguments.

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