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Huygens property of parabolic functions and a uniqueness theorem

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It is well known that the zero function is the only non-negative parabolic function (i.e., solution of the heat equation) on $\boldsymbol{R}^n \times (0, T)$ which vanishes continuously on $\boldsymbol{R}^n \times \{0\}$ (cf. [2, Part 1, Chap. XVI], [6, Chap. VIII]). In this paper, considering the Huygens property of minimal parabolic functions (see Proposition 2), we shall show the following generalization of the above result.

THEOREM. *Let Ω be a Lipschitz cone in $\boldsymbol{R}^n$ $(n \geq 1)$ and let $0 < T \leq \infty$. If u is a non-negative parabolic function in the cylinder $\Omega \times (0, T)$ and vanishes continuously on the parabolic boundary $\partial\Omega \times [0, T] \cup \Omega \times \{0\}$, then $u \equiv 0$.*

Here, we say that a domain Ω is a *Lipschitz cone* if there exists a domain E in the unit sphere S^{n-1} such that $\Omega = \{x \neq 0; x/\|x\| \in E\}$ and $\Omega \cap \{x; \|x\| < 1\}$ is a Lipschitz domain, where $\|x\|$ denotes the euclidean norm of x in $\boldsymbol{R}^n$.

In Section 5, we remark that the above assertion is also valid for solutions of parabolic equations and for a slightly more general domain Ω.

§1. The Huygens property

In this section $D = \Omega \times (0, T)$ will be a cylinder with a domain Ω in $\boldsymbol{R}^n$ $(n \geq 1)$ and $0 < T \leq \infty$.

A solution of the heat equation on D is said to be *parabolic* on D. We denote by $H^+(D)$ the set of all non-negative parabolic functions on D. Also we denote by $\partial_p D$ the parabolic boundary of D, i.e., $\partial_p D = \partial\Omega \times [0, T) \cup \Omega \times \{0\}$, and by G_D the Green function of D with respect to the heat equation (cf. [2, p.298]).

For a non-negative function u and $0 < s < T$, we define a function u_s by

$$u_s(x, t) = \begin{cases} u(x, t) & \text{on} \quad \Omega \times (0, s] \\ \displaystyle\int_\Omega G_D((x, t), (y, s))\, u(y, s)\, dy & \text{on} \quad \Omega \times (s, T). \end{cases}$$

Then we have

LEMMA 1. *For $u \in H^+(D)$ and $0 < s < T$, $u_s \in H^+(D)$ and $u_s \leq u$ on D.*

PROOF. Let $\{\Omega_m\}_{m=1}^{\infty}$ be a regular exhaustion of Ω, that is, Ω_m is relatively compact, $\partial\Omega_m$ is sufficiently smooth and $\bar{\Omega}_m \subset \Omega_{m+1}$ for each m, and $\bigcup_{m=1}^{\infty} \Omega_m = \Omega$. Define $D_m = \Omega_m \times (0, T)$ and

$$v_m(x, t) = \begin{cases} u(x, t) & \text{on } \quad \Omega_m \times (0, s] \\ \displaystyle\int_{\Omega m} G_{D_m}((x, t), (y, s))u(y, s)dy & \text{on } \quad \Omega_m \times (s, T). \end{cases}$$

Then clearly v_m is parabolic on $\Omega_m \times (0, s) \bigcup \Omega_m \times (s, T)$ and continuous on D_m. This implies $v_m \in H^+(D_m)$ (cf. [2, p.276]). Since $v_m \leq u$ on $\partial_p D_m$, the maximum principle implies $v_m \leq u$ on D_m. Since $G_{D_m} \uparrow G_D$, $v_m \uparrow u_s$ as $m \to \infty$. Hence letting $m \to \infty$ we have $u_s \leq u$ on D. The parabolicity of u_s follows from the Harnack convergence theorem for parabolic functions (cf. [2, p.276]). This completes the proof.

In [6, Chap.IX], D.V. Widder discussed the semi-group property of parabolic functions in a strip, which he called the Huygens property. Following Widder, we say that $u \in H^+(D)$ with $u \not\equiv 0$ has the *Huygens property* if $u = u_s$ for any $t_u < s < T$, where $t_u = \inf\{t > 0$; there exists $x \in \Omega$ such that $u(x, t) > 0\}$.

PROPOSITION 2. *Every non-zero minimal parabolic function has the Huygens property.*

Here $u \in H^+(D)$ is called *minimal* if every $v \in H^+(D)$ satisfying $0 \leq v \leq u$ is a constant multiple of u.

PROOF. Let $u \in H^+(D)$, $u \not\equiv 0$, be minimal and $0 < s < T$. By Lemma 1, $u_s = c_s u$ with some constant $c_s \geq 0$. If $s > t_u$ there exists $y \in \Omega$ with $u_s(y, s) = u(y, s) \neq 0$. This implies $c_s = 1$ and the proof is completed.

§2. The Harnack and the boundary Harnack principles

In this section Q will be a cylinder $R \times (0, T)$ with a bounded Lipschitz domain R in $\boldsymbol{R}^n$ and $T > 0$. Let $\{m, r_0\}$ be the *Lipschitz character* of R, i.e., m and r_0 are two positive numbers and for each $X \in \partial R$ there exist a coordinate system of $\boldsymbol{R}^n$ and a Lipschitz function φ on $\boldsymbol{R}^{n-1}$ with $\| \nabla \varphi \|_\infty \leq m$ such that in these coordinates,

$$B \cap R = B \cap \{(x', x_n);\ x_n > \varphi(x')\}$$

and

$$B \cap \partial R = B \cap \{(x', \varphi(x'))\},$$

where $B = B(X, r_0)$, the open ball in $\boldsymbol{R}^n$ of radius $r_0 > 0$ centered at X.

From the Harnack and the boundary Harnack principles for parabolic functions (cf. [1], [3], [4], [5]), we derive the following two lemmas which we shall use later. We denote by $d(R)$ the diameter of R and $\delta(x) = \mathrm{dist}(x, \partial R)$.

LEMMA 3. *There exists a constant $c = c(n, m, r_0, d(R))$ with the following property: If $u \in H^+(Q)$, then*

$$u(y, s) \leq \exp\left[c\left(\frac{1}{t-s} + \frac{t-s}{\delta}\right)\right]u(x, t)$$

for all $x, y \in R$ and $0 < s < t < T$, where $\delta = \min\{1, s, \delta(x)^2, \delta(y)^2\}$.

PROOF. We first remark that there exists a constant $c_1 = c(m, r_0)$ with $0 < c_1 \leq 1$ such that for any $z \in R$, there exists a z_0 with $\delta(z_0) \geq c_1$ satisfying $\mathrm{dist}(\overline{zz_0}, \partial R) \geq c_1\delta(z)$, where $\overline{zz_0}$ is the closed segment $[z, z_0]$. We can take a constant $c_2 = c(m, r_0)$ with $0 < c_2 < c_1$ and a connected compact set K satisfying $\{z \in R;\ \delta(z) > c_1\} \subset K \subset \{z \in R;\ \delta(z) > c_2\}$. Since K is compact there also exist an integer $n_0 \geq 1$ depending only on n, c_2 and $d(R)$, and n_0 points $\{z_1, \dots, z_{n_0}\}$ in K such that $\bigcup_{i=1}^{n_0} B(z_i, c_2/2) \supset K$.

Let (x, t), (y, s) be as in the lemma. We choose $x_0, y_0 \in K$ satisfying $\mathrm{dist}(\overline{xx_0}, \partial R) \geq c_1\delta(x)$ and $\mathrm{dist}(\overline{yy_0}, \partial R) \geq c_1\delta(y)$. Further we can choose a subset $\{y_j\}_{j=1}^k \subset \{z_i\}$ such that $\mathrm{dist}(\overline{y_jy_{j+1}}, \partial R) \geq c_2/2$ $(0 \leq j \leq k)$, where $y_{k+1} = x_0$. Put $y_{-1} = y$, $y_{k+2} = x$ and $s_j = s + (j+1)(t-s)/(k+3)$. Then $\{(y_j, s_j)\}_{j=-1}^{k+2}$ are points in Q. By the Harnack principle ([1, Theorem 5], [3, Theorem 0.1]), for $j = -1, 0, \dots, k+2$,

$$u(y_j, s_j) \leq \exp\left[c_0\left(\frac{(k+3)d(R)^2}{t-s} + \frac{t-s}{(k+3)\delta_j} + 1\right)\right]u(y_{j+1}, s_{j+1})$$

with some constant $c_0 = c(n) > 0$, where $\delta_j = \min\{1, s_j, \mathrm{dist}(\overline{y_jy_{j+1}}, \partial R)^2\}$. Since $\delta_j \geq (c_2/2)^2\delta$, we have the desired inequality with $c = c_0c_2^{-1}(n_0+3)^2(d(R)+1)^2$ by multiplying the above inequalities.

LEMMA 4. *Let $(x_0, t_0) \in Q$ with $T/3 < t_0 < T$ and let $Q' = R' \times (0, s_0)$, $0 < s_0 < t_0$, be a subcylinder of Q. Let F be a compact subset of ∂R such that $\partial R' \cap \partial R$ is compactly contained in F. Put $\Gamma = \bar{R} \times \{0\} \cup F \times (0, T)$. Then there exists a constant $c = c(n, m, r_0, d(R), x_0, T, \mathrm{dist}(R', \partial R \setminus F))$ independent of t_0 and s_0 such that for any $u \in H^+(Q)$ vanishing continuously on Γ, we have*

$$u(y, s) \leq \exp\left(\frac{c}{t_0 - s_0}\right)u(x_0, t_0) \qquad \textit{for all} \qquad (y, s) \in Q'.$$

PROOF. Let $Q_1 = R' \times (0, T/4)$. By [5, Theorem 3.1] there is a constant $c_1 = c(n, m, r_0, x_0, T, \mathrm{dist}(R', \partial R \setminus F)) > 1$ such that

$$\sup_{(y,s) \in Q_1} u(y, s) \leq c_1 u(x_0, t_0).$$

Hence the required inequality holds with $c = T\log c_1$ if $s_0 \leq T/4$. Next, assume $T/4 < s_0 < t_0$. Set

$$2\eta = \min\{[(t_0 - s_0)/2]^{1/2},\ r_0,\ \mathrm{dist}(R', \partial R\backslash F),\ \delta(x_0),\ 1\}$$

and

$$Q_2 = \bigcup\{B(X, \eta) \times (s - \eta^2, s + \eta^2) \cap Q';\ X \in F,\ T/4 \leq s < s_0\},$$
$$Q_3 = \{x \in R;\ \delta(x) \geq \eta/(m+1)\} \times [T/4,\ s_0 + 2\eta^2).$$

Then $Q' \backslash Q_1 \subset Q_2 \cup Q_3 \subset Q$. By the Carleson estimate ([3, Theorem 0.3], [5, Theorem 3.1]), we have

$$\sup_{(y,s)\in Q_2} u(y, s) \leq c_2 \sup_{(y,s)\in Q_3} u(y, s)$$

with $c_2 = c(n, m, r_0) > 0$. Further, by Lemma 3, we have

$$\sup_{(y,s)\in Q_3} u(y, s) \leq \exp\left[c_3\left(\frac{1}{t_0 - s_0} + \frac{T}{\delta_1}\right)\right] u(x_0, t_0)$$

with $c_3 = c(n, m, r_0, d(R)) > 0$, where $\delta_1 = \min\{1,\ T/4,\ (\eta/(m+1))^2\}$. By the choice of η, there is $c_4 = c(r_0, T, \mathrm{dist}(R', \partial R\backslash F), \delta(x_0)) > 0$ such that if $t_0 - s_0 < c_4$ then $\delta_1 = (t_0 - s_0)/8(m+1)^2$. Thus we have

$$\sup_{(y,s)\in Q_2 \cup Q_3} u(y, s) \leq \exp\left(\frac{c_5}{t_0 - s_0}\right) u(x_0, t_0)$$

with $c_5 = c(n, m, r_0, d(R), T, x_0, \mathrm{dist}(R', \partial R\backslash F)) > 0$. This completes the proof.

§3. Compact convex set $H_0^+(D, \mu)$

In this section let $D = \Omega \times (0, T)$, where $0 < T \leq \infty$ and Ω is a domain in $\boldsymbol{R}^n$ such that $\Omega \cap B(0, n)$ are Lipschitz domains for all $n \geq 1$.

We denote by $H(D)$ the space of all parabolic functions on D endowed with the topology of uniform convergence on compact sets and set

$$H_0^+(D) = \{u \in H(D);\ u \geq 0 \text{ and continuously vanishes on } \partial_p D\}.$$

A positive measure μ on D is called a *reference measure* if every point of D can be connected to some point of $\mathrm{supp}(\mu)$, the support of μ, by a polygonal line having strictly increasing t-coordinate. For a reference measure μ on D we put

$$H_0^+(D, \mu) = \{u \in H_0^+(D);\ \int_D u\,d\mu \leq 1\}.$$

Then we have

LEMMA 5. *$H_0^+(D, \mu)$ is a compact convex set in $H(D)$.*

PROOF. The convexity is evident. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $H_0^+(D, \mu)$. Since μ is a reference measure, the Harnack principle (cf. [2, p.277]) shows that there exists a subsequence which converges to some element u_∞ in $H(D)$. What remains is to prove that u_∞ continuously vanishes on $\partial_p D$. Let $(Y, s) \in \partial_p D$. We choose $t_1, t_2 > 0$ with $0 \leq s < t_1 < t_2 < T$ and $x_0 \in \Omega$ with $\|x_0 - Y\| < 1$. Set $Q = \{x \in \Omega; \|x - y\| < 1\} \times (0, t_1)$. Since $u_n \in H_0^+(D)$, by Lemma 4,

$$u_n(x, t) \leq c u_n(x_0, t_2) \quad \text{for} \quad (x, t) \in Q$$

with some $c > 0$ independent of n. We denote by v the parabolic function on Q with boundary values 1 on $\partial_p Q \cap D$ and 0 on $\partial_p Q \cap \partial_p D$. Then $u_n(x, t) \leq c u_n(x_0, t_2) v(x, t)$ and hence $u_\infty(x, t) \leq c u_\infty(x_0, t_2) v(x, t)$ on Q. Thus u_∞ continuously vanishes at (Y, s). This completes the proof.

§4. Proof of the Theorem

Let Ω be a Lipschitz cone and let $D = \Omega \times (0, T)$ with $0 < T \leq \infty$. We set $\Omega(r) = \{x \in \Omega;\ \|x\| < r\}$ for $r > 0$.

We first show

LEMMA 6. *Let $x_0 \in \Omega(1)$ and let $0 < t_0 < T$. Then there exists a constant $c = c(x_0, t_0, \Omega) > 0$ such that for any $u \in H_0^+(D)$ and any $0 < \tau < 1$ with $t_0 + 2\tau < T$,*

$$u(y, s) \leq \exp(c(\|y\|^2 + 1)/\tau) u(x_0, t_0 + \tau)$$

whenever $(y, s) \in \Omega \times (0, t_0 + \tau/2)$.

PROOF. Let u and τ be as in the lemma. We may assume that $u \not\equiv 0$. By putting $u = 0$ on $\Omega \times (-\infty, 0]$, we may also assume that $u \in H^+(\Omega \times (-\infty, T))$. Put $Q = \Omega(2) \times (0, t_0 + 2\tau)$. By Lemma 3, there exists a constant $c_0 = c(\Omega, x_0) > 0$ such that for any $r \geq 1$,

$$u\left(x_0,\ t_0 + 2\tau - \frac{4\tau}{3r^2}\right) \leq \exp(c_0 r^2/\tau) u\left(x_0/r,\ t_0 + 2\tau - \frac{\tau}{r^2}\right)$$

and by Lemma 4, there is a constant $c_0' = c(\Omega, x_0, t_0) > 0$ such that for any $r \geq 1$ and for any $(y, s) \in \Omega(1) \times (0, t_0 + 2\tau - 3\tau/2r^2)$

$$u(y, s) \leq \exp(c_0' r^2/\tau) u\left(x_0,\ t_0 + 2\tau - \frac{4\tau}{3r^2}\right).$$

Since the change of variables $(x, t) \to (x/r, (t - a)/r^2 + a)$, $a = t_0 + 2\tau$, preserves the parabolicity of a function, by the above inequalities, we have

$$u(r x_0,\ t_0 + 2\tau/3) \leq \exp(c_0 r^2/\tau) u(x_0,\ t_0 + \tau)$$

and

$$u(y, s) \leq \exp(c_0' r^2/\tau) u(rx_0, t_0 + 2\tau/3)$$

for $(y, s) \in \Omega(r) \times (0, t_0 + \tau/2)$. Hence if $y \in \Omega(r) \backslash \Omega(r-1)$ and $0 < s < t_0 + \tau/2$, we have

$$u(y, s) \leq \exp(cr^2/2\tau) u(x_0, t_0 + \tau) \leq \exp(c(\|y\|^2 + 1)/\tau) u(x_0, t_0 + \tau)$$

with $c = 2(c_0 + c_0')$.

Next we shall show

LEMMA 7. *For any reference measure μ, $H_0^+(D, \mu) = \{0\}$.*

PROOF. Let u be any extremal point of $H_0^+(D, \mu)$ and assume $u \not\equiv 0$. Then $u(x, t) > 0$ whenever $(x, t) \in \Omega \times (t_u, T)$. Let $x_0 \in \Omega(1)$ and $0 < \tau < 1$ with $t_u + 2\tau < T$. Further we choose $\eta > 0$ with $\eta < \min\{\tau, \tau/4c\}$, where c is the constant in Lemma 6 with $t_0 = t_u$. Then if $t_u < s < t_u + \eta/2$ we have

$$\begin{aligned} &G_D((x_0, t_u + \eta), (y, s)) u(y, s) \\ &\quad \leq c_0 \eta^{-n/2} \exp\left(-\frac{\|x_0 - y\|^2}{4\eta} + \frac{c(\|y\|^2 + 1)}{\tau} \right) u(x_0, t_u + \tau) \end{aligned}$$

with some constant $c_0 > 0$. By the choice of η,

$$\int_\Omega \exp(-\|x_0 - y\|^2/4\eta + c(\|y\|^2 + 1)/\tau) dy < \infty.$$

On the other hand, we see easily that u is minimal. Hence by proposition 2

$$u(x_0, t_u + \eta) = \int_\Omega G_D((x_0, t_u + \eta), (y, s)) u(y, s) dy$$

for any $t_u < s < t_u + \eta/2$. By the above estimate of the integrand, we can apply the Lebesgue convergence theorem, and obtain $u(x_0, t_u + \eta) = 0$, since $\lim_{s \to t_u} u(y, s) = 0$. This is a contradiction, and hence 0 is the only extremal point of $H_0^+(D, \mu)$. The Krein-Milman theorem gives $H_0^+(D, \mu) = \{0\}$.

PROOF OF THE THEOREM. It is sufficient to remark that $H_0^+(D) = \{0\}$ if and only if $H_0^+(D, \mu) = \{0\}$ for any reference measure μ.

§5. A generalization

Let L be a parabolic operator on $\boldsymbol{R}^n \times (-\infty, \infty)$ of the form

$$Lu(x, t) = \sum_{i,j=1}^n D_{x_i}\left(a_{ij}(x, t) D_{x_j} u(x, t) \right) - u_t(x, t),$$

where the matrix $(a_{ij}(x, t))$ is bounded, measurable, symmetric, and uniformly positive definite, i.e., there exists $\lambda \geq 1$ such that for all $x \in \boldsymbol{R}^n$, $\xi \in \boldsymbol{R}^n$ and $t \in (-\infty, \infty)$

$$\frac{1}{\lambda}\|\xi\|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \leq \lambda\|\xi\|^2.$$

A solution of $Lu = 0$ on a domain D will be called *L-parabolic* on D.

By slightly modifying the arguments in the previous sections, we can also prove the following

THEOREM 8. *Let Ω be an unbounded domain in $\boldsymbol{R}^n$ such that $\Omega_k = \{x/k; x \in \Omega, \|x\| \leq k\}$, $k = 1, 2, \cdots$, are all Lipschitz domains whose Lipschitz characters are uniformly bounded. If u is a non-negative L-parabolic function on the cylinder $D = \Omega \times (0, T)$, $0 < T \leq \infty$, and vanishes continuously on the parabolic boundary $\partial_p D$, then $u \equiv 0$.*

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