HIROSHIMA MATH. J.
19 (1989), 397–407

Lie algebras with the minimal condition on centralizer ideals

Falih A. M. ALDOSRAY and Ian STEWART

(Received August 19, 1988)

Ascending or descending chain conditions on various classes of subalgebras of Lie algebras have been investigated by numerous authors. A survey of results up to 1974 may be found in chapters 8, 9, and 11 of Amayo and Stewart [2]. Subsequent work includes Aldosray [1], Ikeda [6], Kubo [8], Kubo and Honda [9], Tôgô [13, 14], and Stewart [12]. Here we introduce a new chain condition: Min-c, the minimal (equivalently maximal) condition on centralizer ideals. It is perhaps curious that this chain condition has hitherto been neglected, because an analogous condition—the minimal condition for annihilator ideals—is prominent in the theory of associative rings. See e.g. Faith [3], Herstein [5], Goldie [4]. However, a crucial ingredient, prime ideals, was not supplied for Lie algebras until the work of Kawamoto [7], later extended by Aldosray [1].

Here we develop a theory of Lie algebras with Min-c that is analogous to, but differs in several respects from, the associative theory. The main results are as follows. In §2 we show that abelian-by-finite Lie algebras satisfy Min-c; that like the associative case, Min-c is not closed under extensions or quotients; and that unlike the associative case Min-c is not closed under taking ideals. In §3 we show that any hypercentral ideal of a Lie algebra with Min-c must be soluble (though not necessarily nilpotent). In §4 we study prime ideals of semisimple Lie algebras with Min-c. We show that for such algebras, maximal centralizer ideals are the same as minimal prime ideals. The main result of this section is that in a semisimple Lie algebra with Min-c, every centralizer ideal is the intersection of a finite number of maximal centralizer ideals, that is, minimal prime ideals. We conclude by stating some open problems.

1. Preliminaries

Throughout the paper, Lie algebras are of finite or infinite dimension over a field of arbitrary characteristic, unless stated otherwise. We use the notation and terminology of Amayo and Stewart [2]. For convenience we summarise the required notation here. If H is a subalgebra of L we write $H \leq L$; if it is an ideal we write $H \lhd L$. Angular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. If X, $Y \subset L$ then X^Y is the smallest subspace containing X that is Y-invariant. If α is an ordinal then L^{α}, $L^{(\alpha)}$, $\zeta_{\alpha}(L)$ respectively denote the α-th term of the lower central series, derived series, and upper central series of L. We require the following classes of Lie algebras:

$\mathscr{A}$	abelian ($L^2 = 0$)
$\mathscr{N}$	nilpotent ($L^n = 0$ for finite n)
$\mathrm{E}\mathscr{A}$	soluble ($L^{(n)} = 0$ for finite n)
$\mathscr{A}^d$	soluble of derived length d ($L^{(d)} = 0$)
$\mathscr{Z}$	hypercentral ($\zeta_{\alpha}(L) = L$ for some α)
$\mathrm{L}\mathscr{N}$	locally nilpotent
Min-$\lhd$	minimal condition for ideals
Max-$\lhd$	maximal condition for ideals.

L is *ideally finite* if it is generated by finite-dimensional ideals. If $\mathscr{X}$ is a class of Lie algebras then $L \in \mathrm{E}\mathscr{X}$ if there exists a finite series

$$0 = L_0 \lhd L_1 \lhd \cdots \lhd L_n = L$$

whose quotients $L_{i+1}/L_i \in \mathscr{X}$. If $I \lhd L \in \mathscr{X}$ then $I \in \mathrm{I}\mathscr{X}$ and $L/I \in \mathrm{Q}\mathscr{X}$. We say that $\mathscr{X}$ is E-*closed* if $\mathscr{X} = \mathrm{E}\mathscr{X}$ and similarly for I and Q. If $I \subset L$ then the *centralizer* $C_L(I) = \{x \in L \mid [I, x] = 0\}$. It is an ideal if I is. The *soluble radical* $\sigma(L)$ is the sum of the soluble ideals of L. If $\sigma(L) = 0$, or equivalently if L has no nonzero abelian ideals, we say L is *semisimple*. An ideal P of L is *prime* if whenever A and B are ideals with $[A, B] \subset P$ then either $A \subset P$ or $B \subset P$. The *radical* Rad I of an ideal I is the intersection of all prime ideals containing I. See Kawamoto [7].

We now introduce some new terminology. Say that I is a *centralizer ideal* of L if there exists an ideal J of L such that $I = C_L(J)$. Say that L satisfies the *minimal condition on centralizer ideals* (Min-c) if every descending chain $I_1 \supset I_2 \supset \cdots$ of centralizer ideals terminates finitely. Similarly L satisfies the *maximal condition on centralizer ideals* (Max-c) if every ascending chain $I_1 \subset I_2 \subset \cdots$ of centralizer ideals terminates finitely. We use the same notation for the corresponding classes: note that (Lemma 2.1(a)) Min-c and Max-c turn out to be the same.

If $I \lhd L$ then I is a *complement ideal* if there exists a nonzero ideal J of L such that $I \cap J = 0$ and, if $K \lhd L$, $K \supsetneq I$, then $K \cap J \neq 0$. Say that L satisfies the *maximal condition on complement ideals* if every increasing chain of complement ideals terminates finitely; and that L *has no infinite direct sum of nonzero ideals* if every direct sum of nonzero ideals contains a finite number of terms.

2. Closure properties

In this section we establish some elementary properties of the class Min-c, showing in particular that it is *not* closed under the operations Q, E, or I. It is easy to show that if $I \lhd L$ then $I \subset C_L C_L(I)$ and $C_L C_L C_L(I) = C_L(I)$. Hence C_L defines an order reversing bijection, or *Galois duality*, between the set

$$\mathscr{C} = \{C \lhd L \mid C = C_L(I) \text{ for some } I \lhd L\}$$

and itself. We begin by establishing some properties analogous to those known to hold for rings with the minimal condition on annihilator ideals.

LEMMA 2.1. *Let L be a Lie algebra. Then*

(a) $L \in$ Min-c *if and only if* $L \in$ Max-c.

(b) $L \in$ Min-c *if and only if every ideal I of L contains a finitely generated ideal J of L such that* $C_L(I) = C_L(J)$.

PROOF. (a) follows from Galois duality, and (b) can be proved exactly as in Faith [3].

LEMMA 2.2. *Let L be a Lie algebra. Then L has no infinite direct sum of non-zero ideals if and only if L satisfies the maximal condition on complement ideals.*

PROOF. Let $I_1 \subset I_2 \subset \cdots$ be a strictly ascending chain of complement ideals in L. Then there exist non-zero ideals J_m such that $I_m \cap J_m = 0$ and $I_{m+1} \cap J_m \neq 0$. Let $K_m = I_{m+1} \cap J_m$. Then $J_n \supset K_m$ $(n \geq m)$, and the sum $K_1 + \cdots + K_m$ is direct. Hence L contains an infinite direct sum of non-zero ideals.

Conversely, suppose there exists an infinite direct sum of non-zero ideals $I_1 \oplus I_2 \oplus \cdots$. Let $J_m = I_m \oplus I_{m+1} \oplus \cdots$. Then J_2 has a complement ideal $K_2 \supset I_1$. Since $K_2 \cap J_2 = 0$, we have $(K_2 \oplus I_2) \cap J_3 = 0$, hence J_3 has a complement ideal $K_3 \supset K_2 \oplus I_2$. Continuing in this way we obtain an infinite ascending chain of complement ideals K_m.

LEMMA 2.3. *Let L be a semisimple Lie algebra, and $I \neq L$. Then I is a centralizer ideal in L if and only if I is a complement ideal in L.*

PROOF. Suppose that $I \neq L$ is a centralizer ideal. Then $C_L(I) \neq 0$ and $I \cap C_L(I) = 0$ since L is semisimple. Moreover if $K \lhd L$, $K \cap C_L(I) = 0$, then $[K, C_L(I)] \subset K \cap C_L(I) = 0$ so $K \subset C_L(C_L(I)) = I$. Therefore I is a complement ideal in L. Now suppose that I is a complement ideal in L. There exists a non-zero $J \lhd L$ such that $I \cap J = 0$. Therefore $I \subset C_L(J)$. If $C_L(J) \supsetneqq I$ then $C_L(J) \cap J \neq 0$ since I is a complement ideal; but this is impossible since L is semisimple. Therefore $I = C_L(J)$ and I is a centralizer ideal.

COROLLARY 2.4. *Let L be semisimple. Then $L \in$* Max-c *if and only if L satisfies the maximal condition on complement ideals.*

COROLLARY 2.5. *If L is a semisimple ideally finite Lie algebra over a field of characteristic zero, then L has* Min-c *if and only if L is finite-dimensional.*

It is easy to show that Min-c is closed under taking finite direct sums. However it is known, Herstein [5], that for associative rings the minimal condition for annihilators is not preserved under quotients or extensions. Not surprisingly, the same is true for Min-c in Lie algebras:

EXAMPLE 2.6. Min-c *is neither* E*-closed nor* Q*-closed.*

(a) E*-closure.* Let F be a field, let $P = F[x_1, x_2, \dots]$ be a polynomial algebra in an infinite number of indeterminates x_i, let I be the associative ideal of P generated by all x_i^2, and let $A = P/I$. Considered as an abelian Lie algebra, A has derivations $\delta_i: f \mapsto x_i f$ $(f \in A)$. The δ_i commute. Let $H = \langle \delta_i | i \geq 1 \rangle$ and form the split extension $L = A \dotplus H$. Then L is metabelian. Every abelian algebra lies in Min-c, so A and $L/A \in$ Min-c. However, $L \notin$ Min-c. For let $I_j = \langle x_1 x_2 \dots x_j \rangle^L$, which is the associative ideal of A generated by $x_1 x_2 \dots x_j$. Then $C_L(I_j) = A + \langle \delta_1, \dots, \delta_j \rangle$. Then $L \notin$ Max-c $=$ Min-c. Hence Min-c is not E-closed.

(b) Q*-closure.* Let P be as above and define $\varepsilon_i: f \mapsto x_i f$ $(f \in P)$. The ε_i commute. Let $K = \langle \varepsilon_i | i \geq 1 \rangle$. Consider P as an abelian Lie algebra with the ε_i as derivations and form $M = P \dotplus K$. Then P, K are abelian. We claim that $M \in$ Min-c since the only centralizer ideals in M are 0, P, M. To see this, let $C = C_M(I)$ be a centralizer ideal. It is easy to see that if $C \neq 0$ then $C \cap P \neq 0$, whence $C \cap P = P$ since P has no divisors of zero. Then either $C = P$ or $C = M$. This establishes the claim.

Let I be as in (a). Then $I \lhd M$ and $M/I \cong L$ as in (a). Thus $M \in$ Min-c but $M/I \notin$ Min-c. Hence Min-c is not Q-closed.

This example also shows that a maximal centralizer ideal of a Lie algebra in Min-c need not be prime, because P is a maximal centralizer ideal in M but M/P is abelian. We will see below that for semisimple algebras, maximal centralizers are prime.

However, the following result is true:

THEOREM 2.7. *Let L be a Lie algebra, $A \lhd L$. If A is abelian and $L/A \in$* Min-$\lhd \cap$ Max-$\lhd$, *then $L \in$* Min-c.

PROOF. Let $I_1 \supset I_2 \supset \cdots$ be a descending chain of centralizer ideals in Then $(I_1 + A)/A \supset (I_2 + A)/A \supset \cdots$ is a descending chain of ideals of $L/A \in$

Min-$\lhd$. Therefore there exists $m \in \mathbb{N}$ such that $I_n + A = I_m + A$ for all $n \geq m$. Also $(C_L(I_1) + A)/A \subset (C_L(I_2) + A)/A \subset \cdots$ is an ascending chain of ideals in $L/A \in$ Max-$\lhd$, so there exists $m' \in \mathbb{N}$ such that $(C_L(I_{m'}) + A)/A = (C_L(I_n) + A)/A$ for all $n \geq m'$. Replacing m and m' by $\max(m, m')$ we may assume $m = m'$. Then $C_A(C_L(I_m) + A) = C_A(C_L(I_n) + A)$ and $C_A(C_L(I_m)) = C_A(C_L(I_n))$ for all $n \geq m$, implying that $A \cap I_m = A \cap I_n$ for all $n \geq m$. As in Amayo and Stewart [2] Theorem 1.7.3, p. 26 we have $I_n = I_m$ for all $n \geq m$, so $L \in$ Min-c.

COROLLARY 2.8. *The class $\mathscr{A}\mathscr{F}$ is contained in* Min-c.

EXAMPLE 2.9. Let I be an ideal of L and let $C = C_L(K)$ be a centralizer ideal of L, K being an ideal of L. Then clearly $C \cap I = C_I(K)$. However, $C \cap I$ need not be a centralizer ideal of I. For example, let $L = A \dot{+} H$ be as in Example 2.6(a), and let $K = \langle x_1 \rangle^L + \langle \delta_1 \rangle$. Then $K = C_L(K)$, but $K \cap A = \langle x_1 \rangle^L$ is not a centralizer ideal in A since A is abelian.

In rings, the minimal condition for annihilators is closed under taking ideals, Herstein [5]. The reason for this is that the annihilator of a subset is an ideal. This is not the case for centralizers in Lie algebras:

EXAMPLE 2.10. Min-c is not I-closed, even when $L \in \mathscr{A}^3$ and the ideal is of codimension 1.

Let $P = F[t]$, a polynomial algebra in one indeterminate t. Let K be the Lie algebra of differential operators

$$\varepsilon: f \mapsto tf$$

$$D^i: f \mapsto d^i f/dt^i$$

($f \in P$, $i = 0, 1, 2, \ldots$) where D^0 is the identity. Then $[D^i, D^j] = 0$, $[D^i, \varepsilon] = iD^{i-1}$ ($i \geq 1$), $[D^0, \varepsilon] = 0$. Form $L = P \dot{+} K$. Clearly $L \in \mathscr{A}^3$.

(i) $L \in$ Min-c. To prove this let $I \lhd L$ and suppose that $f + k \in I$ where $f \in P$, $k \in K$. Then $f = [f + k, D^0] \in I$. If $f \neq 0$ and the degree of f is i then $[f, D^i] = \alpha$ belongs to I, and $\alpha \neq 0$, $\alpha \in F$. Thus $1 \in I$, whence $P \subset I$ by repeated application of ε. On the other hand, if all f are 0 then $I \subset K$, whence easily $I = 0$. Thus every ideal I is either 0 or contains P. If $I \supset P$ then $C_L(I) \subset C_L(P) = P$, whence $C_L(I) = P$ or $C_L(I) = 0$. If $I = 0$ then $C_L(I) = L$. Thus L has only three centralizer ideals, so $L \in$ Min-c.

(ii) Let $H = \langle D^i | i \geq 0 \rangle \lhd K$, and let $M = P \dot{+} H \lhd L$. Then codim $M = 1$. We claim that $M \notin$ Min-c. Let $I_j = F\{1, t, \ldots, t^j\} \lhd M$. Then $C_M(I_j) = P + F\{D^{j+1}, D^{j+2}, \ldots\}$ and these ideals form a strictly decreasing chain.

However, the structure of ideals of algebras in Min-c is subject to a chain condition weaker than Min-c, which shows that the failure of I-closure of Min-c is due to "bad nilpotent sections". More precisely, we shall prove:

THEOREM 2.11. *Let $I \lhd L \in$ Min-c. Let $K_1 \subset K_2 \subset \cdots$ be an ascending chain of centralizer ideals in I. Then there exists m such that $K_i^\omega \subset K_m$ for all i.*

PROOF. $K_i^\omega \lhd L$ by Schenkman [11]. Therefore $C_L(K_i^\omega) \lhd L$. Therefore $C_I(K_i)^L \subset C_L(K_i^\omega)$. Let $D_i = C_L(C_I(K_i)^L) \supset C_L C_L(K_i^\omega) \supset K_i^\omega$. The chain $\{K_i\}$ increases, so $\{C_I(K_i)\}$ decreases, so $\{C_I(K_i)^L\}$ decreases, so $\{D_i\}$ increases; also D_i is a centralizer ideal in L. Therefore there exists m such that $D_m = D_{m+1} = \cdots$. That is, $K_i^\omega \subset D_m$ for all $i \geq m$ (and hence for all i). Now $K_i^\omega \subset I$, so

$$K_i^\omega \subset C_I(C_I(K_m)^L) \subset C_I C_I(K_m) = K_m$$

since K_m is a centralizer ideal of I.

We define classes Fin-ω and Fin-cω as follows. A Lie algebra $L \in$ Fin-ω if for every proper ideal K of L, every increasing chain $\{K_i\}$ of ideals of L with $K_i^\omega \subset K$ stops. A Lie algebra $L \in$ Fin-cω if for every proper centralizer ideal K of L, every increasing chain of centralizer ideals $\{K_i\}$ of L with $K_i^\omega \subset K$ stops. Then Fin-$\omega \subset$ Fin-cω. For example, if $L \in$ Min-c and every ideal J of L is perfect, that is, $J = J^2$, then $L \in$ Fin-ω.

COROLLARY 2.12. *Let $I \lhd L \in$ Min-c, and suppose that $I \in$ Fin-cω. Then $I \in$ Min-c.*

3. Local nilpotence

If L is locally nilpotent and satisfies Min-$\lhd$ then L must be soluble, by Amayo and Stewart [2] Lemma 8.1.2, p. 163. We have not been able to decide whether this result extends to Min-c. However, hypercentral algebras with Min-c must be soluble. Indeed we have a stronger result:

THEOREM 3.1. *If $L \in$ Min-c then every hypercentral ideal of L is soluble.*

PROOF. Let $Z \lhd L$ be hypercentral. We claim that there is an ideal V of L that is maximal with respect to $V \subset Z$, $Z + C_L(V)/C_L(V) \in \mathrm{E}\mathfrak{A}$. To show V exists, let

$$\mathscr{V} = \{U \lhd L \mid U \subset Z, Z + C_L(U)/C_L(U) \in \mathrm{E}\mathfrak{A}\}\,.$$

Then $0 \in \mathscr{V}$ so $\mathscr{V} \neq \phi$. Let

$$\mathscr{C} = \{C_L(U) | U \in \mathscr{V}\}.$$

By Min-c, $\mathscr{C}$ has a minimal element $C_L(U)$. If $X \in \mathscr{V}$ and $U \subset X$ then $C_L(U) = C_L(X)$. Thus $V = \bigcup \{X \in \mathscr{V} | X \supset U\}$ is a maximal element of $\mathscr{V}$. In fact we can show that V is the unique maximal element, but that is not required for the present proof.

Suppose for a contradiction that $V \neq Z$. Let $W/V = \zeta_1(Z/V)$ so that $W > V$. Then $W \lhd L$ and $[Z, W] \subset V$. We have $Z^{(m)} \subset C_L(V)$ for some m. Then

$$[Z^{(m+1)}, W] \subset [Z^{(m)}, [Z^{(m)}, W]] \subset [Z^{(m)}, V] = 0.$$

Thus $Z^{(m+1)} \subset C_L(W)$ and $Z + C_L(W)/C_L(W) \in \mathrm{E}\mathscr{A}$, a contradiction.

Therefore $V = Z$, so $Z + C_L(Z)/C_L(Z) \in \mathrm{E}\mathscr{A}$, whence $Z/\zeta_1(Z) \in \mathrm{E}\mathscr{A}$, and $Z \in \mathrm{E}\mathscr{A}$.

EXAMPLE 3.2. Hypercentral algebras with Min-c need not be nilpotent.

Let $A = \langle a_i | i \in \mathbb{N}\rangle$ be abelian and define a derivation $\sigma: a_i \mapsto a_{i-1}$ $(i \geq 1)$, $a_0 \mapsto 0$. Let $L = A \dotplus \langle\sigma\rangle$. Then $L \in \mathscr{Z} \cap \text{Min-}\lhd \subset \mathscr{Z} \cap \text{Min-c}$, by Amayo and Stewart [2] p. 119, but $L \notin \mathscr{N}$.

4. Prime ideals

LEMMA 4.1. *If L is semisimple and M is a maximal (proper) centralizer ideal of L, then M is a prime ideal.*

PROOF. Let $A, B \lhd L$, $M \subset A$, $M \subset B$, and $[A, B] \subset M$. Suppose for a contradiction that $A, B \neq M$. Let $C = C_L(M)$, which by Galois duality is a minimal centralizer ideal. In particular $C \neq 0$. Then $C_L(A) \subset C$ so either $C_L(A) = 0$ or $C_L(A) = C$. Suppose $C_L(A) = C$. Then $A \subset C_L(C) = M$, a contradiction. Hence $C_L(A) = 0$. Similarly $C_L(B) = 0$.

If $A \cap C = 0$ then $[A, C] = 0$ so $C \subset C_L(A)$, which is not possible. Hence $A \cap C \neq 0$. Now $[B, A \cap C] \subset M \cap C = 0$ by semisimplicity, so $C_L(B) \supset A \cap C \neq 0$, a contradiction.

Even when $L \in \text{Min-c}$ is semisimple, not every prime ideal is a maximal centralizer. For an example let $L = \text{End}(V)/S$ where V is an infinite-dimensional vector space, $\text{End}(V)$ is its algebra of endomorphisms, and S is the set of scalar multiples of the identity. Then by Amayo and Stewart [2] Theorem 8.4.1,

p. 172, L has a unique well-ordered chain of ideals, all of whose factors are simple. Each of these ideals, other than L itself and $T + S/S$, must be prime. But the only centralizer ideals of L are 0 and L.

There is nevertheless a partial converse to Lemma 4.1:

LEMMA 4.2. *Let L be semisimple with* Min-c *and let P be a prime ideal which is a centralizer ideal. Then P is a maximal centralizer.*

PROOF. Let $P = C_L(I)$. Suppose $C_L(J) > P$, $J \neq 0$. Then $[C_L(J), J] = 0$ so either $C_L(J) \subset P$ or $J \subset P$. Thus by assumption $J \subset P$. But if $J \subset P$ then $[J, P] = 0$ so $J \subset \zeta_1(P) = 0$ by semisimplicity.

To pursue the matter to a more satisfactory conclusion, say that a prime ideal of L is a *minimal prime ideal* if it does not properly contain any prime ideal. Then we have:

COROLLARY 4.3. *If L is semisimple with* Min-c, *then every prime centralizer ideal is a minimal prime ideal.*

PROOF. Let P be a prime centralizer ideal of L. By Lemma 4.2 P is a maximal centralizer. Suppose that $P \supset Q$ for a prime ideal Q. We show that Q is also a centralizer ideal. If $Q \neq C_L C_L(Q)$, then there exists $x \in C_L C_L(Q) \backslash Q$. Then $[x^L, C_L(Q)] = 0$, but $x^L \not\subset Q$ and $C_L(Q) \not\subset Q$ by semisimplicity. This contradicts Q being prime. Thus $Q = C_L C_L(Q)$ is a centralizer. By Lemma 4.2 Q is a maximal centralizer. Therefore $P = Q$, so P is a minimal prime ideal.

We prove the converse below in Lemma 4.5. First we prove:

LEMMA 4.4. *If L is semisimple with* Min-c *and $I \lhd L$, then $C_L(I) \cap \mathrm{Rad}\, I =$ 0. In particular if I is a centralizer ideal of L then $I = \mathrm{Rad}\, I$.*

PROOF. Let $R = \mathrm{Rad}\, I$, and suppose that $X = R \cap C_L(I) \neq 0$. Then $C_L(X) < L$, and $I \subset C_L(X)$. Choose $0 \neq Y \subset X$ such that $Y \lhd L$ and $C_L(Y)$ is maximal with respect to $C_L(Y) \supset C_L(X)$. We claim that $C_L(Y)$ is a maximal centralizer. If not, there exists a larger maximal centralizer $C_L(Z)$, which is prime. But now $C_L(Y \cap Z) \supset C_L(Z) > C_L(Y)$. The only possibility is that $Y \cap Z = 0$. But then $Z \subset C_L(Y) \subset C_L(Z)$ contradicting the semisimplicity of L. Then $Y \subset X = R \cap C_L(I) \subset R \subset C_L(Y)$ and Y is abelian, also contradicting the semisimplicity of L.

The second statement now follows immediately.

LEMMA 4.5. *If L is semisimple with* Min-c, *then*

(a) *L has only a finite number of maximal centralizer ideals $M_1, \dots, M_n$.*

(b) *$M_1 \cap \cdots \cap M_n = 0$.*

(c) *A prime ideal of L is minimal prime if and only if it is a maximal centralizer ideal.*

PROOF. (a) Let $\{M_\alpha\}_{\alpha \in A}$ be distinct maximal centralizer ideals, A being an index set. As in Herstein [5] Lemma 4.7, the sum $\Sigma C_L(M_\alpha)$ is direct. By Lemmas 2.2 and 2.3, A is finite.

(b) Taking $A = \{1, \dots, n\}$, let $J = \bigcap_{i=1}^n M_i$ and suppose for a contradiction that $J \neq 0$. By assumption $C_L(J)$ is contained in some maximal centralizer ideal M of L. By definition $J \subset M$. Therefore $C_L(J) \supset C_L(M)$, so $C_L(M) \subset M$. Therefore $C_L(M)^2 = 0$, so $C_L(M) = 0$ by semisimplicity. This is a contradiction, so $J = 0$.

(c) Let P be a minimal prime ideal of L. Since $\bigcap M_i = 0$ we have $[M_1, \dots, M_n] \subset P$, whence $M_i \subset P$ for some i. But M_i is prime by Lemma 4.1, and P is minimal prime, so $P = M_i$ which is a centralizer ideal. The converse follows from Corollary 4.3.

We now prove our main result, an analogue of a theorem of Goldie [4]:

THEOREM 4.6. *If L is semisimple with* Min-c, *then every centralizer ideal in L is the intersection of a finite number of maximal centralizer ideals, that is, minimal prime ideals.*

PROOF. Let

$$I = \bigcap\nolimits_{i=1}^k P_i$$

where the P_i are the minimal prime ideals containing I. By Lemma 4.5(a) only finitely many P_i occur so we may take k finite. Let

$$K = \bigcap\nolimits_{i=2}^k P_i$$

so that $K \subset I$. Since L is semisimple, $I \cap C_L(I) = 0$. Therefore $P_1 \cap (K \cap C_L(I)) = 0$ and $[P_1, K \cap C_L(I)] = 0$, so $K \cap C_L(I) \subset C_L(P_1)$. But $K \cap C_L(I) \neq 0$, otherwise $K \subset I$, hence $C_L(P_1) \neq 0$. Similarly we can show that $C_L(P_i) \neq 0$ for all i. Now suppose that $P_i \supset M$ where M is a minimal prime ideal of L. If $P_i \neq M$, then there exists $x \in P_i \backslash M$. Therefore $[x, C_L(P_i)] = 0$. Hence $[x, C_L(P_i)] \subset M$ and $x \notin M$, so $C_L(P_i) \subset M$, which implies $C_L(P_i) \subset P_i$. This contradicts the semisimplicity of L. Therefore $P_i = M$, and P_i is a minimal prime ideal of L. By Lemma 4.5 each P_i is a centralizer ideal in L and by

Lemma 4.2 each P_i is a maximal centralizer. As already noted, by Lemma 4.5(a) only a finite number of P_i occur.

In closing we mention four open questions.

QUESTION 1. If $L \in \mathrm{L}\mathscr{N} \cap$ Min-c, is L soluble?

QUESTION 2. If $L \in$ Min-c, is $L/\zeta_1(L) \in$ Min-c?

QUESTION 3. If $L/I \in$ Min-c for all proper ideals I of L, is $L \in$ Min-c?

QUESTION 4. If L is semisimple with a finite number of maximal centralizer ideals, is $L \in$ Min-c? If not, is $L \in$ Min-c if further the intersection of the maximal centralizers is 0?

Acknowledgements

The first author is grateful to the Mathematics Institute, University of Warwick, for hospitality during the preparation of this paper. The autho thank the referee for suggesting several improvements.

References

[1] F. A. M. Aldosray, On Lie algebras with finiteness conditions, Hiroshima Math. J. **13** (1983) 665–674.
[2] R. K. Amayo and I. N. Stewart, Infinite-dimensional Lie algebras, Noordhoff, Leyden, 1974.
[3] C. Faith, Rings with ascending chain condition on annihilators, Nagoya Math. J. **27** (1966) 179–191.
[4] A. W. Goldie, Semi-prime rings with maximum conditions, Proc. London Math. Soc. **10** (1960) 201–220.
[5] I. N. Herstein, Topics in Ring Theory, Chicago U. Press, 1969.
[6] T. Ikeda, Chain conditions for abelian, nilpotent and soluble ideals in Lie algebras, Hiroshima Math. J. **9** (1979) 465–467.
[7] N. Kawamoto, On prime ideals of Lie algebras, Hiroshima Math. J. **4** (1974) 679–684.
[8] F. Kubo, Finiteness conditions for abelian ideals and nilpotent ideals in Lie algebras, Hiroshima Math. J. **8** (1978) 301–303.
[9] F. Kubo and M. Honda, Quasi-artinian Lie algebras, Hiroshima Math. J. **14** (1984) 563–570.
[10] D. H. McLain, A characteristically simple group, Proc. Camb. Philos. Soc. **50** (1954) 641–642.
[11] E. Schenkman, A theory of subinvariant Lie algebras, Amer. J. Math. **73** (1951) 433–474.
[12] I. N. Stewart, The minimal condition for subideals of Lie algebras implies that every ascendant subalgebra is a subideal, Hiroshima Math. J. **9** (1979) 35–36.

[13] S. Tôgô, The minimal condition for ascendant subalgebras of Lie algebras, Hiroshima Math. J. **7** (1977) 683–687.

[14] S. Tôgô, Maximal conditions for ideals in Lie algebras, Hiroshima Math. J. **9** (1979) 469–471.

Department of Mathematics,
Umm Al-Qura University,
Makkah,
PO Box 3711,
Saudi Arabia
and
Mathematics Institute,
University of Warwick,
*Coventry CV*4 7*AL,*
England