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On the exterior Dirichlet problem for semilinear elliptic equations with coefficients unbounded on the boundary

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Introduction

Let a be a fixed positive constant and let $\Omega_b \equiv \{x \in \boldsymbol{R}^N;\ a < |x| < b\}$, where $N \geqq 2$ and b is a positive constant with $a < b$. And we put $\Omega \equiv \Omega_\infty = \lim_{b \to \infty} \Omega_b$. Consider the problem:

$$(*)_b \qquad \Delta u = (|x| - a)^{-\lambda} G(x) u^\beta \quad \text{in} \quad \Omega_b\,, \qquad u = 0 \quad \text{on} \quad |x| = a\,,$$

where β is a real constant, λ is a positive constant and $G(x)$ is a locally Hölder continuous function satisfying some conditions stated below. Note that since $\lambda > 0$, the coefficient of u^β is unbounded on the boundary $\partial\Omega$. So, in general, it is not clear that the problem $(*)_b$ has a solution. When $b = \infty$, the problem $(*)_\infty = (*)$ with $\lambda = 0$ has been studied by many authors and various results on the existence and asymptotic behavior as $|x| \to \infty$ of positive solutions have been obtained. Among them we refer to [2, 3, 6–12, 14]. The first aim of this paper is to obtain global positive solutions of $(*)$ belonging to $C^2(\Omega) \cap C(\bar{\Omega})$ under the condition $\lambda < \beta + 1$. We note that the condition $\lambda < \beta + 1$ is necessary for the existence of solutions of $(*)$ when $G(x) = G(|x|)$. More exactly, we show the existence of infinitely many positive solutions of $(*)$ with some growth properties under $\lambda < \beta + 1$ and the integral conditions

$$\int_a^\infty r^{1-\lambda} (\log (r/a))^\beta G^*(r)\, dr < \infty \qquad (N = 2)\,,$$

$$\int_a^\infty r^{1-\lambda} G^*(r)\, dr < \infty \qquad (N \geqq 3)\,,$$

where $G^*(r) = \max_{|x|=r} |G(x)|$.

The second aim is to show that for any given b ($a < b \leqq \infty$) there exists a solution $u(x)$ of $(*)_b$ belonging to $C^2(\Omega_b)$ which blows up (when $b = \infty$, we say that it grows up.), that is $u(x) \to +\infty (|x| \to b)$, when $\beta > 1$ and $G(x) > 0$, $x \in \Omega_b$.

Our plan in this paper is as follows. In Section 1, we construct global positive solutions for the Cauchy problems of related ordinary differential equations. In Section 2, applying the results in Section 1, we obtain solutions of $(*)$ such that if $N = 2$, they have logarithmic order at ∞ and if $N \geqq 3$, they are bounded. In the final section, we discuss the existence of solutions of $(*)_b$ which blow up or grow up as $|x| \to b$ faster than the solutions as in Section 2.

1. Initial value problem for related ordinary differential equations

We consider the following Cauchy problem:

$$\begin{cases} (p(r)y')' = (r-a)^{-\lambda}q(r)y^{\beta}, & r > a, \\ y(a) = 0, \quad y'(a) = \eta > 0, \end{cases} \tag{1.1}$$

where $' \equiv d/dr$, β is a real constant, λ is a positive constant, $p(r) \in C^1[a, \infty)$, $p(r) > 0$, $r \in [a, \infty)$, and $q(r) \in C[a, \infty)$, $q(a) \neq 0$. We call $y(r)$ a solution of (1.1) if $y(r)$ belongs to $C^2(a, \infty) \cap C^1[a, \infty)$ and satisfies (1.1). Then we have:

THEOREM 1.1. *Let* $R(r) \equiv \int_a^r ds/p(s) \to \infty \ (r \to \infty)$ *and*

$$M \equiv \int_a^{\infty} (r-a)^{-\lambda}|q(r)|(R(r))^{\beta}\, dr < \infty . \tag{1.2}$$

(i) *If* $\beta > 1$, *then there exists* $\eta_0 > 0$ *such that for any* η $(0 < \eta < \eta_0)$, (1.1) *has a solution* $y(r)$ *with the condition*

$$c_1 \eta R(r) \leqq y(r) \leqq c_2 \eta R(r), \qquad r \geqq a \tag{1.3}$$

for some $c_1 > 0$, $c_2 > 0$ *which are independent of* η.

(ii) *If* $\beta < 1$, *then there exists* $\eta_0 > 0$ *such that for any* η $(\eta > \eta_0)$, *the same conclusion as in* (i) *holds.*

(iii) *If* $\beta = 1$ *and* $M < 1/2$, *then for any* $\eta > 0$ *the same conclusion as in* (i) *holds. Moreover, if* $q(r)$ *is of definite sign, we can replace* $M < 1/2$ *by* $M < 1$.

(iv) *For each solution* $y(r)$ *in* (i) $\sim$ (iii), $\lim_{r \to \infty} y(r)/R(r)$ *exists and is positive.*

PROOF OF (i). Let us put $\eta_0 \equiv (1/p(a))(2^{\beta+1}M)^{1/(1-\beta)}$, where M is the number in (1.2). Here we denote by C the set of all continuous functions on $[a, \infty)$ which is a Fréchet space equipped with the usual metric topology, and for η $(0 < \eta < \eta_0)$ we set $Y \equiv \{y \in C;\ (1/2)p(a)\eta R(r) \leqq y(r) \leqq 2p(a)\eta R(r),\ r \geqq a\}$. Then we can easily verify that Y is a closed convex subset in C. Define $\mathscr{F}: Y \to C$ by

$$(\mathscr{F}y)(r) \equiv p(a)\eta R(r) + \int_a^r (R(r) - R(t))(t-a)^{-\lambda}q(t)(y(t))^{\beta}\, dt, \qquad r \geqq a . \tag{1.4}$$

Then, we show for $0 < \eta < \eta_0$ that $(\mathrm{P})_1$ $\mathscr{F}: Y \to Y$, $(\mathrm{P})_2$ $\mathscr{F}$ is continuous, $(\mathrm{P})_3$ $\mathscr{F}Y$ is relatively compact in C. First of all, from (1.2) it follows that $\lambda < \beta + 1$. In fact, since $p(a) \neq 0$, we have $(r-a)/2p(a) \leqq R(r) \leqq 2(r-a)/p(a)$, $0 < r - a < \rho$, for some $\rho > 0$. Therefore, $M \geqq \int_a^{\rho'+a} (r-a)^{-\lambda}|q(r)|((r-a)/2p(a))^\beta\, dr \geqq \mathrm{Const} \int_a^{\rho'+a} (r-a)^{\beta-\lambda}\, dr$ for some $0 < \rho' \leqq \rho$ because $q(a) \neq 0$. This implies $\beta - \lambda > -1$, i.e., $\lambda < \beta + 1$.

Now we prove $(\mathrm{P})_1$. Since $0 < R(r) \leqq \mathrm{Const}(r-a)$ near $r = a$ and $\lambda < \beta + 1$, it is apparent that $\mathscr{F}y \in C$ if $y \in Y$. For simplicity we put $p(a)\eta \equiv \tilde{\eta}$, $p(a)\eta_0 \equiv \tilde{\eta}_0$. From (1.4), if $y \in Y$, $0 < \eta < \eta_0$, we have

$$
\begin{aligned}
(1.5)\qquad \mathscr{F}y(r) &\leqq \tilde{\eta}R(r) + \int_a^r (R(r) - R(t))(t-a)^{-\lambda}|q(t)|(2\tilde{\eta})^\beta (R(t))^\beta\, dt \\
&\leqq \left\{\tilde{\eta} + (2\tilde{\eta})^\beta \int_a^\infty (t-a)^{-\lambda}|q(t)|(R(t))^\beta\, dt\right\} R(r) \\
&= (\tilde{\eta} + 2^\beta M \tilde{\eta}^\beta)R(r) \leqq (3/2)\tilde{\eta}R(r)\,,
\end{aligned}
$$

using $\eta_0 = (1/p(a))(2^{\beta+1}M)^{1/(1-\beta)}$. On the other hand, we have

$$
\begin{aligned}
(1.6)\qquad \mathscr{F}y(r) &\geqq \tilde{\eta}R(r) - \int_a^r (R(r) - R(t))(t-a)^{-\lambda}|q(t)|(2\tilde{\eta})^\beta (R(t))^\beta\, dt \\
&\geqq (\tilde{\eta} - (2\tilde{\eta})^\beta M)R(r) \geqq (1/2)\tilde{\eta}R(r)\,.
\end{aligned}
$$

This shows that $\mathscr{F}y \in Y$.

Next, we prove $(\mathrm{P})_2$. Let $\{y_n\} \subset Y$ be a sequence converging to some $y \in Y$. Then for each fixed $r_0 (> a)$ and for any $\varepsilon > 0$, there exists n_0 such that $\sup_{a \leqq r \leqq r_0} |y_n(r) - y(r)| < \varepsilon$ for $n \geqq n_0$. We take δ as $0 < \delta < \beta - \lambda + 1$. Then we can verify that

$$
(1.7)\qquad |(y_n(t))^\beta - (y(t))^\beta| \leqq \beta(4\tilde{\eta}R(t))^{\beta-\delta}|y_n(t) - y(t)|^\delta\,, \qquad t > a\,.
$$

Hence we get for $n \geqq n_0$, $a < r \leqq r_0$,

$$
\begin{aligned}
&|(\mathscr{F}y_n)(r) - (\mathscr{F}y)(r)| \\
&= \left|\int_a^r (R(r) - R(t))(t-a)^{-\lambda} q(t)((y_n(t))^\beta - (y(t))^\beta)\, dt\right| \\
&\leqq R(r_0) \max_{a \leqq t \leqq r_0} |q(t)| \beta (4\tilde{\eta})^{\beta-\delta} \int_a^{r_0} (t-a)^{-\lambda}(R(t))^{\beta-\delta}|y_n(t) - y(t)|^\delta\, dt \\
&\leqq M_1(r_0)\varepsilon^\delta\,,
\end{aligned}
$$

where $M_1(r_0) \equiv R(r_0)\max_{a \leqq t \leqq r_0}|q(t)|\beta(4\tilde{\eta})^{\beta-\delta} \int_a^{r_0} (t-a)^{-\lambda}(R(t))^{\beta-\delta}\,dt < \infty$. This implies that $\mathscr{F}\colon Y \to Y$ is continuous.

We next prove $(\mathrm{P})_3$. Let $y \in Y$ and $r_0 > a$. Since $0 \leqq (\mathscr{F}y)(r) \leqq 2\tilde{\eta}R(r) \leqq 2\tilde{\eta}R(r_0)$, $a \leqq r \leqq r_0$, $y \in Y$, $\mathscr{F}Y$ is locally uniformly bounded. Furthermore, we have for $y \in Y$

$$(\mathscr{F}y)'(r) = \left\{\tilde{\eta} + \int_a^r (t-a)^{-\lambda}q(t)(y(t))^\beta\,dt\right\}\Big/p(r),$$

and so

$$|(\mathscr{F}y)'(r)| \leqq \left\{\tilde{\eta} + (2\tilde{\eta})^\beta \int_a^{r_0} (t-a)^{-\lambda}|q(t)|(R(t))^\beta\,dt\right\}\Big/\min_{a \leqq r \leqq r_0} p(r),$$

$$a \leqq r \leqq r_0.$$

Thus, $\mathscr{F}Y$ is locally equi-continuous. By the Ascoli-Arzelá theorem, we see that $\mathscr{F}Y$ is relatively compact in C.

Consequently, from $(\mathrm{P})_1 \sim (\mathrm{P})_3$ applying the Schauder-Tychonoff fixed point theorem, we can assert that there exists $y \in Y$ such that $\mathscr{F}y(r) = y(r)$, $r \geqq a$. This function $y(r)$ satisfies (1.1), (1.3) with $c_1 = p(a)/2$, $c_2 = 2p(a)$.

PROOF OF (ii). Putting $\eta_0 = (2^{\beta+1}M)^{1/(1-\beta)}/p(a)$ if $\beta > 0$ and $\eta_0 = 2M^{1/(1-\beta)}/p(a)$ if $\beta < 0$, setting for $\eta > \eta_0$, C, Y and $\mathscr{F}$ the same as in (i), we can prove (1.5), (1.6) and then $(\mathrm{P})_1$, $(\mathrm{P})_2$, $(\mathrm{P})_3$. The remaining part of the proof is the same as in (i).

PROOF OF (iii). Let C be as in (i). Now let us put

$$Y_0 \equiv \{y \in C;\, (1-2M)(1-M)^{-1}p(a)\eta R(r) \leqq y(r) \leqq (1-M)^{-1}p(a)\eta R(r),\, r \geqq a\}$$

in the general case of $q(r)$,

$$Y_- \equiv \{y \in C;\, (1-M)p(a)\eta R(r) \leqq y(r) \leqq p(a)\eta R(r),\, r \geqq a\}$$

if $q(r) \leqq 0$, $r \geqq a$, and

$$Y_+ \equiv \{y \in C;\, p(a)\eta R(r) \leqq y(r) \leqq (1-M)^{-1}p(a)\eta R(r),\, r \geqq a\}$$

if $q(r) \geqq 0$, $r \geqq a$.

We define $\mathscr{F}$ by (1.4) with $\beta = 1$. Then, for any $\eta > 0$ $\mathscr{F}$ satisfies $(\mathrm{P})_1 \sim (\mathrm{P})_3$ for each case $Y = Y_0$, Y_+, Y_-. We prove these facts in the case of $Y = Y_0$. Here we note that $M < 1/2$.

Putting $p(a)\eta = \tilde{\eta}$, we have for $y \in Y_0$

$$\mathscr{F}y(r) \leqq \tilde{\eta}R(r) + \int_a^r R(r)|q(t)|(t-a)^{-\lambda}(1-M)^{-1}\tilde{\eta}R(t)\,dt$$
$$\leqq \tilde{\eta}R(r)\{1 + M/(1-M)\} = (1-M)^{-1}\tilde{\eta}R(r), \qquad r \geqq a,$$
$$\mathscr{F}y(r) \geqq \tilde{\eta}R(r) - \int_a^r R(r)|q(t)|(t-a)^{-\lambda}(1-M)^{-1}\tilde{\eta}R(t)\,dt$$
$$\geqq \tilde{\eta}R(r)\{1 - M/(1-M)\} = (1-2M)(1-M)^{-1}\tilde{\eta}R(r), \qquad r \geqq a.$$

This shows that $\mathscr{F}y \in Y_0$. The proofs of $(P)_2$, $(P)_3$ are completely the same as in (i).

In the cases of $Y = Y_+$ and $Y = Y_-$ under the condition $M < 1$, we can examine necessary properties to apply the Schauder–Tychonoff theorem.

PROOF OF (iv). Finally we prove the existence of $\lim_{r\to\infty} y(r)/R(r)$ for each solution $y(r)$ obtained above. Since

$$y(r)/R(r) = p(a)\eta + \{1/R(r)\}\int_a^r (R(r) - R(t))q(t)(t-a)^{-\lambda}(y(t))^\beta\,dt,$$

$$\lim_{r\to\infty} y(r)/R(r) = p(a)\eta + \int_a^\infty q(t)(t-a)^{-\lambda}(y(t))^\beta\,dt$$

holds from L'Hospital's rule.

The right-hand side is finite from (1.2) and (1.3). The positivity of this limit follows from (1.3) in each case (i) $\sim$ (iii). Q.E.D.

REMARK 1.1. The condition $\lambda < \beta + 1$ is necessary for the existence of solution of (1.1). In fact, integrating over $[r_1, r_2]$ $(a < r_1 < r_2)$ both sides of the equation in (1.1), and letting $r_1 \to a$, we have

$$p(r_2)y'(r_2) - p(a)\eta = \lim_{r_1\to a}\int_{r_1}^{r_2} q(r)(r-a)^{-\lambda}(y(r))^\beta\,dr.$$

Therefore, by this and $q(a) \neq 0$, the integral $\int_a^{r_2} (y(r))^\beta (r-a)^{-\lambda}\,dr$ should exist. And since $y'(a) = \eta > 0$, we can assert that there exists $\rho > 0$ such that $\eta(r-a)/2 \leqq y(r) \leqq 3\eta(r-a)/2$, $a < r < a + \rho$. Thus,

$$\int_a^{a+\rho} (r-a)^{\beta-\lambda}\,dr \leqq K\int_a^{a+\rho} (y(r))^\beta (r-a)^{-\lambda}\,dr < \infty,$$

where $K \equiv (2/\eta)^\beta$ if $\beta \geqq 0$ and $K \equiv (2/3\eta)^\beta$ if $\beta < 0$. This shows that $\lambda < \beta + 1$. Q.E.D.

We apply Theorem 1.1 to the following Cauchy problem:

$$\begin{cases} y'' + \{(N-1)/r\}y' = G(r)(r-a)^{-\lambda}y^{\beta}, & r > a, \quad a > 0, \\ y(a) = 0, \quad y'(a) = \eta > 0, \end{cases} \tag{1.8}$$

where N is a positive integer with $N \geqq 2$. Then we get:

COROLLARY 1.1. *Suppose $N = 2$. Assume $G(r) \in C[a, \infty)$, $G(a) \neq 0$, and*

$$M \equiv \int_a^{\infty} r|G(r)|(r-a)^{-\lambda}(\log(r/a))^{\beta}\, dr < \infty . \tag{1.9}$$

Then (i) ~ (iv) *in Theorem* 1.1 *hold. Moreover the solution $y(r)$ satisfy*

$$c_1\eta \log(r/a) \leqq y(r) \leqq c_2\eta \log(r/a), \quad r \geqq a \tag{1.10}$$

for some constants $c_1 > 0$, $c_2 > 0$, and

$$\lim_{r\to\infty} y(r)/\log r \text{ exists and is positive} . \tag{1.11}$$

COROLLARY 1.2. *Suppose $N \geqq 3$. Assume $G(r) \in C[a, \infty)$ $G(a) \neq 0$, and*

$$M \equiv \{1/(N-2)\} \int_a^{\infty} r(1-(a/r)^{N-2})^{\beta}|G(r)|(r-a)^{-\lambda}\, dr < \infty . \tag{1.12}$$

Then (i) ~ (iv) *in Theorem* 1.1 *hold. Moreover the solution $y(r)$ of* (1.8) *satisfy*

$$c_1\eta\{1-(a/r)^{N-2}\} \leqq y(r) \leqq c_2\eta\{1-(a/r)^{N-2}\}, \quad r \geqq a \tag{1.13}$$

for some constants $c_1 > 0$, $c_2 > 0$ and

$$\lim_{r\to\infty} y(r) \text{ exists and is positive} . \tag{1.14}$$

PROOF OF COROLLARIES 1.1 AND 1.2. By the change of variable $z = r^{N-2}y$, problem (1.8) reduces to

$$\begin{cases} (r^{3-N}z')' = r^{1-(N-2)\beta}G(r)(r-a)^{-\lambda}z^{\beta}, & r > a, \\ z(a) = 0, \quad z'(a) = a^{N-2}\eta . \end{cases} \tag{1.15}$$

Thus, putting $p(r) = r^{3-N}$ and

$$R(r) = \log(r/a) \quad (N = 2), \quad = (1/(N-2))(r^{N-2} - a^{N-2}) \quad (N \geqq 3)$$

and applying Theorem 1.1, we have the assertions. Q.E.D.

REMARK 1.2. The condition (1.12) in the cases (i), (ii) in Corollary 1.2 can be replaced by $\lambda < \beta + 1$ and $\int_a^{\infty} r^{1-\lambda}|G(r)|\, dr < \infty$. In fact, from

$$(1-(a/r)^{N-2})^{\beta}(r-a)^{-\lambda} = O((r-a)^{\beta-\lambda}) \quad (\text{as } r \to a)$$
$$= O(r^{-\lambda}) \qquad (\text{as } r \to \infty),$$

we see the condition (1.12) directly.

2. Dirichlet problem for semilinear elliptic equations in the exterior domain

Let $a > 0$ and put $\Omega = \{x \in \boldsymbol{R}^N; |x| > a\}$, where $N \geqq 2$. In this section we consider the problem:

$$\Delta u = G(x)(|x| - a)^{-\lambda} u^{\beta} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{2.1}$$

where β is a real constant and λ is a positive constant. We call $u(x)$ a positive solution of (2.1) if $u(x)$ belongs to $C^2(\Omega) \cap C(\bar{\Omega})$, satisfies (2.1) and $u(x) > 0$, $x \in \Omega$. In what follows we use the notation $G^*(r) \equiv \max_{|x|=r} |G(x)|$. Now we have:

THEOREM 2.1. *Let $\beta \neq 1$ and $\lambda < \beta + 1$. Assume that $G(x) \in C^{\theta}_{\mathrm{loc}}(\bar{\Omega})$ $(0 < \theta < 1)$, $G(x) \not\equiv 0$, $x \in \partial\Omega$.*

(i) *If $N = 2$ and*

$$\int_a^{\infty} r^{1-\lambda}(\log(r/a))^{\beta} G^*(r)\, dr < \infty, \tag{2.2}$$

then there exist infinitely many positive solutions of (2.1) *with the condition*

$$c_1 \log(|x|/a) \leqq u(x) \leqq c_2 \log(|x|/a), \qquad x \in \Omega, \tag{2.3}$$

for some $c_1, c_2 > 0$.

(ii) *If $N \geqq 3$ and*

$$\int_a^{\infty} r^{1-\lambda} G^*(r)\, dr < \infty, \tag{2.4}$$

then there exist infinitely many positive solutions of (2.1) *with the condition*

$$c_1\{1 - (a/|x|)^{N-2}\} \leqq u(x) \leqq c_2\{1 - (a/|x|)^{N-2}\}, \qquad x \in \Omega \tag{2.5}$$

for some $c_1, c_2 > 0$.

THEOREM 2.2. *Let $\beta = 1$, $\lambda < 2$ and G satisfy the same assumption as in Theorem* 2.1.

(i) *If $N = 2$ and*

$$\int_a^{\infty} r(\log(r/a))(r-a)^{-\lambda} G^*(r)\, dr < 1, \tag{2.6}$$

then the same conclusion as (i) *in Theorem* 2.1 *holds.*

(ii) *If* $N \geqq 3$ *and*

$$\int_a^\infty r(1-(a/r)^{N-2})(r-a)^{-\lambda}G^*(r)\,dr < N-2\,, \tag{2.7}$$

then the same conclusion as (ii) *in Theorem* 2.1 *holds.*

Since the proofs of Theorems 2.1 and 2.2 are essentially the same, we only prove Theorem 2.1.

PROOF OF THEOREM 2.1. It suffices to prove in the case $\beta > 1$.

(i) By Corollary 1.1, there exist $\eta_0 > 0$, $\zeta_0 > 0$ such that for any η $(0 < \eta < \eta_0)$, ζ $(0 < \zeta < \zeta_0)$ the problems

$$y'' + (1/r)y' + G^*(r)(r-a)^{-\lambda}y^\beta = 0\,, \qquad r > a\,; \quad y(a) = 0\,, \quad y'(a) = \eta\,, \tag{2.8}$$

$$z'' + (1/r)z' - G^*(r)(r-a)^{-\lambda}z^\beta = 0\,, \qquad r > a\,; \quad z(a) = 0\,, \quad z'(a) = \zeta \tag{2.9}$$

have solutions $y(r;\eta)$, $z(r;\zeta)$, respectively, with the conditions

$$A_1\eta \log(r/a) \leqq y(r;\eta) \leqq A_2\eta\log(r/a)\,, \qquad r > a\,, \tag{2.10}$$

$$B_1\zeta\log(r/a) \leqq z(r;\zeta) \leqq B_2\zeta\log(r/a)\,, \qquad r > a \tag{2.11}$$

for some $A_i > 0$, $B_i > 0$, $i = 1, 2$, with $A_1 \leqq B_2$.

Let $\hat{\eta} \equiv \min\{\eta_0, \zeta_0\}$. Then for any η $(0 < \eta < \hat{\eta})$, putting $\zeta = A_1 B_2^{-1}\eta$, we see that $0 < \zeta < \hat{\eta}$. Now for $y(r;\eta)$, $z(r;\zeta)$ the solutions of (2.8), (2.9) in this case, we get

$$\begin{aligned}(A_1B_1/B_2)\eta\log(r/a) &\leqq z(r;\zeta) \leqq A_1\eta\log(r/a) \\ &\leqq y(r;\eta) \leqq A_2\eta\log(r/a)\,, \qquad r > a\,.\end{aligned} \tag{2.12}$$

We put $v(x) \equiv y(|x|;\eta)$, $w(x) \equiv z(|x|;\zeta)$, $x \in \Omega$. Then $v, w \in C^2(\Omega) \cap C^1(\bar{\Omega})$, and we have

$$(A_1B_1/B_2)\eta\log(|x|/a) \leqq w(x) \leqq v(x) \leqq A_2\eta\log(|x|/a)\,, \qquad x \in \bar{\Omega}\,, \tag{2.12'}$$

and

$$\begin{cases}\Delta v(x) + G^*(|x|)(|x|-a)^{-\lambda}(v(x))^\beta = 0\,, & x \in \Omega\,; \quad v(\xi) = 0\,, \quad \xi \in \partial\Omega\,, \\ \Delta w(x) - G^*(|x|)(|x|-a)^{-\lambda}(w(x))^\beta = 0\,, & x \in \Omega\,; \quad w(\xi) = 0\,, \quad \xi \in \partial\Omega\,.\end{cases} \tag{2.13}$$

Therefore,

$$\Delta v(x) \leqq G(x)(|x|-a)^{-\lambda}(v(x))^\beta\,, \quad x \in \Omega\,; \quad v(\xi) = 0\,, \quad \xi \in \partial\Omega\,, \tag{2.14}$$

$$\Delta w(x) \geqq G(x)(|x|-a)^{-\lambda}(w(x))^\beta\,, \quad x \in \Omega\,; \quad w(\xi) = 0\,, \quad \xi \in \partial\Omega\,. \tag{2.15}$$

This shows that $v(x)$ is a supersolution and $w(x)$ is a subsolution of (2.1). Now for $n = 1, 2, \dots$, if we put

$$\Omega_n \equiv \{x \in \boldsymbol{R}^2; a + 1/(n+1) < |x| < a + n\},$$

then $\Omega_n \to \Omega$ as $n \to \infty$. From Theorem 1 in H. Amann [1], there exists a maximal solution $u_n^* \in C^{2+\theta}(\overline{\Omega_n})(0 < \theta < 1)$ of a problem

$$(2.16)_n \qquad \Delta U = G(x)(|x| - a)^{-\lambda} U^\beta, \quad x \in \Omega_n; \quad U(\xi) = v(\xi), \quad \xi \in \partial\Omega_n$$

such that

$$(2.17) \qquad w(x) \leqq u_n(x) \leqq u_n^*(x) \leqq v(x), \quad x \in \Omega_n$$

for every solution u_n of $(2.16)_n$ with the condition

$$w(x) \leqq u_n(x) \leqq v(x), \quad x \in \Omega_n.$$

Now we define $\tilde{u}_n(x) = u_n^*(x)$ for $x \in \Omega_n$, $\tilde{u}_n(x) = v(x)$ for $x \in \bar{\Omega} \backslash \Omega_n$. Then since $\Omega_n \subset \Omega_{n+1}$, we have

$$\Delta \tilde{u}_{n+1}(x) = G(x)(|x| - a)^{-\lambda}(\tilde{u}_{n+1}(x))^\beta, \quad x \in \Omega_n; \quad \tilde{u}_{n+1}(\xi) \leqq v(\xi), \quad \xi \in \partial\Omega_n.$$

This shows that $\tilde{u}_{n+1}(x)$ is a subsolution of $(2.16)_n$ with $w(x) \leqq \tilde{u}_{n+1}(x) \leqq v(x)$, $x \in \Omega_n$. Thus, again from the result of [1; Theorem 1] there exists a solution $\hat{u}_n(x)$ of $(2.16)_n$ such that

$$w(x) \leqq \tilde{u}_{n+1}(x) \leqq \hat{u}_n(x) \leqq v(x), \quad x \in \Omega_n.$$

From this and the maximality of u_n^* in (2.17) we can assert that $\tilde{u}_{n+1}(x) \leqq \tilde{u}_n(x)$, $x \in \Omega_n$. Thus, we get

$$(2.18) \qquad w(x) \leqq \tilde{u}_{n+1}(x) \leqq \tilde{u}_n(x) \leqq v(x), \quad x \in \Omega.$$

This shows that $\{\tilde{u}_n(x)\}_{n=1}^\infty$ is monotone decreasing and locally uniformly boundeded in Ω. Then by the standard theory of Schauder estimates and L^p-estimates (cf. [11]) to such a sequence of solutions, we can conclude that $u(x) \equiv \lim_{n\to\infty} \tilde{u}_n(x)$ exists in $\bar{\Omega}$ and by choosing a suitable subsequence $\{\tilde{u}_{n_k}\} \subset \{\tilde{u}_n\}$ we get

$$(2.19) \qquad \lim_{n_k \to \infty} \|\tilde{u}_{n_k} - u\|_{C^2(D)} = 0$$

for any bounded domain $D \subset\subset \Omega$. Therefore $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$. From $(2.16)_n$, (2.18) and (2.19) we have

$$\Delta u = G(x)(|x| - a)^{-\lambda} u^\beta, \quad w(x) \leqq u(x) \leqq v(x), \quad x \in \Omega.$$

From the definition of v, w this function u is a solution of (2.1) with the condition (2.3). Now the existence of infinitely many solutions of (2.1) is seen as follows. For the above $\eta > 0$, we put $0 < \eta_1 = (1/2)(A_1B_1/A_2B_2)\eta$. Then replacing η by η_1 in the above arguments, we have a solution $u_1(x)$ of (2.1) such that $u_1(x) \leqq (1/2)u(x)$, $x \in \Omega$. Continuing this procedure and argument, we have a sequence $\{u_n\}$ of solutions of (2.1) with the property $u_{n+1}(x) \leqq (1/2)u_n(x)$, $x \in \Omega$, $n = 1, 2, \dots$. The assertion will follow apparently.

(ii) The proof is the same as in the case (i). Q.E.D.

3. Blow-up and grow-up of positive solutions

In this section we consider the problem $(*)_b$ when $G(x) > 0$ $(x \in \Omega_b)$ and $\beta > 1$. We first prepare some lemmas.

LEMMA 3.1. *Suppose that $\lambda < \beta + 1$, $\beta > 1$ and $q(r) > 0$ $(r \geqq a)$ in* (1.1). *Then for any $\eta > 0$ there exists a unique solution of* (1.1) *in some interval $[a, a + \delta_\eta]$.*

PROOF. For the existence we note that since $\lambda < \beta + 1$, we can take $\delta_\eta > 0$ such that

$$\int_a^{a+\delta_\eta} q(t)(t-a)^{-\lambda}(R(t))^\beta\, dt \leqq 2^{-\beta}(p(a)\eta)^{1-\beta}$$

for any $\eta > 0$, where $R(r) = \int_0^r ds/p(s)$ $(r \geqq a)$. Let us put

$$Y \equiv \{y \in C[a, a+\delta_\eta];\ p(a)\eta R(r) \leqq y(r) \leqq 2p(a)\eta R(r),\ a \leqq r \leqq a+\delta_\eta\},$$

$$(\mathscr{F}y)(r) \equiv p(a)\eta R(r) + \int_a^r (R(r) - R(t))q(t)(t-a)^{-\lambda}(y(t))^\beta\, dt, \quad a \leqq r \leqq a+\delta_\eta.$$

Then from Schauder's fixed point theorem, $\mathscr{F}$ has a fixed point $y(r)$ in Y. This function $y(r)$ is a solution of (1.1) in $[a, a + \delta_\eta]$.

To prove the uniqueness, let $y_i(r)$ $(i = 1, 2)$ be solutions of (1.1) in $[a, a+\delta_\eta]$. Then there exists $M_1 > 0$ such that $|y_i(r)| \leqq M_1 R(r)$, $a \leqq r \leqq a+\delta_\eta$ $(i = 1, 2)$. Therefore we get

$$|(y_1(r))^\beta - (y_2(r))^\beta| \leqq \beta M_1^{\beta-1}(R(r))^{\beta-1}|y_1(r) - y_2(r)|, \quad a \leqq r \leqq a+\delta_\eta,$$

hence,

$$|y_1(r) - y_2(r)| \leqq \beta M_1^{\beta-1}R(r)\int_a^r q(t)(t-a)^{-\lambda}(R(t))^{\beta-1}|y_1(t) - y_2(t)|\, dt,$$

$$a \leqq r \leqq a+\delta_\eta.$$

Now, if we put $\Psi(r) \equiv |y_1(r) - y_2(r)|/R(r)$, $a < r \leqq a+\delta_\eta$, $\Psi(a) = 0$, then we see that $\Psi(r) \in C[a, a+\delta_\eta]$ and

$$(3.1) \qquad \Psi(r) \leqq \beta M_1^{\beta-1} \int_a^r q(t)(t-a)^{-\lambda}(R(t))^\beta \Psi(t)\, dt\,, \quad a \leqq r \leqq a+\delta_\eta\,.$$

Here we choose δ ($0 < \delta \leqq \delta_\eta$) such that

$$\int_a^{a+\delta} q(t)(t-a)^{-\lambda}(R(t))^\beta\, dt < (2\beta M_1^{\beta-1})^{-1}\,.$$

Thus, by (3.1) we can assert that $0 \leqq \Psi(r) \leqq (1/2)\max_{a \leqq t \leqq a+\delta} \Psi(t)$, ($a \leqq r \leqq a+\delta$). This shows that $\Psi(r) = 0$, i.e., $y_1(r) = y_2(r)$ for $a \leqq r \leqq a+\delta$. Q.E.D.

In what follows we denote by $y(r;\eta)$ the solution of (1.1) obtained in Lemma 3.1 and by $[a, T_\eta)$ the right maximal interval of existence for $y(r;\eta)$.

LEMMA 3.2. *Let $\beta > 1$, $\phi \in C[0,\infty)$ and $0 < t_1 < t_2 < \infty$. Suppose that $\phi(t) > \varepsilon$ in $[t_1, t_2+\varepsilon]$ for some $\varepsilon > 0$. Then, there exists a positive constant $\tilde{M} > 0$ such that the solution $z(t)$ of the equation $\ddot{z} = \phi(t)z^\beta$ satisfying $z(\tilde{t}) > \tilde{M}$ and $\dot{z}(\tilde{t}) > \tilde{M}$ for some $\tilde{t} \in [t_1, t_2]$ has the finite right maximal interval $[\tilde{t}, \tilde{t}_+)$ of existence, where a dot denotes differentiation with respect to t. Moreover, $\tilde{t}_+ - \tilde{t} \leqq \varepsilon/2$ and $\lim_{t\to\tilde{t}_+-0} z(t) = \infty$ hold.*

This lemma is essentially the same as [13; Lemma 3.1]. So the proof is omitted. Here, we note that the constant $\tilde{M} > 0$ does not depend on $\tilde{t}$ in $[t_1, t_2]$.

LEMMA 3.3. *Let λ, β and $q(r)$ be as in Lemma 3.1, and $\tilde{\eta} > 0$ be fixed. Then,*

(i) *$y(r;\eta) \to y(r;\tilde{\eta})$, $y'(r;\eta) \to y'(r;\tilde{\eta})$ uniformly in any bounded closed interval $I \subset [a, T_{\tilde{\eta}})$ as $\eta \to \tilde{\eta}$,*

(ii) $\lim_{\eta\to\tilde{\eta}} T_\eta = T_{\tilde{\eta}}$.

PROOF. *Step 1.* We prove the following assertion:

Assume that $a < \hat{r} < T_{\hat{\eta}}$ for some $\hat{\eta} > \tilde{\eta}$. Then, $y(r;\eta)$ exists in $[a, \hat{r}]$ for each η in $(0, \hat{\eta}]$. Furthermore, $y(r;\eta)$ and $y'(r;\eta)$ converge to $y(r;\tilde{\eta})$ and $y'(r;\tilde{\eta})$ uniformly in $[a, \hat{r}]$ as $\eta \to \tilde{\eta}$, respectively.

We first note that by [13; Lemma 0.2]

(3.2) $\quad y(r;\eta') \geqq y(r;\eta'')$ if they exist in some interval $[a, r_1]$ and $\eta' > \eta''$.

This implies that

$$(3.3) \qquad T_{\eta'} \leqq T_{\eta''} \qquad \text{for} \qquad \eta' > \eta'' > 0\,.$$

Hence we have $T_\eta \geqq T_{\hat{\eta}} > \hat{r}$ for every $\eta \in (0, \hat{\eta}]$. Furthermore, by (3.2) we can choose a constant $K_1 > 0$, which is independent of η in $(0, \hat{\eta}]$, such that

$0 \leqq y(r;\eta) \leqq y(r;\hat{\eta}) \leqq K_1 R(r)$, $a \leqq r \leqq \hat{r}$. Now define $\Phi(r)$ by $\Phi(r) = |y(r;\eta) - y(r;\tilde{\eta})|/R(r)$ for $a < r < \hat{r}$ and $\Phi(a) = p(a)|\eta - \tilde{\eta}|$. Then, we obtain analogously to (3.1)

$$\Phi(r) \leqq \Phi(a) + \beta K_1^{\beta-1} \int_a^r q(t)(t-a)^{-\lambda}(R(t))^{\beta}\Phi(t)\,dt\,, \quad a \leqq r \leqq \hat{r}\,.$$

Hence, by the Gronwall inequality,

$$\Phi(r) \leqq p(a)|\eta - \tilde{\eta}|\exp\left(\beta K_1^{\beta-1} \int_a^r q(t)(t-a)^{-\lambda}(R(t))^{\beta}\,dt\right), \quad a \leqq r \leqq \hat{r}\,.$$

This shows that

$$|y(r;\eta) - y(r;\tilde{\eta})| \leqq p(a)|\eta - \tilde{\eta}|\exp\left(\beta K_1^{\beta-1} \int_a^{\hat{r}} q(t)(t-a)^{-\lambda}(R(r))^{\beta}\,dt\right)R(\hat{r})\,,$$
$$a \leqq r \leqq \hat{r}\,,$$

and so

$$\lim_{\eta\to\tilde{\eta}} y(r;\eta) = y(r;\tilde{\eta}) \qquad \text{uniformly in } [a, \hat{r}]\,. \tag{3.4}$$

Next, integrating (1.1) over $[a, r]$ and choosing ν such that $0 < \nu < \beta + 1$, we get

$$\begin{aligned}
&|y'(r;\eta) - y'(r;\tilde{\eta})| \\
&\leqq \left(p(a)|\eta - \tilde{\eta}| + \int_a^r q(t)(t-a)^{-\lambda}|(y(t;\eta))^{\beta} - (y(t;\tilde{\eta}))^{\beta}|\,dt\right)\Big/p(r) \\
&\leqq \left(p(a)|\eta - \tilde{\eta}| + 4^{\beta-\nu}\beta K_1^{\beta-1} \int_a^{\hat{r}} q(t)(t-a)^{-\lambda}(R(t))^{\beta-\nu}\,dt\right. \\
&\qquad \left.\times \max_{a\leqq r\leqq\hat{r}} |y(r;\eta) - y(r;\tilde{\eta})|^{\nu}\right)\Big/p(r)\,, \quad a \leqq r \leqq \hat{r}\,.
\end{aligned}$$

From this and (3.4), we conclude that $\lim_{\eta\to\tilde{\eta}} y'(r;\eta) = y'(r;\tilde{\eta})$ uniformly in $[a, \hat{r}]$.

Step 2. We show that T_η is right continuous at $\eta = \tilde{\eta}$.

Since $\lim_{\eta\to\tilde{\eta}+0} T_\eta \leqq T_{\tilde{\eta}}$ by (3.3), it is enough to show that $\lim_{\eta\to\tilde{\eta}+0} T_\eta \geqq T_{\tilde{\eta}}$. Let now $\{\eta(k)\}$ be an arbitrary sequence such that $\eta(k) > \tilde{\eta}$ and $\lim_{k\to\infty} \eta(k) = \tilde{\eta}$. Take $\hat{\eta}$ such that $\hat{\eta} > \sup_{k\geqq 1} \eta(k)$, and put $\hat{r} = (a + T_{\hat{\eta}})/2$. Then $a < \hat{r} < T_{\eta(k)}$ for $k \geqq 1$. Applying the assertion in Step 1, we have

$$\lim_{k\to\infty} y(\hat{r};\eta(k)) = y(\hat{r};\tilde{\eta}) \qquad \text{and} \qquad \lim_{k\to\infty} y'(\hat{r};\eta(k)) = y'(\hat{r};\tilde{\eta})\,.$$

Hence, by [5; p. 14, Theorem 3.2], there is a function $y(r)$ and a subsequence $\{\eta'(k)\}$ of $\{\eta(k)\}$ such that

$$(3.5)\qquad \begin{cases} (p(r)y'(r))' = q(r)(r-a)^{-\lambda}(y(r))^{\beta}, & \hat{r} < r < r_+, \\ y(\hat{r}) = y(\hat{r}; \tilde{\eta}), \quad y'(\hat{r}) = y'(\hat{r}; \tilde{\eta}), \end{cases}$$

$$(3.6)\qquad r_+ \leqq \liminf_{k\to\infty} T_{\eta'(k)},$$

where $[\hat{r}, r_+)$ is the right maximal interval of existence for $y(r)$. On the other hand, since $y(r; \tilde{\eta})$ is a solution of (3.5) in $[\hat{r}, T_{\tilde{\eta}})$, by the uniqueness of the solution of (3.5) we have $y(r) = y(r; \tilde{\eta})$ and $r_+ = T_{\tilde{\eta}}$. Therefore, by (3.3) and (3.6), we have $\lim_{\eta\to\tilde{\eta}+0} T_\eta = \liminf_{k\to\infty} T_{\eta'(k)} \geqq T_{\tilde{\eta}}$. Thus we get $\lim_{\eta\to\tilde{\eta}+0} T_\eta = T_{\tilde{\eta}}$.

Step 3. We now prove the assertion (i). Let I be a bounded closed interval in $[a, T_{\tilde{\eta}})$. Then, since by (3.3) and Step 2 we can choose $\eta_0 > \tilde{\eta}$ such that $I \subset [a, T_\eta)$ for $0 < \eta < \eta_0$, the assertion follows from Step 1.

Step 4. Finally, we prove the continuity of T_η at $\eta = \tilde{\eta}$. By Step 2, it is enough to show that $\lim_{\eta\to\tilde{\eta}-0} T_\eta = T_{\tilde{\eta}}$. Moreover, we may assume that $0 < T_{\tilde{\eta}} < \infty$. In fact, if $T_{\tilde{\eta}} = +\infty$, then by (3.3) $T_\eta = +\infty$ for every $\eta \in (0, \tilde{\eta})$. By the change of variable $t = \int_a^r ds/p(s)$, $a \leqq r < \infty$, the function $z(t; \eta) \equiv y(r(t); \eta)$ satisfies

$$(3.7)\qquad \begin{cases} \ddot{z}(t; \eta) = \phi(t)(z(t; \eta))^{\beta}, & 0 < t < \hat{T}_\eta, \\ z(0; \eta) = 0, \quad \dot{z}(0; \eta) = p(a)\eta, \end{cases}$$

where $\phi(t) = p(r(t))q(r(t))/(r(t)-a)^{\lambda}$, $r(t)$ is the inverse of $t = t(r)$ and $\hat{T}_\eta = \int_a^{T_\eta} ds/p(s)$. Furthermore, the right maximal intervals of existence for $y(r; \eta)$ and $z(t; \eta)$ are $[a, T_\eta)$ and $[0, \hat{T}_\eta)$, respectively. Therefore, it is enough to show that the left continuity of $\hat{T}_\eta$ at $\eta = \tilde{\eta}$. For any $\varepsilon \in (0, \hat{T}_{\tilde{\eta}})$, choose $\varepsilon_1 \in (0, \varepsilon)$ such that $\phi(t) > \varepsilon_1$ for $t \in [\hat{T}_{\tilde{\eta}} - \varepsilon_1, \hat{T}_{\tilde{\eta}} + \varepsilon_1]$. Then, by Lemma 3.2, we can find $K_2 > 0$ such that for any function $z(t)$ satisfying the equation in (3.7) $\lim_{t\to t_2-0} z(t) = \infty$ holds for some $t_2 \in [t_1, t_1 + \varepsilon_1/2]$, provided $z(t_1) \geq K_2$ and $\dot{z}(t_1) \geqq K_2$ for some t_1 in $[\hat{T}_{\tilde{\eta}} - \varepsilon_1, \hat{T}_{\tilde{\eta}}]$. On the other hand, since $z(t; \tilde{\eta}) \to \infty$ and $\dot{z}(t; \tilde{\eta}) \to \infty$ as $t \to \hat{T}_{\tilde{\eta}} - 0$, there is $t_1^* \in [\hat{T}_{\tilde{\eta}} - \varepsilon_1, \hat{T}_{\tilde{\eta}})$ such that $z(t_1^*; \tilde{\eta}) > K_2$ and $\dot{z}(t_1^*; \tilde{\eta}) > K_2$. By the assertion (i), choosing $\delta > 0$ small enough, we have $z(t_1^*; \eta) > K_2$ and $\dot{z}(t_1^*; \eta) > K_2$ for every η in $(\tilde{\eta} - \delta, \tilde{\eta})$. Hence, $\lim_{t\to t_2-0} z(t; \eta) = \infty$ for some $t_2 < t_1^* + \varepsilon_1/2$, and so we see that $\hat{T}_\eta = t_2 \leqq t_1^* + \varepsilon_1/2 < \hat{T}_{\tilde{\eta}} + \varepsilon_1/2 < \hat{T}_{\tilde{\eta}} + \varepsilon$ for $\tilde{\eta} - \delta < \eta < \tilde{\eta}$. Thus we obtain $\lim_{\eta\to\tilde{\eta}-0} \hat{T}_\eta = \hat{T}_{\tilde{\eta}}$.

Q.E.D.

LEMMA 3.4. *Let* λ, β *and* $q(r)$ *be as in Lemma* 3.1. *Then*,

(i) *there exists* $\eta_\infty > 0$ *such that if* $\eta > \eta_\infty$, T_η *is finite*,

(ii) $\lim_{\eta\to\eta^*+0} T_\eta = +\infty$ *holds, where* $\eta^* \equiv \inf\{\eta > 0;\ T_\eta \text{ is finite}\}$,

(iii) $\lim_{\eta\to+\infty} T_\eta = a$.

PROOF. We put $A \equiv \{\eta > 0;\ T_\eta \text{ is finite}\}$. As we have shown in the proof of Lemma 3.3, it suffices to prove the assertions for the solution $z(t;\eta)$ of (3.7), that is, to prove that (i) $A \neq \varnothing$, (ii) $\hat{T}_{\eta^*} = +\infty$, and (iii) $\lim_{\eta\to\infty} \hat{T}_\eta = 0$.

(i) We fix $t_0 > 0$. If there exists $\eta > 0$ with $\lim_{t\to t_0-0} z(t;\eta) = \infty$, then $\hat{T}_\eta \leqq t_0$ and this implies $A \neq \varnothing$. Therefore we assume here that for any $\eta > 0$, $\hat{T}_\eta > t_0$ holds. Then, from Lemma 3.2, there exists $\tilde{M} > 0$ such that any solution $z(t)$ of the problem: $\ddot{z}(t) = \phi(t)(z(t))^\beta$, $z(t_0) \geqq \tilde{M}$, $\dot{z}(t_0) \geqq \tilde{M}$, has a finite right maximal interval of existence, where $\phi(t)$ is the same as in (3.7). Put $\eta_\infty = \max\{\tilde{M}/p(a), \tilde{M}/(p(a)t_0)\}$. Then, from (3.7) we have $\dot{z}(t_0;\eta) \geqq p(a)\eta \geqq \tilde{M}$, $z(t_0;\eta) \geqq p(a)\eta t_0 \geqq \tilde{M}$ for $\eta > \eta_\infty$. Thus, when $\eta > \eta_\infty$, the value $\hat{T}_\eta$ for $z(t;\eta)$ is finite, i.e., when $\eta > \eta_\infty$, $\eta \in A$ holds.

(ii) By (ii) of Lemma 3.3, $\hat{T}_\eta$ is continuous in η and so A is open in $[0, \infty)$. Hence, we have $\eta^* \notin A$ and $\hat{T}_{\eta^*} = +\infty$.

(iii) We fix $t_1 > 0$ arbitrarily and put $t_1' = t_1/2$. We choose $\varepsilon > 0$ such that $0 < \varepsilon < t_1'$, $\phi(t) > \varepsilon$ for $t \in [t_1', t_1]$. Then for the solution $z(t;\eta)$ of (3.7), the values $z(t_1';\eta)$, $\dot{z}(t_1';\eta)$ can be made as large as possible by choosing $\eta > 0$ sufficiently large. Therefore, from Lemma 3.2, we see that for $\eta > 0$ large enough, $\hat{T}_\eta < t_1$ should hold. From the arbitrariness of $t_1 > 0$ we conclude that $\hat{T}_\eta \to 0$ $(\eta \to \infty)$. Q.E.D.

Now, we are ready to state our theorem.

THEOREM 3.1. *Let* $\lambda < \beta + 1$, $\beta > 1$ *and* $q(r) > 0$, $a \leqq r < \infty$.

(i) *For any* $b > a$, *there exists* $\eta > 0$ *such that*

$$T_\eta = b \qquad \text{and} \qquad \lim_{r\to b-0} y(r;\eta) = \infty\,.$$

(ii) *If* $R(r) = \int_a^r ds/p(s) \to \infty$ *as* $r \to \infty$ *and*

$$\int_a^\infty q(r)(r-a)^{-\lambda}(R(r))^\beta\, dr < \infty\,, \tag{3.8}$$

then there exists $\eta_* > 0$ *such that, for any* η $(0 < \eta < \eta_*)$, $T_\eta = \infty$ *and* $y(r;\eta)/R(r)$ *converges to a positive constant as* $r \to \infty$.

(iii) *Under the same assumptions as in* (ii), *for some* $\eta > 0$, $T_\eta = \infty$ *and* $y(r;\eta)/R(r)$ *tends to* ∞ *as* $r \to \infty$.

PROOF. The statement (i) follows from (ii) of Lemma 3.3 and Lemma 3.4; (ii) is a consequence of (i) and (iv) of Theorem 1.1. For the proof of (iii), put

$$A \equiv \{\eta > 0;\ T_\eta \text{ is finite}\}\,,$$

$$B \equiv \{\eta > 0;\ T_\eta = \infty \text{ and } \lim_{r\to\infty} y(r;\eta)/R(r) \text{ exists and is positive}\}\,.$$

Then, by Lemma 3.4 and (3.3), we have $A = (\eta^*, \infty)$ for some $\eta^* > 0$. We now prove that B is open in $(0, \infty)$. Let $\eta \in B$ and denote $l_\eta \equiv \lim_{r\to\infty} y(r;\eta)/R(r)$. Then by the monotonicity of $y(r;\eta)$ in $\eta > 0$, we have $\eta' \in B$ for $\eta' < \eta$. Since $(p(r)y'(r;\eta))' > 0$ for $r > a$ the limit of $p(r)y'(r;\eta)$ as $r \to \infty$ exists and is equal to l_η by L'Hospital's rule, and hence

$$0 < p(r)y'(r;\eta) < l_\eta\,, \qquad a < r < \infty\,. \tag{3.9}$$

Take $a_1(>a)$ such that

$$\int_{a_1}^{\infty} q(r)(r-a)^{-\lambda}(R(r))^{\beta}\,dr < (1/2)(3l_n/2)^{-\beta} l_n\,. \tag{3.10}$$

Then, we can choose $\delta_1 > 0$ satisfying

$$p(r)y'(r;\eta') < l_\eta\,, \qquad a \leqq r \leqq a_1$$

for $\eta' \in (\eta, \eta + \delta_1)$ by (i) of Lemma 3.3 and (3.9). Furthermore, we have

$$p(r)y'(r;\eta') < (3/2)l_\eta\,, \quad r > a_1\,, \quad \eta' \in (\eta, \eta+\delta_1)\,. \tag{3.11}$$

In fact, assume that there is an $a_2(>a_1)$ with the property

$$p(r)y'(r;\eta') < (3/2)l_\eta\,, \quad a \leqq r < a_2\,; \quad p(a_2)y'(a_2;\eta') = (3/2)l_\eta\,.$$

Noting that $y(r;\eta') = \int_a^r y'(t;\eta')\,dt < (3/2)l_\eta \int_a^r dt/p(t) = (3/2)l_\eta R(r)$ for $a \leqq r < a_2$, by the integration of (1.1) from a_1 to a_2 and by (3.10) we obtain

$$\begin{aligned}(3/2)l_\eta = p(a_2)y'(a_2;\eta') &= p(a_1)y'(a_1;\eta') + \int_{a_1}^{a_2} q(t)(t-a)^{-\lambda}(y(t;\eta'))^{\beta}\,dt \\ &\leqq l_\eta + ((3/2)l_\eta)^{\beta}\int_{a_1}^{a_2} q(t)(t-a)^{-\lambda}(R(t))^{\beta}\,dt < (3/2)l_\eta\,.\end{aligned}$$

This contradiction means that (3.11) holds. Therefore, combining the monotonicity of $p(r)y'(r;\eta')$ in $r > a$ and (3.11), we have the existence of $\lim_{r\to\infty} p(r)y'(r;\eta') = \lim_{r\to\infty} y(r;\eta')/R(r) > 0$. This implies that $\eta' \in B$ if $\eta' \in (\eta, \eta+\delta_1)$. Thus B is open.

Now, putting $\eta^* \equiv \inf A$ and $\eta_* \equiv \sup B$, we get η_*, $\eta_* \notin A \cup B$, $\eta_* \leqq \eta^*$, $T_{\eta_*} = T_{\eta^*} = \infty$, and $\lim_{r\to\infty} y(r;\eta_*)/R(r) = \lim_{r\to\infty} y(r;\eta^*)/R(r) = \infty$. Q.E.D.

We apply Theorem 3.1 to the problem (1.8) with $\beta > 1$. In the following corollary $y(r; \eta)$ and $[a, T_\eta)$ denote the solution of (1.8) and the right maximal interval of existence for $y(r; \eta)$, respectively.

COROLLARY 3.1. *Let $\lambda < \beta + 1$, $\beta > 1$ and assume that $G(r) \in C[a, \infty)$ and $G(r) > 0$, $r \in [a, \infty)$.*

(i) *For any $b > a$, there exists $\eta > 0$ such that*

$$T_\eta = b \qquad \text{and} \qquad \lim_{r \to b-0} y(r; \eta) = \infty .$$

(ii) *Suppose moreover that*

$$(3.12) \qquad \begin{aligned} &\int_a^\infty r^{1-\lambda}(\log(r/a))^\beta G(r)\, dr < \infty \qquad (N = 2), \\ &\int_a^\infty r^{1-\lambda} G(r)\, dr < \infty \qquad (N \geqq 3). \end{aligned}$$

Then there exists $\eta_0 > 0$ such that, for any $\eta \in (0, \eta_0)$, $T_\eta = \infty$ and

$$(3.13) \qquad \begin{aligned} &\lim_{r \to \infty} y(r; \eta)/\log r \text{ exists and is positive, if } N = 2, \\ &\lim_{r \to \infty} y(r; \eta) \text{ exists and is positive, if } N \geqq 3. \end{aligned}$$

(iii) *Under the same assumptions as in* (ii), *there exist η_*, $\eta^*(\eta^* \geqq \eta_* > 0)$ such that, for any $\eta \in [\eta_*, \eta^*]$, $T_\eta = \infty$ and*

$$(3.14) \quad \lim_{r \to \infty} y(r; \eta)/\log r = \infty \quad (N = 2), \quad \lim_{r \to \infty} y(r; \eta) = \infty \quad (N \geqq 3).$$

PROOF. Using the same change of the dependent variable as in the proofs of Corollaries 1.1 and 1.2, we can prove the assertions by Theorem 3.1.

Q.E.D.

Finally, we consider the problem $(*)_b$.

THEOREM 3.2. *Let $\beta > 1$ and $\lambda < \beta + 1$. Assume that $G(x) \in C^\theta_{\mathrm{loc}}(\bar{\Omega})$ $(0 < \theta < 1)$, $G(x) > 0$, $x \in \bar{\Omega}$.*

(i) *For any $b > a$ there exists a solution $u(x)$ of $(*)_b$ which belongs to $C^2(\Omega_b)$ and satisfies*

$$u(x) > 0, \quad x \in \Omega_b \qquad \text{and} \qquad \lim_{|x| \to b-0} u(x) = \infty .$$

(ii) *Suppose moreover that $\max_{|x|=r} G(x)/\min_{|x|=r} G(x)$ $(r \geqq a)$ is bounded and*

(3.15)
$$\int_a^\infty r^{1-\lambda}(\log(\lambda/a))^\beta G^*(r)\,dr < \infty \qquad (N = 2),$$
$$\int_a^\infty r^{1-\lambda} G^*(r)\,dr < \infty \qquad (N \geqq 3)$$

where $G^*(r) = \max_{|x|=r} G(x)$. *Then, there exists a positive solution* $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ *of* $(*)$ *such that*

(3.16)
$$\lim_{|x|\to\infty} u(x)/\log|x| = \infty \qquad (N = 2),$$
$$\lim_{|x|\to\infty} u(x) = \infty \qquad (N \geqq 3)\,.$$

PROOF. We give the proof of (i) in the case of $N = 2$. By (i) of Corollary 3.1, for any $b > a$, there exists a function $y(r; \eta) \in C^2(a, b) \cap C^1[a, b)$ such that

$$y''(r; \eta) + (1/r)y'(r; \eta) = G_*(r)(r-a)^{-\lambda}(y(r; \eta))^\beta\,, \quad y(r; \eta) > 0\,, \quad a < r < b\,,$$
$$y(a; \eta) = 0\,, \quad y'(a; \eta) = \eta > 0 \qquad \text{and} \qquad \lim_{r\to b-0} y(r; \eta) = \infty\,,$$

where $G_*(r) = \min_{|x|=r} G(x)$. Putting $v(x) = y(|x|; \eta)$, $a \leqq |x| < b$, we have

$$\begin{cases} \Delta v(x) = G_*(|x|)(|x|-a)^{-\lambda}(v(x))^\beta \leqq G(x)(|x|-a)^{-\lambda}(v(x))^\beta\,, & x \in \Omega_b\,, \\ v(x) = 0\,, \quad |x| = a \qquad \text{and} \qquad \lim_{|x|\to b-0} v(x) = \infty\,. \end{cases}$$

Now, take a constant τ such that

$$0 < \tau < 1 \quad \text{and} \quad G^*(r) \leqq \tau^{1-\beta} G_*(r)\,, \quad a \leqq r \leqq b\,.$$

So, the function $w(x) = \tau v(x)$ satisfies $w(x) \leqq v(x)$, $x \in \Omega_b$, and

$$\begin{cases} \Delta w(x) = \tau \Delta v(x) = \tau^{1-\beta} G_*(|x|)(|x|-a)^{-\lambda}(\tau v(x))^\beta \\ \qquad \geqq G^*(|x|)(|x|-a)^{-\lambda}(w(x))^\beta \geqq G(x)(|x|-a)^{-\lambda}(w(x))^\beta\,, \quad x \in \Omega_b\,, \\ w(x) = 0\,, \quad |x| = a\,, \quad \text{and} \quad \lim_{|x|\to b-0} w(x) = \infty\,. \end{cases}$$

Therefore, we get a solution $u(x)$ of $(*)_b$ satisfying $w(x) \leqq u(x) \leqq v(x)$, $x \in \Omega_b$, by the similar supersolution-subsolution method in the proof of Theorem 2.1. This function $u(x)$ is a desired solution of $(*)_b$.

The rest of the assertions are proved by essentially the same method as above, so the proof of it is omitted. Q.E.D.

REMARK 3.1. Recently, Usami [14] has proved the existence of entire positive solutions of $\Delta u = G(x)u^\beta$ in $\boldsymbol{R}^N$ satisfying (3.16). We have used the same idea as in [14].

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