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On oscillations of neutral equations with mixed arguments

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1. Introduction and preliminaries

Consider the neutral differential equation

$$\frac{d}{dt}[y(t) + py(t-\tau)] + qy(t-\sigma) = 0 \tag{1}$$

where p, τ, q and σ are real numbers. The main results in this paper are the following:

THEOREM 1. *The following statements are equivalent*:

(a) *Every bounded solution of Eq.* (1) *oscillates.*

(b) *The characteristic equation associated with Eq.* (1)

$$F(\lambda) = \lambda + \lambda p e^{-\lambda\tau} + q e^{-\lambda\sigma} = 0 \tag{2}$$

has no roots in $(-\infty, 0]$.

THEOREM 2. *The following statements are equivalent*:

(a) *Every unbounded solution of Eq.* (1) *oscillates.*

(b) *The characteristic equation* (2) *associated with Eq.* (1) *has no roots in* $(0, \infty)$ *and* 0 *is not a double root of Eq.* (2).

An immediate corollary of the above theorems is the following result which was proved in [3].

COROLLARY *Every solution ofEq.* (1) *oscillates if and only if its characteristic equation* (2) *has no real roots.*

As is customary a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory.*

In the sequel all functional inequalities that we write are assumed to hold eventually, that is for sufficiently large t.

We now list some preliminary results which will be useful in our study of Eq. (1).

The first result we will make use of is extracted from [5].

LEMMA 1. *Let r and μ be positive constants. Assume that $x(t)$ is a positive solution of the inequality*

$$\dot{x}(t) + rx(t - \mu) \leq 0$$

and $y(t)$ is a positive solution of the inequality

$$\dot{y}(t) - ry(t + \mu) \geq 0 .$$

Then

$$x(t - \mu) \leq \frac{4}{(r\mu)^2} x(t)$$

and

$$y(t + \mu) \leq \frac{4}{(r\mu)^2} y(t) .$$

For a proof of the next lemma see [4].

LEMMA 2. *Let $y(t)$ be a solution of Eq. (1) for $t \geq t_0$ and let α and β be any constants. Then*

$$x(t) = \int_{t-\alpha}^{t-\beta} y(s)\, ds$$

is also a solution for $t \geq t_0 + \max\{\alpha, \beta\}$.

The next result deals with the characteristic equation (2).

LEMMA 3. *Assume the characteristic equation (2) has no roots in $(-\infty, 0]$. Then there exists $m > 0$ such that for all $\lambda \geq 0$*

$$\lambda + \lambda p e^{\lambda\tau} - q e^{\lambda\sigma} \leq -m \qquad \text{if} \quad q > 0$$

while

$$-\lambda - \lambda p e^{\lambda\tau} + q e^{\lambda\sigma} \leq -m \qquad \text{if} \quad q < 0 .$$

Also, if (2) has no roots in $(0, \infty)$, there exists $m > 0$ such that for all $\lambda \geq 0$

$$-\lambda - \lambda p e^{-\lambda\tau} - q e^{-\lambda\sigma} \leq -m \qquad \text{if} \quad q > 0$$

while

$$\lambda + \lambda p e^{-\lambda\tau} + q e^{-\lambda\sigma} \leq -m \qquad \textit{if} \quad q < 0 .$$

The next lemma, which follows from [1], shows that if Eq. (1) has a nonoscillatory solution then it also has a nonoscillatory solution with "nice" properties which are useful in the study of Eq. (1).

LEMMA 4. *Assume* $q \neq 0$ *and let* $y(t)$ *be an eventually positive solution of Eq.* (1). *Define* $z(t) = y(t) + py(t-\tau)$ *and* $w(t) = z(t) + pz(t-\tau)$. *Then*

$$w(t) > 0, \quad \dot{w}(t) < 0, \quad \ddot{w}(t) > 0 \quad \textit{and} \quad \lim_{t\to\infty} w(t) = 0$$

if $y(t)$ *is bounded, while*

$$w(t) > 0, \quad \dot{w}(t) > 0, \quad \ddot{w}(t) > 0 \quad \textit{and} \quad \lim_{t\to\infty} w(t) = \infty$$

if $y(t)$ *is unbounded. Moreover* $z(t)$ *is a differentiable solution of Eq.* (1) *and* $w(t)$ *is a twice differentiable solution of Eq.* (1).

For the following see Grammatikopoulos, Sficas and Stavroulakis [2].

LEMMA 5. *Let* $v(t)$ *be a positive and continuously differentiable function. Assume that there exists positive numbers* A *and* α *such that either*

$$v(t-\alpha) < Av(t) \tag{3}$$

or

$$v(t+\alpha) < Av(t) . \tag{4}$$

Set

$$\Lambda = \{\lambda \geq 0 \colon \dot{v}(t) + \lambda v(t) \leq 0\} \qquad \textit{if (3) holds}$$

and

$$\Lambda = \{\lambda \geq 0 \colon -\dot{v}(t) + \lambda v(t) \leq 0\} \qquad \textit{if (4) holds} .$$

Then $(A > 1)$ *and*

$$\lambda_0 = \frac{\ln A}{\alpha} \notin \Lambda .$$

PROOF. We will prove the lemma when (3) holds. The case when (4) holds is similar and will be omitted. Assume that (3) holds and, for the sake of contradiction, assume that $\lambda_0 \in \Lambda$. Then

$$\frac{d}{dt}[e^{\lambda_0 t} v(t)] = e^{\lambda_0 t}[\dot{v}(t) + \lambda_0 v(t)] \leq 0$$

which implies that the function $e^{\lambda_0 t}v(t)$ is decreasing. Hence

$$e^{\lambda_0(t-\alpha)}v(t-\alpha) \geq e^{\lambda_0 t}v(t)$$

or

$$v(t-\alpha) \geq e^{\lambda_0 \alpha}v(t) = e^{\ln A}v(t) = Av(t)$$

which contradicts (3) and completes the proof of the lemma.

The following "Duality Lemma" from [1] will enable us to reduce the required number of cases we have to consider in our proofs of the theorems.

(DUALITY) LEMMA. *Suppose that p is a nonzero real number. Then $y(t)$ is a solution of Eq.* (1) *if and only if $y(t)$ is a solution of*

$$\frac{d}{dt}\left[y(t) + \frac{1}{p}y(t-(-\tau)\right] + \frac{q}{p}y(t-(\sigma-\tau)) = 0\,.$$

2. Proof of Theorem 1

PROOF. (a) $\Rightarrow$ (b). If it is false the characteristic equation (2) would have a root $\lambda_0 \in (-\infty, 0]$ and therefore Eq. (1) would have the nonoscillatory bounded solution

$$y(t) = e^{\lambda_0 t}\,.$$

But this contradicts the hypothesis that every bounded solution of Eq. (1) oscillates.

(b) $\Rightarrow$ (a). Assume, for the sake of contradiction, that Eq. (1) has a bounded eventually positive solution $y(t)$. First assume $p = 0$. Then (1) and (2) reduce to

$$\dot{y}(t) + qy(t-\sigma) = 0 \tag{5}$$

and

$$\lambda + qe^{-\lambda\sigma} = 0\,. \tag{6}$$

As (6) has no real roots in $(-\infty, 0]$ it follows that $q \neq 0$ and when $q > 0$ then $\sigma \neq 0$. Hence we have the following cases to consider:

(i) $q > 0$ and $\sigma > 0$
(ii) $q > 0$ and $\sigma < 0$
(iii) $q < 0$.

Case (i): $q > 0$ and $\sigma > 0$. Define

$$\Lambda = \{\lambda \geq 0\colon \dot{y}(t) + \lambda y(t) \leq 0\}\,.$$

Clearly $0 \in \Lambda$ and so Λ is a nonempty interval. We will show that Λ has the following contradictory properties.

(P_1) There exist positive numbers λ_1 and λ_2 such that $\lambda_1 \in \Lambda$ and $\lambda_2 \notin \Lambda$.

(P_2) $\lambda \in \Lambda \Rightarrow \lambda + m \in \Lambda$ where m is as defined in Lemma 3.

Observe that $\dot{y}(t) + qy(t) \leq 0$ which implies that $\lambda_1 = q \in \Lambda$. Applying Lemmas 1 and 5 to (5) we obtain

$$\lambda_2 = \frac{\ln \frac{4}{(\sigma q)^2}}{\sigma} \notin \Lambda .$$

Let $\lambda \in \Lambda$ and set $\varphi(t) = e^{\lambda t} y(t)$. Then $\dot{\varphi}(t) = e^{\lambda t}[\dot{y}(t) + \lambda y(t)] \leq 0$ which implies $\varphi(t)$ is nonincreasing. Now

$$\begin{aligned} \dot{y}(t) + (\lambda + m)y(t) &= -qy(t-\sigma) + (\lambda + m)y(t) \\ &= -qe^{-\lambda(t-\sigma)}\varphi(t-\sigma) + (\lambda + m)e^{-\lambda t}\varphi(t) \\ &\leq e^{-\lambda t}\varphi(t)[-qe^{\lambda\sigma} + \lambda + m] \leq e^{-\lambda t}\varphi(t)[-m + m] = 0 , \end{aligned}$$

which shows $\lambda + m \in \Lambda$.

Case (ii): $q > 0$ and $\sigma < 0$. We have $F(0) = q > 0$ and $F(-\infty) = -\infty$ which implies that the characteristic equation has a root in $(-\infty, 0]$. This is a contradiction.

Case (iii): $q < 0$. Here

$$\dot{y}(t) = -qy(t-\sigma) > 0$$

which implies $\lim_{t\to\infty} y(t) = \ell \in (0, \infty)$. But then $\lim_{t\to\infty} \dot{y}(t) = -q\ell > 0$ which implies that $\ell = \infty$. This is a contradiction and the proof is complete when $p = 0$.

Next, observe that if $\tau = 0$ and $p \neq -1$, Eq. (1) reduces to

$$\dot{y}(t) + \frac{q}{1+p} y(t-\sigma) = 0 \tag{7}$$

for which the result has just been established. On the other hand, when $\tau = 0$ and $p = -1$ Eqs. (1) and (2) reduce to

$$qy(t-\sigma) = 0 \tag{8}$$

and

$$qe^{-\lambda\sigma} = 0 \tag{9}$$

respectively. As (9) has no roots in $(-\infty, 0]$, it follows that $q \neq 0$ and so (8) implies that $y(t) \equiv 0$ which is a contradiction.

Because of the Duality Lemma we may and do assume that $\tau > 0$.

For subsequent use, define $z(t) = y(t) + py(t-\tau)$ and $w(t) = z(t) + pz(t-\tau)$. Then, it follows from Lemma 4 that

$$w(t) > 0, \quad \dot{w}(t) < 0, \quad \ddot{w}(t) > 0 \quad \text{and} \quad \lim_{t\to\infty} w(t) = 0 .$$

REMARK 1. By integrating Eq. (1) from $t-\alpha$ to ∞ with $y(t)$ replaced by $w(t)$ one sees that

$$x(t) = \int_{t-\alpha}^{\infty} w(s)\, ds$$

is a solution of Eq. (1).

The remaining part of the proof will be accomplished by considering the following eight cases:

	p	q	τ	σ
1.	+	+	+	+
2.	+	+	+	−, 0
3.	+	−	+	+
4.	+	−	+	−, 0
5.	−	+	+	+
6.	−	+	+	−, 0
7.	−	−	+	+
8.	−	−	+	−, 0

Case 1: $p > 0$, $q > 0$, $\tau > 0$ and $\sigma > 0$. Since $F(-\infty) = +\infty$ it follows that $\sigma > \tau$. Set

$$w_n(t) = \begin{cases} w(t), & n = 0 \\ w_{n-1}(t) + pw_{n-1}(t-\tau), & n = 1, 2, \ldots . \end{cases} \tag{10}$$

It follows from Lemma 4 or from the fact that Eq. (1) is linear and autonomous that $w_n(t)$ is a twice differentiable solution of Eq. (1). Then for $n = 1, 2, \ldots.$ we have

$$\dot{w}_n(t) = -qw_{n-1}(t-\sigma) \tag{11}$$

$$w_n(t) > 0 , \quad \dot{w}_n(t) < 0 , \quad \ddot{w}_n(t) > 0 \tag{12}$$

and

$$\dot{w}_n(t) + p\dot{w}_n(t-\tau) + qw_n(t-\sigma) = 0 . \tag{13}$$

The proof of (11), (12) and (13) is by induction.

Set

$$\Lambda_n = \{\lambda \geq 0 \colon \dot{w}_n(t) + \lambda w_n(t) \leq 0\} .$$

The proof will be accomplished by showing that Λ_n has the contradictory properties:

(P_1) There exist positive numbers λ_1 and λ_2 such that $\lambda_1 \in \Lambda_n$ and $\lambda_2 \notin \Lambda_n$ for $n = 1, 2, \dots$.

(P_2) There exists a positive μ, independent of n, such that $\lambda \in \Lambda_n$ with $\lambda \geq \lambda_1 \Rightarrow \lambda + \mu \in \Lambda_{n+1}$ for $n = 1, 2, \dots$.

First we will prove (P_1). From (12) and (13) we have

$$(1 + p)\dot{w}_n(t - \tau) + q w_n(t - \sigma) \leq 0 .$$

It follow that

$$\dot{w}_n(t) + \frac{q}{1 + p} w_n(t - (\sigma - \tau)) \leq 0 \tag{14}$$

or

$$\dot{w}_n(t) + \frac{q}{1 + p} w_n(t) \leq 0 .$$

Hence

$$\lambda_1 = \frac{q}{1 + p} \in \bigcap_{n=1}^{\infty} \Lambda_n .$$

Applying Lemma 1 to (14) we obtain

$$w_n(t - (\sigma - \tau)) \leq \frac{4(1 + p)^2}{q^2(\sigma - \tau)^2} w_n(t) .$$

From Lemma 5 we have

$$\lambda_2 = \frac{1}{\sigma - \tau} \ln \frac{4(1 + p)^2}{q^2(\sigma - \tau)^2} \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

Let $\lambda \in \Lambda_n$ and set $\mu = m/(1 + p e^{\lambda_2 \tau})$ and $\varphi_n(t) = e^{\lambda t} w_n(t)$. Then

$$\dot{\varphi}_n(t) = e^{\lambda t}[\dot{w}_n(t) + \lambda w_n(t)] \leq 0$$

which shows φ_n is a nonincreasing function. Now

$$\begin{aligned}
\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) &= -q w_n(t - \sigma) + (\lambda + \mu)[w_n(t) + p w_n(t - \tau)] \\
&= -q \varphi_n(t - \sigma) e^{-\lambda(t - \sigma)} \\
&\quad + (\lambda + \mu)[\varphi_n(t) e^{-\lambda t} + p \varphi_n(t - \tau) e^{-\lambda(t - \tau)}] \\
&\leq \varphi_n(t - \tau) e^{-\lambda t}[-q e^{\lambda \sigma} + \lambda + \lambda p e^{\lambda \tau} + \mu + \mu p e^{\lambda_2 \tau}] \\
&\leq \varphi_n(t - \tau) e^{-\lambda t}[-m + m] = 0 .
\end{aligned}$$

The proof is complete in this case.

Cases 2 *and* 8: $p > 0$, $q > 0$, $\tau > 0$, $\sigma \leq 0$ or $p < 0$, $q < 0$, $\tau > 0$, $\sigma \leq 0$. In these cases $F(0) \cdot F(-\infty) < 0$ which implies that the characteristic equation has a root in $(-\infty, 0]$.

Cases 3 *and* 4: $p > 0$, $q < 0$, $\tau > 0$, $\sigma > 0$ or $p > 0$, $q < 0$, $\tau > 0$, $\sigma \leq 0$. Here

$$\dot{z}(t) = -qy(t - \sigma) > 0 .$$

Integrating the above from t_0 and t and taking the limit as $t \to \infty$ implies that $y(t) \in L^1[t_0, \infty)$ and so $z(t) \in L^1[t_0, \infty)$. As $z(t)$ is also a monotonic function it follows that $\lim_{t \to \infty} z(t) = 0$ which is impossible because $z(t) > 0$ and increasing.

Case 5: $p < 0$, $q > 0$, $\tau > 0$ and $\sigma > 0$. Set

$$w_n(t) = \begin{cases} w(t) , & n = 0 \\ w_{n-1}(t) + p w_{n-1}(t - \tau) , & n = 1, 2, \dots . \end{cases} \tag{15}$$

Then for $n = 1, 2, \dots$ we have

$$\begin{gathered} \dot{w}_n(t) = -q w_{n-1}(t - \sigma) \\ w_n(t) > 0 , \quad \dot{w}_n(t) < 0 , \quad \ddot{w}(t) > 0 \end{gathered} \tag{16}$$

and

$$\dot{w}_n(t) + p \dot{w}_n(t - \tau) + q w_n(t - \sigma) = 0 . \tag{17}$$

Set

$$\Lambda_n = \{\lambda \geq 0 \colon \dot{w}_n(t) + \lambda w_n(t) \leq 0\} , \qquad n = 1, 2, \dots .$$

As in Case 1, the proof will be accomplished by showing that Λ_n has the contradictory properties (P_1) and (P_2).

From (17) we have

$$\dot{w}_n(t) + q w_n(t - \sigma) < 0 . \tag{18}$$

Hence

$$\lambda_1 = q \in \bigcap_{n=1}^{\infty} \Lambda_n .$$

Applying Lemma 1 to (18) yields

$$w_n(t - \sigma) \leq \frac{4}{(q\sigma)^2} w_n(t) .$$

It now follows from Lemma 5 that

$$\lambda_2 = \frac{1}{\sigma} \ln \frac{4}{(q\sigma)^2} \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

Let $\lambda \in \Lambda_n$ and set $\varphi_n(t) = e^{\lambda t} w_n(t)$ and $\mu = m$. Then

$$\begin{aligned}
\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) &= -q w_n(t-\sigma) + (\lambda+\mu)[w_n(t) + p w_n(t-\tau)] \\
&= -q e^{-\lambda(t-\sigma)} \varphi_n(t-\sigma) \\
&\quad + (\lambda+\mu)[e^{-\lambda t}\varphi_n(t) + p e^{-\lambda(t-\tau)}\varphi_n(t-\tau)] \\
&\le e^{-\lambda t}\varphi_n(t)[-q e^{\lambda\sigma} + \lambda + \lambda p e^{\lambda\tau} + \mu + \mu p e^{\lambda\tau}] \\
&\le e^{-\lambda t}\varphi_n(t)[-m + \mu] = 0 .
\end{aligned}$$

The proof is complete in this case.

Case 6: $p < 0$, $q > 0$, $\tau > 0$ and $\sigma \le 0$. The dual of Case 6 is

$$p < 0 , \quad q < 0 , \quad \tau < 0 \quad \text{and} \quad \sigma < 0 \quad \text{with} \quad \sigma < \tau$$

which we will now consider. As in [3], set

$$w_n(t) = \begin{cases} w(t) , & n = 0 \\ -[w_{n-1}(t) + p w_{n-1}(t-\tau)] - q \displaystyle\int_{t-\tau}^{t-\sigma} w_{n-1}(s)\, ds , & n = 1, 2, \dots \end{cases} \tag{19}$$

and define Λ_n as in Case 5. Then for $n = 1, 2, \dots$ we have

$$\dot{w}_n(t) + p\dot{w}_n(t-\tau) + q w_n(t-\sigma) = 0 , \tag{20}$$

$$\dot{w}_n(t) = q w_{n-1}(t-\tau) \tag{21}$$

and

$$w_n(t) > 0 , \quad \dot{w}_n(t) < 0 \quad \text{and} \quad \ddot{w}_n(t) > 0 . \tag{22}$$

From (20) and (22) we obtain

$$-\dot{w}_n(t) - p\dot{w}_n(t-\tau) \le 0 . \tag{23}$$

Applying (21) to (23) yields

$$-q w_{n-1}(t-\tau) - p\dot{w}_n(t-\tau) \le 0 . \tag{24}$$

Integrating the above from $t + \tau$ to t yields

$$-q w_{n-1}(t-\tau)(-\tau) - p w_n(t-\tau) + p w_n(t) \le 0$$

from which it follows that

$$q w_{n-1}(t-\tau) \ge -\frac{p}{\tau} w_n(t) . \tag{25}$$

Combining (25) with (21) gives

$$\dot{w}_n(t) + \frac{p}{\tau} w_n(t) \geq 0$$

which implies

$$\lambda_2 = \frac{p}{\tau} \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

From (19) we have

$$w_n(t) \leq (-p - q(\tau - \sigma)) w_{n-1}(t - \tau) . \tag{26}$$

Applying (26) to (21) yields

$$\dot{w}_n(t) + \frac{-q}{-p - q(\tau - \sigma)} w_n(t) \leq 0$$

which implies

$$\lambda_1 = \frac{-q}{-p - q(\tau - \sigma)} \in \bigcap_{n=1}^{\infty} \Lambda_n .$$

Let $\lambda \geq \lambda_1$ and set $\varphi_n(t) = e^{\lambda t} w_n(t)$ and

$$\mu = m \left[-p e^{\lambda_2 \tau} - \frac{q e^{\lambda_2 \tau}}{\lambda_1} \right]^{-1} .$$

Now

$$\begin{aligned}
&\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) \\
&\quad = q w_n(t - \tau) + (\lambda + \mu) \left[-w_n(t) - p w_n(t - \tau) - q \int_{t-\tau}^{t-\sigma} w_n(s)\, ds \right] \\
&\quad \leq e^{-\lambda t} \varphi_n(t - \tau) \left[q e^{\lambda \tau} - \lambda - \lambda p e^{\lambda \tau} + q e^{\lambda \sigma} - q e^{\lambda \tau} - \mu - \mu p e^{\lambda \tau} + \frac{\mu q}{\lambda} (e^{\lambda \sigma} - e^{\lambda \tau}) \right] \\
&\quad \leq e^{-\lambda t} \varphi_n(t - \tau) \left[-\lambda - \lambda p e^{\lambda \tau} + q e^{\lambda \sigma} + \mu \left(-p e^{\lambda_2 \tau} - \frac{q e^{\lambda_2 \tau}}{\lambda_1} \right) \right] \\
&\quad \leq e^{-\lambda t} \varphi_n(t - \tau) [-m + m] = 0 .
\end{aligned}$$

The proof is complete in this case.

Case 7: $p < 0$, $q < 0$, $\tau > 0$ and $\sigma > 0$. Here $F(0) = q < 0$ and so in order that $F(-\infty) = -\infty$ we must have $\sigma > \tau$.

Let V be the set of all C^2 solutions of Eq. (1) which satisfy

$$v(t) > 0 , \quad \dot{v}(t) < 0 , \quad \ddot{v}(t) > 0 \quad \text{and} \quad \lim_{t \to \infty} v(t) = 0 .$$

Set

$$\Lambda(v) = \{\lambda \geq 0 \colon \dot{v}(t) + \lambda v(t) \leq 0\} .$$

First we establish that for every $v \in V$ the set $\Lambda(v)$ is nonempty and bounded from above. To this end observe that for $v \in V$

$$\dot{v}(t) + p\dot{v}(t - \tau) + qv(t - \sigma) = 0 . \tag{27}$$

Now (27) implies

$$p\dot{v}(t - \tau) + qv(t - \sigma) \geq 0 ,$$

from which it follows

$$\dot{v}(t) + \frac{q}{p} v(t - (\sigma - \tau)) \leq 0 . \tag{28}$$

From (28) we have

$$\dot{v}(t) + \frac{q}{p} v(t) \leq 0 .$$

Hence

$$\frac{q}{p} \in \bigcap\nolimits_{v \in V} \Lambda(v) .$$

Applying Lemma 1 to (28) yields

$$v(t - (\sigma - \tau)) \leq Bv(t)$$

where $B \equiv 4p^2/[(\sigma - \tau)^2 q^2]$. It now follows from Lemma 5 that

$$\lambda^* = \frac{\ln B}{\sigma - \tau} \notin \bigcup\nolimits_{v \in V} \Lambda(v) .$$

Next let $\lambda_0 = q/p$ and set $\mu = m/(e^{\lambda^* \sigma} - pe^{\lambda^* \tau} - 1)$. We will prove by induction that if

$$\lambda_n = \lambda_{n-1} + \mu , \qquad n = 1, 2, \dots ,$$

and if

$$w_n(t) = \begin{cases} w(t) , & n = 0 \\ -[w_{n-1}(t) + pw_{n-1}(t - \tau)] + \lambda_{n-1} \displaystyle\int_{t-\sigma}^{\infty} w_{n-1}(s)\, ds , & n = 1, 2, \dots \end{cases} \tag{29}$$

then $w_n \in V$ and $\lambda_n \in \Lambda(w_n)$. As $\Lambda(w_n)$ is bounded from above, this will be a contradiction and will complete the proof in this case.

Clearly $w_n \in V$ for $n = 1, 2, \ldots$. Next, assume that $\lambda_n \in \Lambda(w_n)$. We will show that $\lambda_{n+1} \in \Lambda(w_{n+1})$. We first derive an inequality which we will utilize to prove the above.

Since $\lambda_n \in \Lambda(w_n)$

$$\dot{w}_n(t) + \lambda_n w_n(t) \leq 0 . \tag{30}$$

Integrating (30) from t to ∞ yields

$$-w_n(t) + \lambda_n \int_t^\infty w_n(s)\, ds \leq 0 .$$

Using the above inequality in (29) gives

$$w_{n+1}(t) \leq -p w_n(t-\tau) + \lambda_n \int_{t-\sigma}^t w_n(s)\, ds . \tag{31}$$

Now let $\varphi_n(t) = e^{\lambda_n t} w_n(t)$ and observe, using (31), that

$$\begin{aligned}
&\dot{w}_{n+1}(t) + (\lambda_n + \mu) w_{n+1}(t) \\
&\quad \leq q w_n(t-\sigma) - \lambda_n w_n(t-\sigma) + (\lambda_n + \mu)\left[-p w_n(t-\tau) + \lambda_n \int_{t-\sigma}^t w_n(s)\, ds\right] \\
&\quad = q e^{-\lambda_n(t-\sigma)} \varphi_n(t-\sigma) - \lambda_n e^{-\lambda_n(t-\sigma)} \varphi_n(t-\sigma) + (\lambda_n + \mu)(-p e^{-\lambda_n(t-\tau)} \varphi_n(t-\tau)) \\
&\qquad + \lambda_n(\lambda_n + \mu) \int_{t-\sigma}^t e^{-\lambda_n s} \varphi_n(s)\, ds \\
&\quad \leq \varphi_n(t-\sigma) e^{-\lambda_n t}[q e^{\lambda_n \sigma} - \lambda_n e^{\lambda_n \sigma} - \lambda_n p e^{\lambda_n \tau} - \lambda_n + \lambda_n e^{\lambda_n \sigma} + \mu(e^{\lambda^* \sigma} - p e^{\lambda^* \tau} - 1)] \\
&\quad \leq \varphi_n(t-\sigma) e^{-\lambda_n t}[-m + m] = 0 .
\end{aligned}$$

The proof is complete.

3. Proof of Theorem 2

PROOF. (a) $\Rightarrow$ (b). Clearly, if there is a root in $(0, \infty)$ then an unbounded nonoscillatory solution exists. Also, $\lambda = 0$ cannot be a double root, for if 0 were a double root then $q = 0$ and $p = -1$ which would reduce Eq. (1) to

$$\frac{d}{dt}[y(t) - y(t-\tau)] = 0$$

which has the unbounded nonoscillatory solution $y(t) = t$.

(b) $\Rightarrow$ (a). Assume, for the sake of contradiction, that Eq. (1) has an unbounded eventually positive solution $y(t)$.

First assume $p = 0$. Then clearly q must be negative, for otherwise $y(t)$ would be bounded. Also $\sigma \neq 0$, for otherwise the characteristic equation would have a positive root. Hence, there remain the following cases to consider.

(i) $q < 0$ and $\sigma > 0$

(ii) $q < 0$ and $\sigma < 0$.

Case (i): $q < 0$ and $\sigma > 0$. We have $F(0) \cdot F(\infty) < 0$ which implies that the characteristic equation has a positive root.

Case (ii): $q < 0$ and $\sigma < 0$. Set

$$\Lambda = \{\lambda \geq 0 \colon -\dot{y}(t) + \lambda y(t) \leq 0\} .$$

As in Theorem 1, we will show that Λ has the contradictory properties (P_1) and (P_2).

From (5) we have $-\dot{y}(t) + (-q)y(t) \leq 0$ which yields $\lambda_1 = -q \in \Lambda$. From Lemmas 1 and 5 it follows that

$$\lambda_2 = \frac{1}{-\sigma} \ln \frac{4}{(\sigma q)^2} \notin \Lambda .$$

Let $\lambda \in \Lambda$ and set $\varphi(t) = e^{-\lambda t} y(t)$. Observe that

$$\dot{\varphi}(t) = -e^{-\lambda t}[-\dot{y}(t) + \lambda y(t)] \geq 0$$

which shows that $\varphi(t)$ is increasing.

Now

$$\begin{aligned} -\dot{y}(t) + (\lambda + m)y(t) &= qy(t-\sigma) + (\lambda + m)y(t) \\ &= q\varphi(t-\sigma)e^{\lambda(t-\sigma)} + (\lambda + m)e^{\lambda t}\varphi(t) \\ &\leq \varphi(t)e^{\lambda t}[qe^{-\lambda\sigma} + \lambda + m] \\ &\leq \varphi(t)e^{\lambda t}[-m + m] = 0 \end{aligned}$$

which completes the proof when $p = 0$.

The case when $\tau = 0$ and $p \neq -1$ follows in a manner analogous to the case when $p = 0$. On the other hand the case $\tau = 0$ and $p = -1$ is trivial. So we will assume $p\tau \neq 0$.

When $q = 0$, Eq. (1) reduces to

$$\frac{d}{dt}[y(t) + py(t-\tau)] = 0$$

which implies $y(t) + py(t-\tau) = c$. Clearly, $y(t)$ cannot be positive and unbounded if $p > 0$. Also, (2) reduces to

$$F(\lambda) = \lambda(1 + pe^{-\lambda\tau}) = 0$$

and it follows that $p > -1$ when $\tau > 0$, for otherwise (2) has a positive root or 0 is a double root of (2). Furthermore, because of the Duality Lemma, we need only consider $-1 < p < 0$ and $\tau > 0$ to complete the proof when $q = 0$. To this end, let $\{t_n\}$ be a sequence of points such that $\lim_{n\to\infty} t_n = \infty$, $y(t_n) = \max_{s \le t_n} y(s)$ and $\lim_{n\to\infty} y(t_n) = \infty$. Observe that

$$c = y(t_n) + py(t_n - \tau) \ge (1 + p)y(t_n) \to \infty \quad \text{as} \quad n \to \infty$$

which is impossible.

Finally, by utilizing the Duality Lemma, one can see that the following cases remain to complete the proof of the theorem.

	p	q	τ	σ
1.	+	+	+	+, 0
2.	+	+	+	−
3.	+	−	+	+, 0
4.	+	−	+	−
5.	−	+	+	+, 0
6.	−	+	+	−
7.	−	−	+	+, 0
8.	−	−	+	−

In the sequel $z(t)$ and $w(t)$ denote the functions defined in Lemma 4.

Cases 1 *and* 2: $p > 0$, $q > 0$, $\tau > 0$, $\sigma \ge 0$ or $p > 0$, $q > 0$, $\tau > 0$, $\sigma < 0$. Here

$$\dot{z}(t) = -qy(t - \sigma) < 0 \,.$$

Hence $\lim_{t\to\infty} z(t)$ exists, which contradicts the assumption that $y(t)$ is unbounded.

Cases 3 *and* 7: $p > 0$, $q < 0$, $\tau > 0$, $\sigma \ge 0$ or $p < 0$, $q < 0$, $\tau > 0$, $\sigma \ge 0$. We have $F(0)\cdot F(\infty) < 0$ which implies that the characteristic equation has a positive root.

Case 4: $p > 0$, $q < 0$, $\tau > 0$ and $\sigma < 0$. The dual of Case 4 is

$$p > 0\,, \quad q < 0\,, \quad \tau < 0 \quad \text{and} \quad \sigma < 0 \quad (\sigma < \tau)$$

which we will now consider. Set

$$w_n(t) = \begin{cases} w(t)\,, & n = 0 \\ w_{n-1}(t) + pw_{n-1}(t - \tau)\,, & n = 1, 2, \ldots \end{cases}$$

and

$$\Lambda_n = \{\lambda \geq 0 \colon -\dot{w}_n(t) + \lambda w_n(t) \leq 0\}\,.$$

It follows that for $n = 1, 2, \ldots$

$$\dot{w}_n(t) + p\dot{w}_n(t-\tau) + qw_n(t-\sigma) = 0\,,$$

$$\dot{w}_n(t) = -qw_{n-1}(t-\sigma)$$

and

$$w_n(t) > 0\,, \quad \dot{w}_n(t) > 0 \quad \text{and} \quad \ddot{w}_n(t) > 0\,.$$

From the above we have

$$\dot{w}_n(t-\tau) + \frac{q}{1+p} w_n(t-\sigma) \geq 0$$

which implies

(32) $$\dot{w}_n(t) + \frac{q}{1+p} w_n(t + (\tau - \sigma)) \geq 0$$

and so

(33) $$-\dot{w}_n(t) + \frac{-q}{1+p} w_n(t) \leq 0\,.$$

Applying Lemmas 1 and 5 to (32) yields

$$\lambda_2 = \frac{1}{\tau - \sigma} \ln \frac{4(1+p)^2}{q^2(\tau-\sigma)^2} \notin \bigcap_{n=1}^{\infty} \Lambda_n$$

while (33) yields

$$\lambda_1 = \frac{-q}{1+p} \in \bigcup_{n=1}^{\infty} \Lambda_n\,.$$

Let $\lambda \in \Lambda_n$ and set $\varphi_n(t) = e^{-\lambda t} w_n(t)$ and $\mu = m/(1 + pe^{-\lambda_2 \tau})$. Now

$$\begin{aligned}
&-\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) \\
&\quad = qw_n(t-\sigma) + (\lambda+\mu)[w_n(t) + pw_n(t-\tau)] \\
&\quad = q\varphi_n(t-\sigma)e^{\lambda(t-\sigma)} + (\lambda+\mu)[e^{\lambda t}\varphi_n(t) + pe^{\lambda(t-\tau)}\varphi_n(t-\tau)] \\
&\quad \leq \varphi_n(t-\sigma)e^{\lambda t}[qe^{-\lambda\sigma} + \lambda + \lambda pe^{-\lambda\tau} + \mu + \mu pe^{-\lambda_2\tau}] \\
&\quad \leq \varphi_n(t-\sigma)e^{\lambda t}[-m + m] = 0
\end{aligned}$$

which completes the proof in this case.

Case 5: $p < 0$, $q > 0$, $\tau > 0$ and $\sigma \geq 0$. First assume that $\sigma < \tau$. Set

$$w_n(t) = \begin{cases} w(t)\,, & n = 0 \\ -[w_{n-1}(t) + pw_{n-1}(t-\tau)]\,, & n = 1, 2, \dots \end{cases}$$

and

$$\Lambda_n = \{\lambda \geq 0\colon -\dot{w}_n(t) + \lambda w_n(t) \leq 0\}\,.$$

It follows that for $n = 1, 2, \dots$

$$\begin{gathered} \dot{w}_n(t) + p\dot{w}_n(t-\tau) + qw_n(t-\sigma) = 0\,, \\ \dot{w}_n(t) = qw_{n-1}(t-\sigma) \end{gathered} \tag{34}$$

and

$$w_n(t) > 0\,, \quad \dot{w}_n(t) > 0 \quad \text{and} \quad \ddot{w}_n(t) > 0\,.$$

From (34) we have

$$\dot{w}_n(t-\tau) + p\dot{w}_n(t-\tau) + qw_n(t-\sigma) \leq 0$$

and from [1] it follows that $p < -1$. Combining these results gives

$$\dot{w}_n(t) - \frac{q}{-(1+p)} w_n(t + (\tau - \sigma)) \geq 0$$

and

$$-\dot{w}_n(t) + \frac{q}{-(1+p)} w_n(t) \leq 0\,.$$

It now follows that

$$\lambda_1 = \frac{q}{-(1+p)} \in \bigcap_{n=1}^{\infty} \Lambda_n$$

and

$$\lambda_2 = \frac{1}{\tau - \sigma} \ln \frac{4(1+p)^2}{q^2(\tau - \sigma)^2} \notin \bigcup_{n=1}^{\infty} \Lambda_n\,.$$

Let $\lambda \in \Lambda_n$ and set $\varphi_n(t) = e^{-\lambda t} w_n(t)$ and $\mu = m/(-p)$. Now

$$\begin{aligned} -\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) &= -qw_n(t-\sigma) + (\lambda + \mu)[-w_n(t) - pw_n(t-\tau)] \\ &\leq \varphi_n(t-\sigma) e^{\lambda t}[-qe^{-\lambda\sigma} - \lambda - \lambda p e^{-\lambda\tau} - \mu - \mu p e^{-\lambda\tau}] \\ &\leq \varphi_n(t-\sigma) e^{\lambda t}[-m + m] = 0\,. \end{aligned}$$

This completes the proof in Case 5 for $\sigma < \tau$.

For $\sigma \geq \tau$, set

$$(35)\quad w_n(t) = \begin{cases} w(t)\,, & n = 0 \\ -[w_{n-1}(t) + pw_{n-1}(t-\tau)] + q\displaystyle\int_{t-\sigma}^{t-\tau} w_{n-1}(s)\,ds\,, & n = 1, 2, \ldots \end{cases}$$

and observe that

$$\dot{w}_{n+1}(t) = qw_n(t-\tau) > 0\,. \tag{36}$$

From (35) we obtain

$$w_{n+1}(t) < [-p + q(\sigma - \tau)]w_n(t-\tau)\,.$$

This together with (36) gives

$$-\dot{w}_{n+1}(t) + \frac{q}{[-p + q(\sigma - \tau)]} w_{n+1}(t) \le 0$$

which implies

$$\lambda_1 = \frac{q}{-p + q(\sigma - \tau)} \in \textstyle\bigcap_{n=1}^{\infty} \Lambda_n\,.$$

Now

$$\dot{w}_n(t) + p\dot{w}_n(t-\tau) + qw_n(t-\sigma) = 0$$

implies

$$\dot{w}_n(t) + pw_n(t-\tau) \le 0\,. \tag{37}$$

Combining (36) and (37) gives

$$qw_{n-1}(t-\tau) + p\dot{w}_n(t-\tau) \le 0\,. \tag{38}$$

Integrating (38) from t to $t + \tau$ we obtain

$$q\tau w_{n-1}(t-\tau) + pw_n(t) - pw_n(t-\tau) \le 0\,.$$

The above implies

$$qw_{n-1}(t-\tau) < -\left(\frac{p}{\tau}\right) w_n(t) > 0 \tag{39}$$

From (36) and (39) we now obtain

$$-\dot{w}_n(t) + \left(-\frac{p}{\tau}\right) w_n(t) > 0$$

which implies

$$\lambda_2 = -\frac{p}{\tau} \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

Let $\lambda \geq \lambda_1$ and set $\varphi_n(t) = e^{-\lambda t} w_n(t)$ and $\mu = m[-p + (q/\lambda_1)]^{-1}$. Observe that

$$\begin{aligned}
&-\dot{w}_{n+1}(t) + (\lambda + \mu) w_{n+1}(t) \\
&= -q w_n(t-\tau) + (\lambda + \mu)\left[-w_n(t) - p w_n(t-\tau) + q \int_{t-\sigma}^{t-\tau} w_n(s)\, ds\right] \\
&\leq -q\varphi_n(t-\tau) e^{\lambda(t-\tau)} + (\lambda + \mu)\Big[-\varphi_n(t) e^{\lambda t} - p\varphi_n(t-\tau) e^{\lambda(t-\tau)} \\
&\qquad + q \int_{t-\sigma}^{t-\tau} e^{\lambda s} \varphi_n(s)\, ds\Big] \\
&\leq \varphi_n(t-\tau) e^{\lambda t}\Big[-q e^{-\lambda\tau} - \lambda - \lambda p e^{-\lambda\tau} - \mu - \mu p e^{-\lambda\tau} + q e^{-\lambda\tau} - q e^{-\lambda\sigma} \\
&\qquad + \frac{\mu q}{\lambda}(e^{-\lambda\tau} - e^{-\lambda\sigma})\Big] \\
&\leq \varphi_n(t-\tau) e^{\lambda t}\left[-\lambda - \lambda p e^{-\lambda\tau} - q e^{-\lambda\sigma} - \mu - \mu p e^{-\lambda\tau} + \frac{\mu q}{\lambda}(e^{-\lambda\tau} - e^{-\lambda\sigma})\right] \\
&\leq \varphi_n(t-\tau) e^{\lambda t}[-m + m] = 0 .
\end{aligned}$$

The proof is complete in this case.

Case 6: $p < 0$, $q > 0$, $\tau > 0$ and $\sigma < 0$. Set

$$w_n(t) = \begin{cases} w(t), & n = 0 \\ -[w_{n-1}(t) + p w_{n-1}(t-\tau)] - q \displaystyle\int_{t-\tau}^{t-\sigma} w_{n-1}(s)\, ds, & n = 1, 2, \ldots \end{cases} \tag{40}$$

and define Λ_n as in Case 5.

Now for $n = 1, 2, \ldots$

$$\dot{w}_n(t) + p\dot{w}_n(t-\tau) + q w_n(t-\sigma) = 0 \tag{41}$$

and

$$\dot{w}_{n+1}(t) = q w_n(t-\tau) > 0 . \tag{42}$$

Again it follows from [4] that $p < -1$. This together with (41) yields

$$\dot{w}_n(t) - \frac{q}{-(1+p)} w_n(t + (-\sigma)) \geq 0$$

and

$$-\dot{w}_n(t) + \frac{q}{-(1+p)} w_n(t) \leq 0 .$$

We now have

$$\lambda_1 = \frac{q}{-(1+p)} \in \bigcap_{n=1}^{\infty} \Lambda_n$$

and

$$\lambda_2 = \frac{1}{-\sigma} \ln \frac{4(1+p)^2}{(q\sigma)^2} \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

To complete the proof in this case repeat exactly the same argument as in Case 5 when $\sigma \geq \tau$.

Case 8: $p < 0$, $q < 0$, $\tau > 0$ and $\sigma < 0$. The proof will follow a format similar to that of Case 7 in Theorem 1.

Let V be the set of all C^2 solutions of Eq. (1) which satisfy

$$v(t) > 0 , \quad \dot{v}(t) > 0 , \quad \ddot{v}(t) > 0 \quad \text{and} \quad \lim_{t\to\infty} v(t) = \infty .$$

Set

$$\Lambda(v) = \{\lambda \geq 0 \colon -\dot{v}(t) + \lambda v(t) \leq 0\} .$$

Observe that

$$\dot{v}(t) + p\dot{v}(t-\tau) + qv(t-\sigma) = 0 . \tag{43}$$

From (43) we obtain

$$\dot{v}(t) - (-q)v(t + (-\sigma)) \geq 0 \tag{44}$$

and so

$$-\dot{v}(t) + (-q)v(t) \leq 0 . \tag{45}$$

From (44) and (45) we obtain

$$-q \in \Lambda(v)$$

and

$$\lambda^* = \frac{1}{-\sigma} \ln \frac{4}{(q\sigma)^2} \notin \Lambda(v) .$$

Now, let $\lambda_0 = -q$ and set $\mu = m/(1 - pe^{-\lambda^* \sigma})$. We will prove by induction that if

$$\lambda_n = \lambda_{n-1} + \mu, \qquad n = 1, 2, \ldots,$$

and if

$$w_n(t) = \begin{cases} w(t), & n = 0 \\ w_{n-1}(t) + pw_{n-1}(t) - \lambda_{n-1} p \displaystyle\int_{t-\tau}^{t-\sigma} w_{n-1}(s)\, ds, & n = 1, 2, \ldots \end{cases}$$

then $w_n \in V$ and $\lambda_n \in \Lambda(w_n)$. As $\Lambda(w_n)$ is bounded from above, this will be a contradiction and will complete the proof in this case. To this end, set $\varphi_n(t) = e^{-\lambda_n t} w_n(t)$ and observe that

$$\begin{aligned} &-\dot{w}_{n+1}(t) + (\lambda_n + \mu) w_{n+1}(t) \\ &= q w_n(t-\sigma) + \lambda_n p[w_n(t-\sigma) - w_n(t-\tau)] \\ &\quad + (\lambda_n + \mu)\left[w_n(t) + pw_n(t-\tau) - \lambda_n p \int_{t-\tau}^{t-\sigma} w_n(s)\, ds \right] \\ &\le q w_n(t-\sigma) + \lambda_n p w_n(t-\sigma) + \lambda_n w_n(t) - \lambda_n^2 p \int_{t-\tau}^{t-\sigma} w_n(s)\, ds \\ &\quad + \mu\left[w_n(t) - \lambda_n p \int_{t-\tau}^{t-\sigma} w_n(s)\, ds \right] \\ &\le \varphi_n(t-\sigma) e^{\lambda_n t}[q e^{-\lambda_n \sigma} + \lambda_n + \lambda_n p e^{-\lambda_n \tau} + \mu(1 - pe^{-\lambda^* \sigma})] \\ &\le \varphi_n(t-\sigma) e^{\lambda_n t}[-m + m] = 0 . \end{aligned}$$

The proof of Theorem 2 is complete.

REMARK 2. In several instances, in the proofs of Theorems 1 and 2, we found points

$$\lambda_1 \in \bigcap_{n=1}^{\infty} \Lambda_n \quad \text{and} \quad \lambda_2 \notin \bigcup_{n=1}^{\infty} \Lambda_n .$$

The values of λ_1 and λ_2 were expressed in terms of the coefficients, delays and advances of Eq. (1). Clearly when they are such that

$$\lambda_1 \ge \lambda_2$$

this is a contradiction. Utilizing this idea we can obtain "easily verifiable" sufficient conditions for the oscillation of all bounded and all unbounded solutions of Eq. (1).

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