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Generalized random ergodic theorems and Hausdorff-measures of random fractals

Yoshiki TSUJII

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§1. Introduction

Random ergodic theorems were investigated by Kakutani [3], Morita [5] and others. In this paper we establish *generalized* random ergodic theorems which contain the results obtained by Kakutani. A prototype of a generalized random ergodic theorem is found in the proof of the result that the Hausdorff measures of almost all random fractals take a common number.

In Section 2 we show Hausdorff measure's constancy of random fractals to understand a typical generalized random ergodic theorem. Random fractals were investigated by Mauldin-Williams [4], Falconer [1] and Graf [2] and they showed the Hausdorff dimensions of almost all random fractals equal to a constant under some condition. In this section we show that the Hausdorff *measures* of almost all random fractals equal to a constant. In the proof we use a result which is a prototype of a generalized random ergodic theorem.

In Section 3 we develop generalized random ergodic theorems which contain the results obrained by Kakutani [3], and in Section 4 we consider generalized random dynamical systems with random *discrete* parameter, which illustrate the idea developed in Section 3.

§2. Hausdorff measures of random fractals

The starting point for the considerations in the present paper is a scheme used by Graf [2] for producing statistically self-similar fractals. To generate a fractal at random, Graf starts with a probability distribution μ on the set of all N-tuples of contractions of a given bounded separable complete metric space X. First he chooses an N-tuples $(S_1, S_2, \dots, S_N)$ of contractions at random with respect to μ and sets

$$A_1 = \bigcup_{i=1}^{N} S_i(X).$$

For every $i \in \{1, \dots, N\}$, he chooses independently an N-tuples $(S_{i1}, \dots, S_{iN})$ at random with respct to μ and sets

$$A_2 = \bigcup_{i=1}^{N} S_i(\bigcup_{k=1}^{N} S_{ik}(X)).$$

He continues this process. Then $K = \bigcup_{n \in \mathbb{N}} \bar{A}_n$ is a random fractal.

More formally we proceed as follows: Let $X \subset \mathbf{R}^d$ be a compact set with $\mathring{X} \neq \emptyset$ where $\mathring{X}$ is the set of all interior points of X. Let d be the Euclidean metric on $\mathbf{R}^d$. Let S be a contraction similality (briefly similality) of X, i.e., S is a map from X into itself such that there exists an $0 < r = r(S) < 1$ satisfying $d(Sx, Sy) = rd(x, y)$ for all $x, y \in X$. The number $r(S)$ is called the contraction ratio of S.

For $0 < \delta < 1$ we denote by $\mathrm{Sim}_\delta(X)$ the set of all contraction similarities S of X such that $r(S) > \delta$. The space $\mathrm{Sim}_\delta(X)$ is equipped with the topology of pointwise convergence and the Borel field of $\mathrm{Sim}_\delta(X)$ is denoted by $\mathscr{E}_0$. Let $\mathbf{N}$ denote the set of nonegative integers. Fix a positive integer $N \geq 2$. Let

$$D = D(N) = \bigcup_{m \in \mathbf{N}} C_m$$

where $C_m = C_m(N) = \{1, 2, \ldots, N\}^m$ for a positive m and $C_0 = \{\emptyset\}$. If $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ and $\tau = (\tau_1, \tau_2, \ldots, \tau_r)$ are in D, let

$$\sigma | n = (\sigma_1, \sigma_2, \ldots, \sigma_n) \qquad \text{for } n \leq m$$

and

$$\sigma * \tau = (\sigma_1, \sigma_2, \ldots, \sigma_m, \tau_1, \tau_2, \ldots, \tau_r).$$

Our fundamental space is $\mathscr{S} = (\mathrm{Sim}_\delta(X)^N)^D$. Every element $\mathfrak{s} \in \mathscr{S} = (\mathrm{Sim}_\delta(X)^N)^D$ can be expreseed as

$$\mathfrak{s} = (\underline{S}_\sigma)_{\sigma \in D}$$

where $\underline{S}_\sigma = (S_{\sigma * 1}(\mathfrak{s}), S_{\sigma * 2}(\mathfrak{s}), \ldots, S_{\sigma * N}(\mathfrak{s})) \in \mathrm{Sim}_\delta(X)^N$ and $S_\emptyset = (S_1(\mathfrak{s}), S_2(\mathfrak{s}), \ldots, S_N(\mathfrak{s}))$.

Let $K(\mathfrak{s}) = \bigcap_{m>0} \bigcup_{\sigma \in C_m} S_{\sigma|1}(\mathfrak{s}) \circ S_{\sigma|2}(\mathfrak{s}) \circ \cdots \circ S_{\sigma|m}(\mathfrak{s})(X)$. This $K(\mathfrak{s})$ is a non-empty compact set and is considered as a fractal set constructed from an N-ary tree of contraction similarities $\mathfrak{s} = (\underline{S}_\sigma)_{\sigma \in D}$. Let $\mathscr{E}$ denote the product Borel field $(\mathscr{E}_0^N)^D$ of $(\mathrm{Sim}_\delta(X)^N)^D$.

Let μ be a Borel probalility measure on $\mathrm{Sim}_\delta(X)^N$, and μ^D denote the corresponding product measure on $\mathscr{S} = (\mathrm{Sim}_\delta(X)^N)^D$.

Let $\mathscr{K}(X)$ be the space of all non-empty compact subsets of X with the usual Hausdorff metric. Graf showed that there exists a Borel set $\mathscr{S}_0 \subset (\mathrm{Sim}_\delta(X)^N)^D$ with $\mu^D(\mathscr{S}_0) = 1$ such that a map ψ: $(\mathrm{Sim}_\delta(X)^N)^D \to \mathscr{K}(X)$ defined by

$$\psi(\mathfrak{s}) = \begin{cases} \bigcap_{m>0} \bigcup_{\sigma \in C_m} S_{\sigma|1}(\mathfrak{s}) \circ S_{\sigma|2}(\mathfrak{s}) \circ \cdots \circ S_{\sigma|m}(\mathfrak{s})(X) & \text{if } \mathfrak{s} \in \mathscr{S}_0 \\ X & \text{if } \mathfrak{s} \notin \mathscr{S}_0 \end{cases}$$

is Borel measurable. Let P_μ be the image measure of μ^D with respect to ψ,i.e., for every Borel set $B \subset \mathscr{K}(X)$, $P_\mu(B) = \mu^D(\psi^{-1}(B))$. P_μ is the unique "μ-self-similar" probability measure on $\mathscr{K}(X)$.

For a set K in $\mathbf{R}^d$, the Hausdorff measure and the Hausdorff dimension of K are defined as follows. For $0 \leq \lambda < \infty$ and $\delta > 0$, let

$$\mathscr{H}^\lambda_\delta(K) = \inf\{\textstyle\sum_{i=1}^\infty |U_i|^\lambda \colon K \subset \bigcup_{i=1}^\infty U_i,\ 0 < |U_i| \leq \delta\}$$

where $|U|$ is the diameter of a set U, and let

$$\mathscr{H}^\lambda(K) = \lim_{\delta\to 0}\mathscr{H}^\lambda_\delta(K).$$

Then $\mathscr{H}^\lambda$ is an outer measure. It is easy to see $\mathscr{H}^\lambda(S(K)) = r(S)^\lambda \mathscr{H}^\lambda(K)$ for any similality S. The restriction of $\mathscr{H}^\lambda$ to the σ-field of all measurable sets is called the Hausdorff λ-dimensional measure. There is a number $0 \leq \dim_{\mathrm{H}}(K) \leq d$, called the Hausdorff dimension of K, such that $\mathscr{H}^\lambda(K) = \infty$ if $\lambda < \dim_{\mathrm{H}}(K)$ and $\mathscr{H}^\lambda(K) = 0$ if $\lambda > \dim_{\mathrm{H}}(K)$.

Graf proved the following theorem:

THEOREM 1 (Graf [2]). *Suppose that, for μ-a.e. $(S_1, S_2, \dots, S_N) \in \mathrm{Sim}_\delta(X)^N$, $S_i(\mathring{X}) \cap S_j(\mathring{X}) = \varnothing$ for $i, j \in \{1, 2, \dots, N\}$ with $i \neq j$. Then*

$$E[\mathscr{H}^\alpha(K)] = \int_{\mathscr{K}(X)} \mathscr{H}^\alpha(K) P_\mu(dK) < \infty$$

and

$$\dim_{\mathrm{H}}(K) = \alpha \qquad \text{for } P_\mu\text{-a.e. } K \in \mathscr{K}(X)$$

where α is such that

$$\int \textstyle\sum_{i=1}^N r(S_i)^\alpha d\mu(S_1, S_2, \dots, S_N) = 1.$$

Furthermore the following statements are equivalent:

(1) $\sum_{i=1}^N r(S_i)^\alpha = 1$ *for μ-a.e. $(S_1, S_2, \dots, S_N) \in \mathrm{Sim}_\delta(X)^N$.*

(2) $\mathscr{H}^\alpha(K) > 0$ *for P_μ-a.e. $K \in \mathscr{K}(X)$.*

(3) $P_\mu(\{K\colon \mathscr{H}^\alpha(K) > 0\}) > 0.$

REMARK. Indeed the above statements with respect to P_μ-a.e. $K \in \mathscr{K}(X)$ are derived from the corresponding statements with respect to μ^D-a.e. $\mathfrak{s} \in \mathscr{S}$.

With respect to the Hausdorff measure $\mathscr{H}^\alpha(K(\mathfrak{s}))$, we have the following theorem.

THEOREM 2. *Let the assumption of Theorem 1 be satisfied. Then there exists a $0 \leq \beta < \infty$ such that*

$$\mathscr{H}^\alpha(K(\mathfrak{s})) = \beta \qquad \text{for } \mu^D\text{-a.e. } \mathfrak{s} \in \mathscr{S}.$$

Hence

$$\mathscr{H}^\alpha(K) = \beta \qquad \text{for } P_\mu\text{-a.e. } K \in \mathscr{K}(X).$$

For the proof of Theorem 2, we require two lemmas where we suppose the assumption of Theorem 1 is satisfied. Define φ_i: $(\mathrm{Sim}_\delta(X)^N)^D \to (\mathrm{Sim}_\delta(X)^N)^D$ for $i = 1, 2, \dots, N$ by

$$\underline{S}_\sigma(\varphi_i(\omega)) = \underline{S}_{i*\sigma}(\omega) \qquad \text{for } \sigma \in D.$$

It is easy to see that each φ_i is μ^D-measure preserving, i.e., $\mu^D(\varphi_i^{-1}(E)) = \mu^D(E)$ for $E \in \mathscr{E}$. Then we have the following lemma.

LEMMA 1.

$$\mathscr{H}^\alpha(K(\omega)) = \sum_{i=1}^N r(S_i(\omega))^\alpha \mathscr{H}^\alpha(K(\varphi_i(\omega))) \qquad \textit{for } \mu^D\textit{-a.e. } \omega,$$

where $(S_1(\omega), S_2(\omega), \dots, S_N(\omega)) = \underline{S}_\varnothing(\omega)$.

PROOF. By the definition of $K(\omega)$, we have

$$\begin{aligned} K(\omega) &= \bigcap_{m>0} \bigcup_{\sigma \in C_m} S_{\sigma|1}(\omega) \circ S_{\sigma|2}(\omega) \circ \cdots \circ S_{\sigma|m}(\omega)(X) \\ &= \bigcup_{i=1}^N S_i(\omega) \Big(\bigcap_{m>0} \bigcup_{\sigma \in C_m} S_{i*\sigma|2}(\omega) \circ S_{i*\sigma|3}(\omega) \circ \cdots \circ S_{i*\sigma|m+1}(\omega)(X)\Big) \\ &= \bigcup_{i=1}^N S_i(\omega) K(\varphi_i(\omega)), \end{aligned}$$

so that

$$\begin{aligned} \mathscr{H}^\alpha(K(\omega)) &\le \sum_{i=1}^N \mathscr{H}^\alpha(S_i(\omega) K(\varphi_i(\omega))) \\ &= \sum_{i=1}^N r(S_i(\omega))^\alpha \mathscr{H}^\alpha(K(\varphi_i(\omega))) \end{aligned}$$

because $S_i(\omega)$ are similalities. Integrating both sides with respect to μ^D, and using the facts that $\underline{S}_\varnothing(\omega)$ and $\{\underline{S}_\sigma(\omega)\}_{\sigma \neq \varnothing}$ are independent and that $\varphi_i(\omega)$ and ω are identically distributed, we obtain

$$\begin{aligned} E[\mathscr{H}^\alpha(K(\omega))] &\le \sum_{i=1}^N E[r(S_i(\omega))^\alpha \mathscr{H}^\alpha(K(\varphi_i(\omega)))] \\ &= \sum_{i=1}^N E[r(S_i)^\alpha] E[\mathscr{H}^\alpha(K(\varphi_i(\omega)))] \\ &= E\Big[\sum_{i=1}^N r(S_i)^\alpha\Big] E[\mathscr{H}^\alpha(K(\omega))] \\ &= E[\mathscr{H}^\alpha(K(\omega))]. \end{aligned}$$

Since $E[\mathscr{H}^\alpha(K(\omega))] < \infty$ by Theorem 1, we have

$$\mathscr{H}^\alpha(K(\omega)) = \sum_{i=1}^N r(S_i(\omega))^\alpha \mathscr{H}^\alpha(K(\varphi_i(\omega))) \qquad \text{for } \mu^D\text{-a.e. } \omega.$$

This completes the proof.

LEMMA 2. *Assume a Borel probability measure* μ *on* $\mathrm{Sim}_\delta(X)^N$ *satisfies that* $\sum_{i=1}^N r(S_i)^\alpha = 1$ *for* μ*-a.e.* $(S_1, S_2, \dots, S_N) \in \mathrm{Sim}_\delta(X)^N$. *For* $f(\omega) \in L^1((\mathrm{Sim}_\delta(X)^N)^D, \mu^D)$, *we define* $Uf(\omega)$ *by*

$$Uf(\mathfrak{s}) = \sum_{i=1}^{N} r(S_i(\mathfrak{s}))^\alpha f(\varphi_i(\mathfrak{s})).$$

Then any integrable U-invariant function (i.e., $Uf(\mathfrak{s}) = f(\mathfrak{s})$ for μ^D-a.e. $\mathfrak{s}$) is equal to a contant a.e. $\mathfrak{s}$.

This lemma is a prototype of a generalized random ergodic theorem which is investigated in Section 3. It can be proved directly, but we will postpone the proof until we prove Corollary 1 in Section 3.

PROOF OF THEOREM 2. If the condition $\sum_{i=1}^{N} r(S_i)^\alpha = 1$ for μ-a.e. $(S_1, S_2, \ldots, S_N) \in \mathrm{Sim}_\delta(X)^N$ does not hold, the statement (3) in Theorem 1 fails, that is, $\mathscr{H}^\alpha(K(\mathfrak{s})) = 0$ for μ^D-a.e. $\mathfrak{s}$.

Assume that $\sum_{i=1}^{N} r(S_i)^\alpha = 1$ for μ-a.e. $(S_1, S_2, \ldots, S_N) \in \mathrm{Sim}_\delta(X)^N$. By Theorem 1 and Lemma 1, $f(\mathfrak{s}) = \mathscr{H}^\alpha(K(\mathfrak{s}))$ is integrable and U-invariant. Hence we have a $0 \le \beta < \infty$ such that $\mathscr{H}^\alpha(K(\mathfrak{s})) = \beta$ for μ^D-a.e. $\mathfrak{s}$, using Lemma 2. This completes the proof.

EXAMPLES. (a) (Graf [2]) Let X be the unit inteval $[0, 1]$. Define contraction maps T_i $(i = 0, 1, 2)$ by $T_i(x) = (x + i)/3$ for $x \in [0, 1]$. Then $r(T_0) = r(T_1) = r(T_2) = 1/3$. Let $\mu = (\varepsilon_{(T_0,T_1)} + \varepsilon_{(T_0,T_2)} + \varepsilon_{(T_1,T_2)})/3$ on $\mathrm{Sim}_\delta([0, 1])^2$ with $0 < \delta < 1/3$. Since $r(S_1)^\alpha + r(S_2)^\alpha = 1$ for μ-a.e. where $\alpha = (\log 2)/(\log 3)$, it follows from Theorems 1 and 2 that

$$\dim_H(K) = \alpha \qquad \text{for } P_\mu\text{-a.e. } K,$$

and

$$\mathscr{H}^\alpha(K) = \beta \qquad \text{for } P_\mu\text{-a.e. } K$$

where β is some constant > 0.

(b) Let X be $[0, 1]^2$. Define contraction maps T_1, T_2, T_3 and $\tilde{T}_1$, $\tilde{T}_2$, $\tilde{T}_3$ by $T_1(x, y) = (x/3, y/3)$, $T_2(x, y) = ((x + 1)/3, (y + 2)/3)$, $T_3(x, y) = ((x + 2)/3, y/3)$ and $\tilde{T}_1(x, y) = (x/2, y/2)$, $\tilde{T}_2(x, y) = ((x + 1)/3, (y + 2)/3)$, $\tilde{T}_3(x, y) = ((x + 5)/6, y/6)$. Then $r(T_1) + r(T_2) + r(T_3) = r(\tilde{T}_1) + r(\tilde{T}_2) + r(\tilde{T}_3) = 1$. Let $\mu = (\varepsilon_{(T_1,T_2,T_3)} + \varepsilon_{(\tilde{T}_1,\tilde{T}_2,\tilde{T}_3)})/2$ on $\mathrm{Sim}_\delta([0, 1]^2)^3$ with $0 < \delta < 1/6$. Since $r^1(S_1) + r^1(S_2) + r^1(S_3) = 1$ for μ-a.e. , it follows that

$$\dim_H(K) = 1 \qquad \text{for } P_\mu\text{-a.e. } K$$

and

$$\mathscr{H}^1(K) = \beta \qquad \text{for } P_\mu\text{-a.e. } K$$

where β is some constant > 0.

§3. Generalized random ergodic theorems

In this section we develop generalized random ergodic theorems which guarantee Lemma 2 in Section 2.

Let $(S, \mathscr{B}\ m)$ be a probability space and $(Y, \mathscr{F}, \nu)$ another probability space where $(Y, \mathscr{F})$ is a standard measurable space. Let $\mathscr{B}_0$ be a sub σ-field of $\mathscr{B}$ and $\{\gamma(s, y): s\in S\}$ a family of $\mathscr{B}_0 \times \mathscr{F}$-measurable probability density functions on Y, i.e., $\gamma(s, y) \geq 0$ and $\int \gamma(s, y)\nu(dy) = 1$ for all $s\in S$. Let $\Phi = \{\varphi_y: y\in Y\}$ be a family of m-measure preserving transformations φ_y defined on S with parameter $y\in Y$ (i.e., $m(\varphi_y^{-1}(B)) = m(B)$, $B\in\mathscr{B}$, $y\in Y$). Assume that Φ is $\mathscr{B}\times\mathscr{F}$-measurable, i.e., $\{(s, y): \varphi_y(s)\in B\}\in\mathscr{B}\times\mathscr{F}$ for $B\in\mathscr{B}$ and that the sub σ-fields $\vee_y\varphi_y^{-1}\mathscr{B}$ and $\mathscr{B}_0$ are *independent*. We call the quadruplet $((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s\in S\}, \Phi = \{\varphi_y: y\in Y\})$ a *generalized random dynamical system*.

REMARK 1. If $\gamma(s, y) \equiv 1$, our formulation is the same as that of Kakutani [3] and Morita [5]. They discussed random dynamical systems of the form

$$\varphi_{y_n(\omega)}\varphi_{y_{n-1}(\omega)}\cdots\varphi_{y_1(\omega)}(s)$$

where $\{y_i(\omega)\}_{i=1}^{\infty}$ are independent identically distributed Y-valued random variables.

REMARK 2. Consider $(\mathrm{Sim}_\delta(X)^N)^D$, $\mathscr{E}$ and μ^D which are introduced in Section 2. Assume that the measure μ on $\mathrm{Sim}_\delta(X)^N$ satisfies $\sum_{i=1}^N r(S_i)^\alpha = 1$ for μ-a.e. $(S_1, S_2,\dots, S_N)\in\mathrm{Sim}_\delta(X)^N$ where $r(S_i)$ is the contraction ratio of S_i. Let $S = (\mathrm{Sim}_\delta(X)^N)^D$, $\mathscr{B} = \mathscr{E}$, $m = \mu^D$, $Y = \{1, 2,\dots, N\}$, $\mathscr{F}$ = the set of all subsets of Y and $\nu(\{i\}) = 1/N$ for $i\in Y$. Let $\gamma(s, i) = Nr(S_i(s))^\alpha$ and $\mathscr{B}_0 = \sigma(\{s: \underline{S}_\varnothing(s) = (S_1(s),\dots, S_N(s))\in E\}\,|\,E\in\mathscr{E}_0^N)$. Then $\{\gamma(s, i): i\in Y\}$ satisfy that $\gamma(s, i) > 0$, $\int\gamma(s, i)\nu(di) = 1$ and $\gamma(s, i)$ is $\mathscr{B}_0$-measurable. The transformations $\{\varphi_i(s): S \to S\}_{i=1}^N$ defined by $\underline{S}_\sigma(\varphi_i(s)) = \underline{S}_{i*\sigma}(s)$ for all $\sigma\in D$ are m-measure preserving and $\vee_i\varphi_i^{-1}\mathscr{B}$ and $\mathscr{B}_0$ are independent. Thus the quadruplet $(((\mathrm{Sim}_\delta(X)^N)^D, \mathscr{B}, \mathscr{B}_0, \mu^D), (Y = \{1, 2,\dots, N\}, \mathscr{F}, \nu), \{Nr(S_i(s))^\alpha: s\in(\mathrm{Sim}_\delta(X)^N)^D\}, \{\varphi_i(s): i\in Y\})$ is a generalized random dynamical system.

A generalized random dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s\in S\}, \Phi = \{\varphi_y: y\in Y\})$ induces a Markov process, a Markov operator and a skew product transformation.

Induced Markov process. Let us put, for $s\in S$ and $B\in\mathscr{B}$,

$$P(s, B) = \int_Y I_B(\varphi_y(s))\gamma(s, y)\nu(dy).$$

Then $\{P(s, B)\}$ are Markov transition probabilities with the invariant distribution $m(B)$. In fact,

$$\begin{aligned}\int_S P(s, B)m(ds) &= \int_S \left[\int_Y I_B(\varphi_y(s))\gamma(s, y)\nu(dy)\right]m(ds)\\ &= \int_Y \left[\int_S I_B(\varphi_y(s))m(ds)\int_S \gamma(s, y)m(ds)\right]\nu(dy)\\ &= m(B)\int_Y \left[\int_S \gamma(s, y)m(ds)\right]\nu(dy)\\ &= m(B)\int_S \left[\int_Y \gamma(s, y)\nu(dy)\right]m(ds) = m(B),\end{aligned}$$

where we use the facts that random functions $\{\gamma(s, y)\}$ and $\{I_B(\varphi_y(s))\}$ defined on $(S, \mathscr{B}, m)$ are independent for fixed $y \in Y$ and that φ_y are m-measure preserving transformations. The Markov process with transition probabilities $P(s, B)$ and the invariant distribution $m(B)$ is called the Markov process induced by a generalized random dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s \in S\}, \Phi = \{\varphi_y: y \in Y\})$.

Induced Markov operater. Let $L^1(S)$ be the space of all integrable functions defined on $(S, \mathscr{B}, m)$. Let us define a bounded linear operator T on $L^1(S)$ by

$$Tf(s) = \int_S P(s, dt)f(t) = \int_Y f(\varphi_y(s))\gamma(s, y)\nu(dy).$$

We call T the induced Markov operator.

Induced skew product transformation. Let Y^* be the infinite product space $\prod_{i=1}^{\infty} Y$ of Y. On the product space $\Omega = S \times Y^*$, we define a probability measure P as follows: For a set $E \times F$ in $\Omega = S \times Y^*$ where $E \in \mathscr{B}$ and $F = F_1 \times F_2 \times \cdots \times F_n \times Y \times Y \times \cdots \subset Y^*$ $(F_i \in \mathscr{F})$, we define $P(E \times F)$ by

$$\begin{aligned}P(E \times F) &= \int_E \left[\int_{F_1} \cdots \left[\int_{F_{n-1}} \left[\int_{F_n} \gamma(\varphi_{y_{n-1}} \cdots \varphi_{y_1}(s), y_n)\nu(dy_n)\right]\right.\right.\\ &\qquad \left.\left.\gamma(\varphi_{y_{n-2}} \cdots \varphi_{y_1}(s), y_{n-1})\nu(dy_{n-1})\right] \cdots \gamma(s, y_1)\nu(dy_1)\right]m(ds)\\ &= \int_S \int_Y \cdots \int_Y I_E(s) I_{F_1}(y_1) \cdots I_{F_n}(y_n)\gamma_n(y_1, y_2, \ldots, y_n; s)\\ &\qquad \prod_{i=1}^{n} \nu(dy_i)m(ds),\end{aligned}$$

denoting

$$\begin{aligned}\gamma_n(y_1, y_2, \ldots, y_n; s) &= \gamma(s, y_1)\gamma(\varphi_{y_1}(s), y_2)\cdots\\ &\qquad \gamma(\varphi_{y_{n-1}}\varphi_{y_{n-2}}\cdots\varphi_{y_1}(s), y_n).\end{aligned}$$

By Kolmogorov's extension theorem we have a unique probability measure P on $\Omega = S \times Y^*$. Let us consider the skew product transformation φ^* defined by

$$\varphi^*(s, y^*) = (\varphi_{y_1}(s), \theta y^*) \quad \text{for } (s, y^*) \in \Omega,$$

where $y^* = (y_1, y_2, \ldots) \in Y^*$ and θ is the shift transformation on Y^*, *i.e.*, $(\theta y^*)_n = y_{n+1}$ for $n \in \mathbf{N}$.

We show that the transformation φ^* on $(S \times Y^*, P)$ is P-measure preserving. It suffices to show that $P(\varphi^{*-1}(E \times F)) = P(E \times F)$ for $E \in \mathscr{B}$ and a cylinder set $F = F_1 \times F_2 \times \cdots \times F_n \times Y \times Y \times \cdots$ where $F_i \in \mathscr{F}$. Since $\gamma(s, y)$ is $\mathscr{B}_0 \times \mathscr{F}$-measurable, σ-fields $\varphi_y^{-1}\mathscr{B}$ and $\mathscr{B}_0$ are independent and $\{\varphi_y\}$ are m-measure preserving transformations, we have

$$\begin{aligned}
P(\varphi^{*-1}(E \times F)) &= P(\{(s, y^*)\colon \varphi_{y_1}(s) \in E,\ y_2 \in F_1, \ldots, y_{n+1} \in F_n\}) \\
&= \int I_E(\varphi_{y_1}s) I_{F_1}(y_2) \cdots I_{F_n}(y_{n+1}) P(d(s, y^*)) \\
&= \int_S \int_{Y^{n+1}} I_E(\varphi_{y_1}s) I_{F_1}(y_2) \cdots I_{F_n}(y_{n+1}) \gamma(s, y_1) \\
&\qquad \gamma_n(y_2, y_3, \ldots, y_{n+1}; \varphi_{y_1}s) \textstyle\prod_{i=1}^{n+1} \nu(dy_i) m(ds) \\
&= \int_{Y^{n+1}} \left[\int_S \gamma(s, y_1) m(ds)\right]\left[\int_S I_E(\varphi_{y_1}s) I_{F_1}(y_2) \cdots I_{F_n}(y_{n+1})\right. \\
&\qquad \left. \gamma_n(y_2, \ldots, y_{n+1}; \varphi_{y_1}s) m(ds)\right] \textstyle\prod_{i=1}^{n+1} \nu(dy_i) \\
&= \int_{Y^{n+1}} \left[\int_S \gamma(s, y_1) m(ds)\right]\left[\int_S I_E(t) I_{F_1}(y_2) \cdots I_{F_n}(y_{n+1})\right. \\
&\qquad \left. \gamma_n(y_2, y_3, \ldots, y_{n+1}; t) m(dt)\right] \textstyle\prod_{i=1}^{n+1} \nu(dy_i) \\
&= \int_{Y^n} \int_S I_E(t) I_{F_1}(y_2) \cdots I_{F_n}(y_{n+1}) \gamma_n(y_2, y_3, \ldots, y_{n+1}; t) m(dt) \\
&\qquad \textstyle\prod_{i=2}^{n+1} \nu(dy_i) \\
&= P(E \times F).
\end{aligned}$$

This P-measure preserving transformation φ^* is called the induced skew product trasformation of a generalized random dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y)\colon s \in S\}, \Phi = \{\varphi_y\colon y \in Y\})$.

Now we give the definition of ergodicity of a generalized random dynamical system and summarize other well known definitions.

Definition 1. A generalized random dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0,$

$m)$, $(Y, \mathscr{F}, \nu)$, $\{\gamma(s, y): s \in S\}$, $\Phi = \{\varphi_y: y \in Y\})$ is called ergodic if $\int_S \int_Y I_{(\varphi_y^{-1}(B) \triangle B)}(s)$ $\gamma(s, y)\nu(dy)m(ds) = 0$ implies $m(B) = 0$ or 1 for $B \in \mathscr{B}$. The $\triangle$ stands for symmetric difference.

DEFINITION 2. A Markov process with transition probabilities $P(s, B)$ and an invariant distribution $m(B)$ is called ergodic if $m(E) > 0$ and $P(s, E) = 1$ for m-a.e. $s \in E$ implies $m(E) = 1$ for $E \in \mathscr{B}$.

DEFINITION 3. A Markov operator T on $L^1(S, m)$ is called ergodic if $Tf(s) = f(s)$ for a.e. $s \in S$ implies $f(s) \equiv$ constant a.e.

DEFINTION 4. A measure preserving transformation φ^* on a probability space $(\Omega, \mathscr{G}, P)$ is called ergodic if $\varphi^{*-1}(E) = E$ implies $P(E) = 0$ or 1 for $E \in \mathscr{G}$.

DEFINTION 5. A Markov operator T on $L^1(S, m)$ is called mixing if

$$\int_S T^n f(s) g(s) m(ds) \longrightarrow \int_S f(s) m(ds) \int_S g(s) m(ds) \quad (n \longrightarrow \infty)$$

for $f(s) \in L^1(S, m)$ and $g(s) \in L^\infty(S, m)$.

DEFINITION 6. A measure preserving transformation φ^* on a probability space $(\Omega, \mathscr{G}, P)$ is called mixing if

$$\int_\Omega f(\varphi^{*n}(\omega)) g(\omega) P(d\omega) \longrightarrow \int_\Omega f(\omega) P(d\omega) \int_\Omega g(\omega) P(d\omega) \quad (n \longrightarrow \infty)$$

for $f(\omega) \in L^1(\Omega, P)$ and $g(\omega) \in L^\infty(\Omega, P)$.

Let $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s \in S\}, \Phi = \{\varphi_y: y \in Y\})$ be a generalized random dynamical system, and T the induced Markov operator. The ergodic theorem for the Markov operator T is as follows:

THEOREM 3. *Let T be the induced Markov operator by a generalized random dynamical system and $f(s) \in L^1(S, \mathscr{B}, m)$. Then there exists an integrable T-invariant function $\bar{f}(s)$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f(s) = \bar{f}(s)$$

in the L^1 sense and a.e. $s \in S$.

PROOF. Because $Tf(s) = \int P(s, dt) f(t)$ where $\{P(s, dt)\}$ are the transition probabilities of the induced Markov process, the well known ergodic theorem for Markov operators yields the result.

Now we show the equivalence of ergodicities in various notions. This is a generalization of the result given by Kakutani [3].

THEOREM 4. *Let* $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s \in S\}, \Phi = \{\varphi_y: y \in Y\})$ *be a generalized random dynamical system. Then the following conditions are mutually equivalent.*

(*a*) *The generalized random dynamical system is ergodic.*
(*b*) *The induced Markov process is ergodic.*
(*c*) *The induced Markov operator is ergodic.*
(*d*) *The induced skew product transformation* φ^* *is ergodic.*

The proof is similar to that of Kakutani [3; Theorem 3], so it is given in Appendix.

COROLLARY 1. *For a generalized random dynamical system* $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s \in S\}, \Phi = \{\varphi_y: y \in Y\})$, let $Y_0 = \{y \in Y: \varphi_y$ *is ergodic and* $\gamma(s, y) > 0$ *for m-a.e.* $s \in S\}$. *If* $\nu(Y_0) > 0$, *then the induced Markov operator T is ergodic.*

PROOF. By Theorem 4, it suffices to show that the generalized random dynamical system is ergodic. Let $B \in \mathscr{B}$ be such that

$$\int_S \int_Y I_{(\varphi_y^{-1}(B)\Delta B)}(s)\gamma(s, y)\nu(dy)m(ds) = 0.$$

Since $\nu(Y_0) > 0$, there exists an element $y \in Y$ such that φ_y is ergodic and $I_{(\varphi_y^{-1}(B)\Delta B)}(s) = 0$ for *m-a.e.* $s \in S$, that is, B is φ_y-invariant. It follows that $m(B) = 0$ or 1, and so the generalized random dynamical system is ergodic.

Using Corollary 1, we prove Lemma 2 in Section 2.

PROOF OF LEMMA 2 IN SECTION 2. As mentioned in Remark 2 in this section, the quadruplet $((\mathrm{Sim}_\delta(X)^N)^D, \mathscr{B}, \mathscr{B}_0, \mu^D), (Y = \{1, 2,\dots, N\}, \mathscr{F}, \nu), \{Nr(S_i(s))^\alpha: s \in (\mathrm{Sim}_\delta(X)^N)^D\}, \{\varphi_i(s): i \in Y\})$ is a generalized random dynamical system. Note that $\gamma(i, s) = Nr(S_i(s))^\alpha \geq N\delta^\alpha > 0$ and $\{\varphi_i: i = 1, 2,\dots, N\}$ are ergodic because μ^D is the product measure of μ. It follows that $\mathrm{Y}_0 = \mathrm{Y}$. By Corollary 1, the induced Markov operator T is ergodic, i.e., any integrable T-invariant function is constant a.e. $s \in (\mathrm{Sim}_\delta(X)^N)^D$. Since the operator U is identical to the operator T, this completes the proof.

THEOREM 5. *Let T and* φ^* *be the induced Markov operator and the induced skew product transformation by a generalized random dynamical system, respectively. Then T is mixing if and only if* φ^* *is mixing.*

PROOF. For $f(s) \in L^1(S)$ and $g(s) \in L^\infty(S)$, it holds that

$$\int_S T^n f(s)g(s)m(ds) = \int_S \left[\int_{Y^n} f(\varphi_{y_n}\varphi_{y_{n-1}}\cdots\varphi_{y_1}(s))\gamma_n(y_1,\ldots, y_n; s)\prod_{i=1}^n \nu(dy_i)\right] g(s)m(ds)$$
$$= \int_{S\times Y^*} f^*(\varphi^{*n}(s, y^*))g^*(s, y^*)P(d(s, y^*))$$

where $f^*(s, y^*) = f(s)$ and $g^*(s, y^*) = g(s)$. Therefore T is mixing if φ^* is mixing. A similar argument used in the proof of (c)$\Rightarrow$(d) of Theorem 4 (in Appendix) shows that φ^* is mixing if T is mixing.

THEOREM 6. *Suppose that a generalized random dynamical system* $((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y)\colon s\in S\}, \Phi = \{\varphi_y\colon y\in Y\})$ *satisfies the conditions* (i) $\bigvee_{i=0}^{\infty}\bigvee_{y_1,y_2,\ldots,y_i}\varphi_{y_1}^{-1}\varphi_{y_2}^{-1}\cdots\varphi_{y_i}^{-1}\mathscr{B}_0 = \mathscr{B}$ *and* (ii) *the* σ-*fields* $\bigvee_{i=0}^{n-1}\bigvee_{z_1,z_2,\ldots,z_i\in Y}\varphi_{z_1}^{-1}\varphi_{z_2}^{-1}\cdots\varphi_{z_i}^{-1}\mathscr{B}_0$ *and* $\varphi_{y_1}^{-1}\varphi_{y_2}^{-1}\cdots\varphi_{y_n}^{-1}\mathscr{B}$ *are independent for any* $y_1,\cdots, y_n\in Y$ *and* $n\in\mathbf{N}$. *Then the induced Markov operator T is mixing.*

PROOF. *Let* $\mathscr{B}_k = \bigvee_{i=0}^{k}\bigvee_{y_1,y_2,\ldots,y_i}\varphi_{y_1}^{-1}\varphi_{y_2}^{-1}\cdots\varphi_{y_i}^{-1}\mathscr{B}_0$. By the assumption that $\bigvee_{k=1}^{\infty}\mathscr{B}_k = \mathscr{B}$, it suffices to show that for $n > k$, $\int_S T^n f(s)g(s)m(ds) = \int_S f(s)m(ds)\int_S g(s)m(ds)$ where $f(s)$ is a $\mathscr{B}$-measurable function in $L^1(S)$ and $g(s)$ is a $\mathscr{B}_{k-1}$-measurable function in $L^\infty(S)$. Since $\gamma_{p+q}(y_1,\ldots, y_{p+q}; s) = \gamma_p(y_1,\ldots, y_p; s)\gamma_q(y_{p+1},\ldots, y_{p+q}; \varphi_{y_p}\cdots\varphi_{y_1}(s))$, we have for such $f(s)$ and $g(s)$

$$\int_S T^n f(s)g(s)m(ds) = \int_S\left[\int_{Y^n} f(\varphi_{y_n}\cdots\varphi_{y_{k+1}}(\varphi_{y_k}\cdots\varphi_{y_1}(s)))\,\gamma_{n-k}(y_{k+1},\cdots, y_n; \varphi_{y_k}\cdots\varphi_{y_1}(s))\,\gamma_k(y_1,\cdots, y_k; s)\prod_{i=1}^n \nu(dy_i)\right]g(s)m(ds)$$
$$= \int_{Y^n}\left[\int_S f(\varphi_{y_n}\cdots\varphi_{y_{k+1}}(\varphi_{y_k}\cdots\varphi_{y_1}(s)))\,\gamma_{n-k}(y_{k+1},\ldots, y_n; \varphi_{y_k}\cdots\varphi_{y_1}(s))m(ds)\right]\left[\int_S g(s)\gamma_k(y_1,\ldots, y_k; s)m(ds)\right]\prod_{i=1}^n \nu(dy_i)$$
$$= \left[\int_S f(t)m(dt)\right]\left[\int_S g(s)\left[\int_{Y^k}\gamma_k(y_1,\ldots, y_k; s)\prod_{i=1}^k \nu(dy_i)\right]m(ds)\right]$$
$$= \int_S f(s)m(ds)\int_S g(s)m(ds)$$

where in the second equality we used the condition that $f(\varphi_{y_n}\cdots\varphi_{y_{k+1}}(\varphi_{y_k}\cdots\varphi_{y_1}(s)))\gamma_{n-k}(y_{k+1},\ldots, y_n; \varphi_{y_k}\cdots\varphi_{y_1}(s))$ and $g(s)\gamma_k(y_1,\ldots, y_k; s)$ are independent functions defined on $(S, \mathscr{B}, m)$ for fixed $y_1, y_2, \cdots, y_n$ because

$\varphi_{y_1}^{-1}\varphi_{y_2}^{-1}\cdots\varphi_{y_k}^{-1}\mathscr{B}$ and $\mathscr{B}_{k-1}$ are independent. This completes the proof.

Corollary 2. *The quadruplet* $(((\mathrm{Sim}_\delta(X)^N)^D, \mathscr{B}, \mathscr{B}_0, \mu^D), (\{1, 2,\dots, N\}, \mathscr{F}, \nu), \{Nr(S_i(s))^\alpha: s\in(\mathrm{Sim}_\delta(X)^N)^D\}, \{\varphi_i(s): i\in Y\})$ *introduced in Section 2 is mixing.*

This follows immediately from Theorem 6.

§4. Discrete parameter case

In this section we consider generalized random dynamical systems with discrete parameter space Y.

Consider a generalized dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s\in S\}, \Phi = \{\varphi_y: y\in Y\})$ which is defined as follows:

$S = [0, 1)$, $\mathscr{B}$ = the Borel field of $S = [0, 1)$ and m = the Lebesgue measure;

$Y = \mathbf{N} = \{1, 2, \cdots\}$, $\mathscr{F}$ = the Borel field of Y and ν = a probability measure on Y;

$\gamma(s, n) = p_n (0 \le s < 1/4$ or $3/4 \le s < 1)$, $\gamma(s, n) = q_n (1/4 \le s < 3/4)$ for $n\in Y = \mathbf{N}$, where p_n, $q_n \ge 0$ and $\sum p_n\nu(\{n\}) = \sum q_n\nu(\{n\}) = 1$;

$\varphi_n(s) = (n+1)s \pmod 1$ if n is odd and $\varphi_n(s) = -ns + 1 \pmod 1$ if n is even for $0 \le s < 1$;

$\mathscr{B}_0 = \{\varnothing, [0, 1/4)\cup[3/4, 1), [1/4, 3/4), [0, 1)\}$.

Then $\mathscr{D}$ is a generalized random dynamical system and φ_n are ergodic.

Let T be the induced Markov operator, i.e., $Tf(s) = \sum_n f(\varphi_n(s))\gamma(s, n)\nu(\{n\})$. Does there exist a non-constant T-invariant function?

The answer is negative if there exists an $n\in\mathbf{N}$ such that $\nu(\{n\}) > 0$ and $p_nq_n > 0$, because of Corollary 1 in Section 3.

On the other hand consider $\mathscr{D}$ such that $\nu(\{1\}) = \nu(\{2\}) = 1/2$ and $\nu(\{n\}) = 0$ for $n \ge 3$; $p_1 = q_2 = 2$ and $q_1 = p_2 = 0$. Note that this $\mathscr{D}$ does not satisfy the assumption of Corollary 1. A function $f(s)$ defined by $f(s) = \alpha$ $(0 \le s < 1/2)$ and $f(s) = \beta$ $(1/2 \le s < 1)$ is a T-invariant function. If $\alpha \ne \beta$, $f(s)$ is non-constant.

Appendix. Proof of Theorem 4

Proof of (a) ⇒ (b). Assume that a generalized random dynamical system $\mathscr{D} = ((S, \mathscr{B}, \mathscr{B}_0, m), (Y, \mathscr{F}, \nu), \{\gamma(s, y): s\in S\}, \Phi = \{\varphi_y: y\in Y\})$ is ergodic. If the induced Markov process is not ergodic, then there exists a $\mathscr{B}$-measurable subset B of S such that $0 < m(B) < 1$ and $P(s, B) = 1$ for a.e. $s\in B$. Then

$$\begin{aligned} m(B) &= \int_B P(s, B)m(ds) \\ &= \int_B \left[\int_Y I_B(\varphi_y(s))\gamma(s, y)\nu(dy)\right] m(ds) \\ &= \iint I_B(\varphi_y(s)) I_B(s)\gamma(s, y)\nu(dy)m(ds). \end{aligned}$$

It follows that

$$\iint I_B(s)(1\text{-}I_B(\varphi_y(s)))\gamma(s, y)\nu(dy)m(ds) = 0$$

and

$$\iint I_B(\varphi_y(s))(1\text{-}I_B(s))\gamma(s, y)\nu(dy)m(ds) = 0,$$

because $m(B) = \iint I_B(s)\gamma(s, y)\nu(dy)m(ds)$ and $m(B) = \int_S P(s, B)m(ds) = \iint I_B(\varphi_y(s))\gamma(s, y)\nu(dy)m(ds)$. This means that $\iint I_{(\varphi_y^{-1}(B)\Delta B)}(s)\gamma(s, y)\nu(dy)m(ds) = 0$. Since $0 < m(B) < 1$, this is a contradiction to the assumption that $\mathscr{D}$ is ergodic.

PROOF OF (b) $\Rightarrow$ (c). See the proof of (b) $\rightarrow$ (d) of Theorem 3 of Kakutani [3].

PROOF OF (c) $\Rightarrow$ (d). It suffices to show that under the assumption that

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_S T^k f(s) g(s) m(ds) = \int_S f(s)m(ds) \int_S g(s)m(ds)$$

for any functions $f(s) \in L^1(S, m)$ and $g(s) \in L^\infty(S, m)$, we have

$$\begin{aligned} \text{(2)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \int_{S\times Y^*} & f^*(\varphi^{*k}(s, y^*)) g^*(s, y^*) P(d(s, y^*)) \\ &= \int_{S\times Y^*} f^*(s, y^*) P(d(s, y^*)) \int_{S\times Y^*} g^*(s, y^*) P(d(s, y^*)) \end{aligned}$$

for any functions $f^*(s, y^*) \in L^1(S \times Y^*, P)$ and $g^*(s, y^*) \in L^\infty(S \times Y^*, P)$. Let $f^*(s, y^*)$ and $g^*(s, y^*)$ be of the form

$$\begin{aligned} f^*(s, y^*) &= f(s) \textstyle\prod_{i=1}^{l} f_i(y_i), \\ g^*(s, y^*) &= g(s) \textstyle\prod_{j=1}^{m} g_j(y_j) \end{aligned}$$

where $f(s) \in L^1(S)$, $g(s) \in L^\infty(S)$; l, m are positive integers; $f_i(y)$, $g_j(y) \in L^\infty(Y)$, i

$=1,\dots, l$, $j=1,\dots, m$. Since the linear combinations of such functions are everywhere dense in $L^1(S\times Y^*, P)$ and $L^\infty(S\times Y^*, P)$ respectively, it suffices to prove (2) only for the case when $f^*(s, y^*)$ and $g^*(s, y^*)$ are of the above forms. Let for $k>m$,

$$\begin{aligned}(*)_k := &\int_{S\times Y^*} f^*(\varphi^{*k}(s, y^*))g^*(s, y^*)P(d(s, y^*))\\ = &\iint_{S\times Y^{k+l}} f(\varphi_{y_k}\varphi_{y_{k-1}}\cdots\varphi_{y_1}(s))\textstyle\prod_{i=1}^{l} f_i(y_{i+k})g(s)\prod_{j=1}^{m} g_j(y_j)\\ &\gamma_{k+l}(y_1, y_2,\dots, y_{k+l}; s)\textstyle\prod_{i=1}^{l+k}\nu(dy_i)m(ds).\end{aligned}$$

Put

$$F(s)=\int_{Y^l} f(s)\textstyle\prod_{i=1}^{l} f_i(y_i)\gamma_l(y_1, y_2,\dots, y_l; s)\prod_{i=1}^{l}\nu(dy_i)$$

and

$$Vh(s)=\int_{Y^m} h(\varphi_{y_m}\varphi_{y_{m-1}}\cdots\varphi_{y_1}(s))\textstyle\prod_{j=1}^{m} g_j(y_j)\gamma_m(y_1, y_2,\dots, y_m; s)\prod_{j=1}^{m}\nu(dy_j).$$

Then it holds that

$$\begin{aligned}(*)_k &= \int_S V(T^{k-m}F(s))g(s)m(ds)\\ &= \int_S (T^{k-m}F(s))(V^*g(s))m(ds)\end{aligned}$$

where V^* denotes the bounded linear dual operator of V. It follows that by (1),

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\textstyle\sum_{k=0}^{n-1}&\int_{S\times Y^*} f^*(\varphi^{*k}(s, y^*))g^*(s, y^*)P(d(s, y^*))\\ &=\lim_{n\to\infty}\frac{1}{n}\textstyle\sum_{k=0}^{n-1}\int_S (T^{k-m}F(s))(V^*g(s))m(ds)\\ &=\int_S F(s)m(ds)\int_S V^*g(s)m(ds).\end{aligned}$$

On the other hand

$$\int_S F(s)m(ds)=\int_{S\times Y^*} f^*(s, y^*)P(d(s, y^*)),$$

and

$$\int_S V^*g(s)m(ds)=\int_S (V1)(g(s))m(ds)$$

$$= \iint_{S\times Y^*} \prod_{j=1}^{m} g_j(y_j)\gamma_m(y_1, y_2, \ldots, y_m; s)\prod_{j=1}^{m} \nu(dy_j)g(s)m(ds)$$

$$= \int_{S\times Y^*} g^*(s, y^*)P(d(s, y^*)).$$

This completes the proof.

PROOF OF (d) ⇒ (a). Assume that φ^* is ergodic on $(S \times Y^*, P)$. If the generalized random dynamical system $\mathscr{D}$ is not ergodic, there exists a $\mathscr{B}$-measurable subset B of S such that $0 < m(B) < 1$ and $\int_S\int_Y I_{(\varphi_y^{-1}(B)\Delta B)}(s)\gamma(s, y)\nu(dy)m(ds) = 0$. Put $B^* = B \times Y^*$, then

$$P(\varphi^{*-1}(B^*) \triangle B^*) = \int_{S\times Y^*} I_{(\varphi^{*-1}(B^*)\Delta B^*)}(s, y^*)P(d(s, y^*))$$

$$= \int_S\int_Y I_{(\varphi_y^{-1}(B)\Delta B)}(s)\gamma(s, y)\nu(dy)m(ds) = 0.$$

This means that B^* is φ^*-invariant and $0 < P(B^*) = m(B) < 1$. This is a contradiction to the assumption that φ^* is ergodic on $(S \times Y^*, P)$.

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Department of Computer Sciences,
Faculty of Science,
Kyoto Sangyo University