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Error bounds for asymptotic expansions of the maximums of the multivariate t- and F-variables with common denominator

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1. Introduction

Let $\boldsymbol{X} = (X_1, \dots, X_p)$ be a scale mixture of a p-dimensional random vector $\boldsymbol{Z} = (Z_1, \dots, Z_p)$ with scale factor $\sigma > 0$, i.e.,

$$\boldsymbol{X} = \sigma \boldsymbol{Z}, \tag{1.1}$$

where $\boldsymbol{Z}$ and σ are independent. Let F_p and Q_p denote the distribution functions of $\boldsymbol{X}$ and $\boldsymbol{Z}$, respectively. Then

$$\begin{aligned} F_p(\boldsymbol{x}) &= P(X_1 \le x_1, \dots, X_p \le x_p) \\ &= E_\sigma[Q_p(\sigma^{-1}\boldsymbol{x})], \end{aligned} \tag{1.2}$$

where $\boldsymbol{x} = (x_1, \dots, x_p)$. The distribution function of $\mathrm{Max}\{X_j\}$ is given by $F_p(x, \dots, x)$. We are concerned with asymptotic expansions of the distribution functions of $\mathrm{Max}\{X_j\}$ and their error bounds in the two important special cases:

(i) $\quad Z_1, \dots, Z_p$ i.i.d. $\sim N(0, 1)$, $\quad \sigma = (\chi_n^2/n)^{1/2}$,

(ii) $\quad Z_1, \dots, Z_p$ i.i.d. $\sim G(\lambda)$, $\quad \sigma = \chi_n^2/n$,

where $G(\lambda)$ denotes the gamma distribution with the probability density function $g(x; \lambda) = x^{\lambda-1}e^{-x}/\Gamma(\lambda)$, if $x > 0$, and $= 0$, *if* $x \le 0$. The random vector $\boldsymbol{X}$ in the case (i) is a multivariate t-variable with common denominator. The random vector $\boldsymbol{X}$ in the case (ii) is essentially equivalent to a multivariate F-variable with common denominator. These distributions are used in simultaneous inferences about the means of normal populations. It may be noted that asymptotic expansions of the distributions of $\mathrm{Max}\{X_j\}$ in the cases (i) and (ii) have been studied by Hartley [6], Nair [7], Dunnett and Sobel [2], Chambers [1], etc. The purpose of this paper is to give a unified derivation of the asymptotic expansions as well as their error bounds.

In Section 2 we give two types of asymptotic approximations for the distribution function of X and their error bounds. The one is newly given, but the other has been given in Fujikoshi and Shimizu [5]. In Section 3 we consider the distribution of $\mathrm{Max}\{X_j\}$ in the case when $Z_1,\dots, Z_p$ are independent and identically distributed. The results obtained are based on further reductions of the general results in Section 2. In Section 4 we obtain asymptotic expansions of the distributions of $\mathrm{Max}\{X_j\}$ and their error bounds in the two cases (i) and (ii).

2. The distribution of X

We assume that the support of $\boldsymbol{Z}$ is either $\Omega = \mathrm{R}^p$ or R^p_+, and Q_p is k times continuously differentialbe on Ω. We consider the following two types of approximations for the function $Q_p(\sigma^{-1}\mathrm{x})$ in (1.2):

$$A_{p,\delta,k}(\boldsymbol{x}, \sigma) = \sum_{j=0}^{k-1}\frac{1}{j!}a_{p,\delta,j}(\boldsymbol{x})(\sigma^{2\delta}-1)^j, \tag{2.1}$$

$$B_{p,\delta,k}(\boldsymbol{x}, \sigma) = \sum_{j=0}^{k-1}\frac{1}{j!}b_{p,\delta,j}(\boldsymbol{x})(\sigma^{\delta}-1)^j, \tag{2.2}$$

where $\delta = -1$ or 1, and

$$a_{p,\delta,j}(\boldsymbol{x}) = (d^j/ds^j)Q_p(s^{-\delta/2}\boldsymbol{x})\big|_{s=1}, \tag{2.3}$$

$$b_{p,\delta,j}(\boldsymbol{x}) = (d^j/ds^j)Q_p(s^{-\delta}\boldsymbol{x})\big|_{s=1}. \tag{2.4}$$

The approximation $A_{p,\delta,k}(\boldsymbol{x}, \sigma)$ is newly introduced, but $B_{p,\delta,k}(\boldsymbol{x}, \sigma)$ has been given in Fujikoshi and Shimizu [5]. In Section 4 we shall see that $A_{p,\delta,k}(\boldsymbol{x}, \sigma)$ in the case of $p = 1$ is the same as the previous one due to Fujikoshi [3] and Fujikoshi and Shimizu [5]. Under the appropriate conditions on the moments of σ we propose the following two types of approximations for the distribution function of $\boldsymbol{X}$:

$$\begin{aligned} A_{p,\delta,k}(\boldsymbol{x}) &= E_\sigma[A_{p,\delta,k}(\boldsymbol{x}, \sigma)] \\ &= Q_p(\boldsymbol{x}) + \sum_{j=1}^{k-1}\frac{1}{j!}a_{p,\delta,j}(\boldsymbol{x})E\{(\sigma^{2\delta}-1)^j\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} B_{p,\delta,k}(\boldsymbol{x}) &= E_\sigma[B_{p,\delta,k}(\boldsymbol{x}, \sigma)] \\ &= Q_p(\boldsymbol{x}) + \sum_{j=1}^{k-1}\frac{1}{j!}b_{p,\delta,j}(\boldsymbol{x})E\{(\sigma^{\delta}-1)^j\}. \end{aligned} \tag{2.6}$$

It will be seen that the approximations $A_{p,\delta,k}(\boldsymbol{x})$ and $B_{p,\delta,k}(\boldsymbol{x})$ are useful for the cases (i) and (ii), respectively. In the following we list all the assumptions used in this paper.

1: *Q_p is k times continuously differentiable on $\Omega = \mathrm{R}^p$ or R^p_+,*

A1(δ): $\bar{a}_{p,\delta,k} = \sup_x|a_{p,\delta,k}(\boldsymbol{x})| < \infty$, A2: $E(\sigma^{2k}) < \infty,\ E(\sigma^{-2k}) < \infty,$

A3(δ): $\bar{a}_{p,\delta,k}(\ell) = \sup_x(1 + \|x\|^{\ell})|a_{p,\delta,k}(x)| < \infty,$

A4: $E(\sigma^{2k+\ell}) < \infty,\ E(\sigma^{-2k}) < \infty,$

B1(δ): $\bar{b}_{p,\delta,k} = \sup_x|b_{p,\delta,k}(x)| < \infty,$

B2: $E(\sigma^{k}) < \infty,\ E(\sigma^{-k}) < \infty,$

B3(δ): $\bar{b}_{p,\delta,k}(\ell) = \sup_x(1 + \|x\|^{\ell})|b_{p,\delta,k}(x)| < \infty,$

B4: $E(\sigma^{k+\ell}) < \infty,\ E(\sigma^{-k}) < \infty.$

LEMMA 2.1. *Suppose that* $Q_p(x)$ *satisfies Assumption* 1.
(i) *Under Assumption* A1(δ) *it holds that*

$$\sup_x|Q_p(\sigma^{-1}x) - A_{p,\delta,k}(x, \sigma)| \le \frac{1}{k!}\bar{a}_{p,\delta,k}(\sigma^2 \vee \sigma^{-2} - 1)^k \le \frac{1}{k!}\bar{a}_{p,\delta,k}\{|\sigma^2 - 1|^k + |\sigma^{-2} - 1|^k\}. \tag{2.7}$$

(ii) *Under Assumption* B1(δ) *it holds that*

$$\sup_x|Q_p(\sigma^{-1}x) - B_{p,\delta,k}(x, \sigma)| \le \frac{1}{k!}\bar{b}_{p,\delta,k}(\sigma \vee \sigma^{-1} - 1)^k \le \frac{1}{k!}\bar{b}_{p,\delta,k}\{|\sigma - 1|^k + |\sigma^{-1} - 1|^k\}. \tag{2.8}$$

PROOF. (ii) has been proved by Fujikoshi and Shimizu [5]. We shall show (i). Letting $s = \sigma^{2\delta}$ and considering Taylor's expansion of $Q_p(s^{-\delta/2}x)$ around $s = 1$, we have

$$Q_p(\sigma^{-1}x) = A_{p,\sigma,k}(x, \sigma) + \Delta_{p,\delta,k}(x, \sigma), \tag{2.9}$$

where

$$\Delta_{p,\delta,k}(x, \sigma) = \frac{1}{k!}(\sigma^{2\delta} - 1)^k \frac{d^k}{ds^k}Q_p(s^{-\delta/2}x)\Big|_{s=1+\theta(\sigma^{2\delta}-1)}$$

and $0 \le \theta \le 1$. We can write

$$\Delta_{p,\delta,k}(x, \sigma) = \frac{1}{k!}a_{p,\delta,k}(t)\{1 + \theta(\sigma^{2\delta} - 1)\}^{-k}(\sigma^{2\delta} - 1)^k, \tag{2.10}$$

where $t = \{1 + \theta(\sigma^{2\delta} - 1)\}^{-\delta/2}x$. Noting that $0 \le \theta \le 1$, we have

$$|1 + \theta(\sigma^{2\delta} - 1)|^{-k}|\sigma^{2\delta} - 1|^k \le (\sigma^2 \vee \sigma^{-2} - 1)^k \le |\sigma^2 - 1|^k + |\sigma^{-2} - 1|^k$$

which proves (i).

THEOREM 2.1. *Suppose that* $X=\sigma Z$ *is a scale mixture of* Z satisfying Assumption 1.

(i) *Under Assumptions* A1(σ) *and* A2 it holds that

$$\sup_x |F_p(x) - A_{p,\sigma,k}(x)| \le \frac{1}{k!}\bar{a}_{p,\sigma,k} E\{(\sigma^2 \vee \sigma^{-2} - 1)^k\} \tag{2.11}$$

$$\le \frac{2}{k!}\bar{a}_{p,\sigma,k} E\{|\sigma^2 - 1|^k + |\sigma^{-2} - 1|^k\}.$$

(ii) *Under Assumptions* B1(δ) *and* B2 *it holds that*

$$\sup_x |F_p(x) - B_{p,\sigma,k}(x)| \le \frac{1}{k!}\bar{b}_{p,\sigma,k} E\{(\sigma \vee \sigma^{-1} - 1)^k\} \tag{2.12}$$

$$\le \frac{1}{k!}\bar{b}_{p,\sigma,k} E\{|\sigma - 1|^k + |\sigma^{-1} - 1|^k\}.$$

PROOF. The results (i) and (ii) follow immediately from (1.2) and Lemma 2.1. The second result (ii) was obtained by Fujikoshi and Shimizu [5].

Next we derive nonuniform error bounds in approximating $F_p(x)$ by $A_{p,\sigma,k}(x)$ or $B_{p,\sigma,k}(x)$, which are improvements on the uniform bounds in the tail part of the distribution of X. The following lemma, which is an extension of Fujikoshi [4] to the multivariate case, is fundamental in our nonuniform error bounds.

LEMMA 2.2. *Suppose that* $Q_p(x)$ *satisfies Assumption* 1.
(i) *Under Assumption* A3(σ) *it holds that*

$$(1 + \|x\|^\ell)|Q_p(\sigma^{-1}x) - A_{p,\delta,k}(x, \sigma)| \tag{2.13}$$

$$\le \frac{1}{k!}\bar{a}_{p,\delta,k}(\ell)(\sigma^\ell \vee 1)(\sigma^2 \vee \sigma^{-2} - 1)^k$$

$$\le \frac{1}{k!}\bar{a}_{p,\delta,k}(\ell)\{\sigma^\ell|\sigma^2 - 1|^k + |\sigma^{-2} - 1|^k\}.$$

(ii) *Under Assumption* B3(σ) *it holds that*

$$(1 + \|x\|^\ell)|Q_p(\sigma^{-1}x) - B_{p'\delta,k}(x, \sigma)| \tag{2.14}$$

$$\le \frac{1}{k!}\bar{b}_{p,\delta,k}(\ell)(\sigma^\ell \vee 1)(\sigma \vee \sigma^{-1} - 1)^k$$

$$\le \frac{1}{k!}\bar{b}_{p,\delta,k}(\ell)\{\sigma^\ell|\sigma - 1|^k + |\sigma^{-1} - 1|^k\}.$$

PROOF. Using (2.9) and (2.10), we have

$$(1 + \|x\|^{\ell})|\Delta_{p,\sigma,k}(x, \sigma)| = \{1 + |1 + \theta(\sigma^{2\delta} - 1)|^{\ell\delta/2}\|t\|^{\ell}\} \times \frac{1}{k!}|a_{p,\delta,k}(t)||(\sigma^{2\delta} - 1)|^{k}|1 + \theta(\sigma^{2\delta} - 1)|^{-k}. \tag{2.15}$$

The first factor of the right-hand side in (2.15) is bounded by

$$\begin{cases} 1 + \sigma^{\ell}\|t\|^{\ell}, & \text{if } \sigma \geq 1, \\ 1 + \|t\|^{\ell}, & \text{if } 0 < \sigma < 1, \end{cases} \leq (1 + \|t\|^{\ell})(1 \vee \sigma^{\ell}).$$

This result and Lemma 2.1(i) imply (i). Similarly we can prove (ii).

Lemma 2.2 implies the following Theorem 2.2.

THEOREM 2.2. *Suppose that $X = \sigma Z$ is a scale mixture of Z satisfying Assumption 1.*

(i) *Under Assumptions* A3(σ) *and* A4 *it holds that*

$$|F_p(x) - A_{p,\delta,k}(x)| \leq \frac{1}{k!}(1 + \|x\|^{\ell})^{-1}\bar{a}_{p,\delta,k}(\ell) \times E\{\sigma^{\ell}|\sigma^2 - 1|^k + |\sigma^{-2} - 1|^k\}. \tag{2.16}$$

(ii) *Under Assumptions* B3(σ) *and* B4 *it holds that*

$$|F_p(x) - B_{p,\delta,k}(x)| \leq \frac{1}{k!}(1 + \|x\|^{\ell})^{-1}\bar{b}_{p,\delta,k}(\ell) \times E\{\sigma^{\ell}|\sigma - 1|^k + |\sigma^{-1} - 1|^k\}. \tag{2.17}$$

The results (2.16) and (2.17) in the special case of $p = 1$ were obtained by Fujikoshi [4].

3. The distribution of Max$\{X_1, \cdots, X_p\}$

The distribution function of Max$\{X_j\}$ can be expressed as

$$P(\mathrm{Max}\{X_j\} \leq x) = P(X_1 \leq x, \cdots, X_p \leq x) = F_p(x, \cdots, x). \tag{3.1}$$

Therefore we can get two types of approximations for $P(\mathrm{Max}\{X_j\} \leq x)$ and their error bounds from Theorems 2.1 and 2.2 by putting $x_1 = \cdots = x_p = x$. Let $a^{[p]}_{p,\delta,k}(x)$, $A^{[p]}_{p,\delta,k}(x)$, $b^{[p]}_{p,\delta,k}(x)$ and $B^{[p]}_{p,\delta,k}(x)$ denote $a_{p,\delta,k}(x)$, $A_{p,\delta,k}(x)$, $b_{p,\delta,k}(x)$ and $B_{p,\delta,k}(x)$ in the case of $x_1 = \cdots = x_p = x$, respectively. Then we can write two types of approximations for $P(\mathrm{Max}\{X_j\} \le x)$ as follows:

$$A_{\delta,k}^{[p]}(x) = \sum_{j=0}^{k-1} \frac{1}{j!} a_{\delta,k}^{[p]}(x) E\{(\sigma^{2\delta} - 1)^j\}, \tag{3.2}$$

$$B_{\delta,k}^{[p]}(x) = \sum_{j=0}^{k-1} \frac{1}{j!} b_{\delta,k}^{[p]}(x) E\{(\sigma^{\delta} - 1)^j\}. \tag{3.3}$$

The quantities appearing in the error bounds are expressed as

$$\begin{aligned} &\bar{a}_{\delta,k}^{[p]} = \sup|a_{\delta,k}^{[p]}(x)|, \quad \bar{b}_{\delta,k}^{[p]} = \sup|b_{\delta,k}^{[p]}(x)|, \\ &\bar{a}_{\delta,k}^{[p]}(\ell) = \sup\{1 + (\sqrt{p}|x|)^{\ell}\}|a_{\delta,k}^{[p]}(x)|, \\ &\bar{b}_{\delta,k}^{[p]}(\ell) = \sup\{1 + (\sqrt{p}|x|)^{\ell}\}|b_{\delta,k}^{[p]}(x)|. \end{aligned} \tag{3.4}$$

Now we consider the case when $Z_1, \cdots, Z_p$ are independent and identically distributed. Let Q denote the distribution function of Z_1. Then

$$a_{\delta,j}^{[p]}(x) = (d^j/ds^j)\{Q(s^{-\delta/2}x)\}^p\big|_{s=1}, \tag{3.5}$$

$$b_{\delta,k}^{[p]}(x) = (d^j/ds^j)\{Q(s^{-\delta}x)\}^p\big|_{s=1}. \tag{3.6}$$

These quantities can be expressed in terms of

$$a_{\delta,j}(x) = (d/ds)Q(s^{-\delta/2}x)\big|_{s=1}, \tag{3.7}$$

$$b_{\delta,j}(x) = (d/ds)Q(s^{-\delta}x)\big|_{s=1}, \tag{3.8}$$

respectively. We denote the correspondence from $(Q, \{a_{\delta,i}(x)\})$ to $a_{\delta,k}^{[p]}(x)$ by Y_j, i.e.,

$$a_{\delta,j}^{[p]}(x) = Y_j(Q, \{a_{\delta,i}(x)\}). \tag{3.9}$$

Then we can write

$$b_{\delta,k}^{[p]}(x) = Y_j(Q, \{b_{\delta,i}(x)\}). \tag{3.10}$$

Letting $Y_j = Y_j(Q, \{q_i\})$, it is seen that

$$\begin{aligned} Y_0 &= Q^p, \\ Y_1 &= pQ^{p-1}q_1, \\ Y_2 &= p(p-1)Q^{p-2}q_1^2 + pQ^{p-1}q_2, \\ Y_3 &= p(p-1)(p-2)Q^{p-3}q_1^3 + 3p(p-1)Q^{p-2}q_1q_2 + pQ^{p-1}q_3, \\ Y_4 &= p(p-1)(p-2)(p-3)Q^{p-3}q_1^4 + 6p(p-1)(p-2)Q^{p-3}q_1^2q_2 \\ &\quad + p(p-1)Q^{p-2}\{3q_2^2 + 4q_1q_3\} + pQ^{p-1}q_4. \end{aligned} \tag{3.11}$$

We note that

$$\bar{a}^{[p]}_{\delta,k} \le \tilde{a}^{[p]}_{\delta,k} = Y_k(1, \{\bar{a}_{\delta,i}\}), \tag{3.12}$$

$$\bar{b}^{[p]}_{\delta,k} \le \tilde{b}^{[p]}_{\delta,k} = Y_k(1, \{\bar{b}_{\delta,i}\}), \tag{3.13}$$

where $\bar{a}_{\delta,i} = \sup|a_{\delta,i}(x)|$ and $\bar{b}_{\delta,i} = \sup|b_{\delta,i}(x)|$. Similar bounds are also obtained for $\bar{a}^{[p]}_{\delta,k}(\ell)$ and $\bar{b}^{[p]}_{\delta,k}(\ell)$.

4. The two special cases

4.1. The case (i). Let $X_j = Z_j/(\chi_n^2/n)^{1/2}, j = 1,\dots, p$, where $Z_1,\dots, Z_p$ i.i.d. $\sim N(0, 1)$ and $(Z_1,\dots, Z_p)$ and σ are independent. Let $\Phi(X)$ and $\phi(X)$ denote the distribution and the probability density functions of the standard normal variable. We use (3.1) as an approximation for $P(\mathrm{Max}\{X_j\} \le x)$. We have seen that $a^{[p]}_{\delta,j}(x)$'s are determined by

$$a^{[p]}_{\delta,j}(x) = Y_j(\Phi(x), \{a_{\delta,i}(x)\}), \tag{4.1}$$

where $a_{\delta,j}(x) = (d^j/ds^j)\Phi(s^{-\delta/2}x)\big|_{s=1}$. Then, by induction, it is proved that

$$\begin{aligned} a_{1,j}(x) &= -2^{-j}H_{2j-1}(x)\phi(x), \\ a_{-1,j}(x) &= (-1)^{j-1}2^{-j}\Big\{x^{2j-1} + \textstyle\sum_{i=1}^{j-1} 1\cdot 3\cdots(2i-1)\binom{j-1}{i} \\ &\qquad \times x^{2j-2i-1}\Big\}\phi(x), \end{aligned} \tag{4.2}$$

where $H_j(x)$ is the Hermite polynomial defined by

$$(d^j/dx^j)\phi(x) = (-1)^j H_j(x)\phi(x).$$

We note that $a_{\delta,j}(x)$'s are the same as the previous ones due to Fujikoshi [3] and Fujikoshi and Shimizu [5], which are introduced by the other methods. For nonnegative integers j and ℓ, let

$$\begin{aligned} m_{1,j}(\ell) &= E[(\chi_n^2/n)^{-\ell}\{(\chi_n^2/n)^{-1} - 1\}^j], \\ m_{1,j} &= m_{1,j}(0), \quad m_{-1,j} = E[\{(\chi_n^2/n) - 1\}^j]. \end{aligned} \tag{4.3}$$

The quantities $m_{-1,j}$'s exist for any j, but the quantities $m_{1,j}(\ell)$'s exist only for $n - 2\ell - 2j > 0$. For $m_{1,j}(\ell)$ and $m_{-1,j}$ of $j = 1,\dots, 6$, see Fujikoshi [4]. We can write (3.1) as

$$A^{[p]}_{\delta,k}(x) = \Phi(x)^p + \textstyle\sum_{j=1}^{k-1} \frac{1}{j!} a^{[p]}_{\delta,j}(x) m_{\delta,j}. \tag{4.4}$$

From Theorems 2.1 and 2.2 it holds that

(i) if $n - 2k > 0$ and k is even,

$$\sup_x |P(\mathrm{Max}\{X_j\} \le x) - A_{\delta,k}^{[p]}(x)| \le \frac{1}{k!} \bar{a}_{\delta,k}^{[p]}\{m_{1,k} + m_{-1,k}\}, \tag{4.5}$$

(ii) if $n - 2\ell - 2k > 0$ and k is even,

$$|P(\mathrm{Max}\{X_j\} \le x) - A_{\delta,k}^{[p]}(x)| \le \frac{1}{k!}\{1 + (px^2)^{\ell}\}^{-1} a_{\delta,k}^{[p]}(2\ell)\{m_{1,k}(\ell) + m_{-1,k}\}. \tag{4.6}$$

It may be noted that the order of error terms is $O(n^{-k/2})$ and $A_{\delta,1}^{[p]}(x)$ is an asymptotic expansion for $P(\mathrm{Max}\{X_j\} \le x)$ up to $O(n^{-k/2})$ since $m_{\delta,j}(\ell) = O(n^{-(j+1)/2})$, if j is odd, and $= O(n^{-j/2})$, if j is even.

4.2. The case (ii). Let $X_j = Z_j/(\chi_n^2/n)$, $j = 1,\dots, p$, where $Z_1,\dots, Z_p$ i.i.d. $\sim G(\lambda)$ and $(Z_1,\dots, Z_p)$ and σ are independent. Let $G(x; \lambda)$ and $g(x; \lambda)$ denote the distribution and the probability density functions of the gamma distribution $G(\lambda)$. We use (3.2) as an approximation for $P(\mathrm{Max}\{X_j\} \le x)$. Here the support of $\mathrm{Max}\{X_j\}$ is R_+ and so we consider only for $x > 0$. It is known (Fujikoshi [3], Fujikoshi and Shimizu [5]) that

$$\begin{aligned} b_{1,j}(x; \lambda) &= (d^j/ds^j)G(s^{-1}x; \lambda)\big|_{s=1} \\ &= -xL_{j-1}^{(\lambda)}(x)g(x; \lambda), \\ b_{-1,j}(x; \lambda) &= (d^j/ds^j)G(sx; \lambda)\big|_{s=1} \\ &= (-1)^{j-1}x\tilde{L}_{j-1}^{(\lambda)}(x)g(x; \lambda), \end{aligned} \tag{4.7}$$

where $L_p^{(\lambda)}(x)$ is the Laguerre polynomial defined by

$$L_p^{(\lambda)}(x) = (-1)^p x^{-\lambda} e^x (d^p/dx^p)(x^{p+\lambda}e^{-x})$$

and

$$\tilde{L}_p^{(\lambda)}(x) = x^p + \textstyle\sum_{i=1}^p (1-\lambda)\cdots(i-\lambda)\binom{p}{i}x^{p-\lambda}.$$

We can write (3.2) as

$$B_{\delta,k}^{[p]}(x; \lambda) = G(x; \lambda)^p + \textstyle\sum_{j=1}^{k-1} \frac{1}{j!} B_{\delta,j}^{[p]}(x; \lambda)m_{\delta,j}, \tag{4.8}$$

where

$$b_{\delta,j}^{[p]}(x; \lambda) = Y_j(G(x; \lambda), \{b_{\delta,i}(x; \lambda)\}). \tag{4.9}$$

From Theorems 2.1 and 2.2 it holds that

(i) if $n-2k>0$ and k is even,

$$\sup_x |P(\mathrm{Max}(X_j)\le x)-B_{\delta,k}^{[p]}(x;\ \lambda)| \le \frac{1}{k!}\bar{b}_{\delta,k}^{[p]}\{m_{1,k}+m_{-1,k}\}, \tag{4.10}$$

(ii) if $n-2\ell-2k>0$ and k is even,

$$\begin{aligned} &|P(\mathrm{Max}\{X_j\}\le x)-B_{\delta,k}^{[p]}(x;\ \lambda)| \\ &\le \frac{1}{k!}\{1+(\sqrt{p}\ x)^{\ell}\}^{-1}\bar{b}_{\delta,k}^{[p]}(\ell;\ \lambda)\{m_{1,k}(\ell)+m_{-1,k}(\ell)\}, \end{aligned} \tag{4.11}$$

where $\bar{b}_{\delta,k}^{[p]}(\ell;\ \lambda)=\sup_{x>0}\{1+(\sqrt{p}\ x)^{\ell}\}|b_{\delta,k}^{[p]}(x;\ \lambda)|$. We note that $B_{\delta,k}^{[p]}(x;\ \lambda)$ is an asymptotic expansion for $P(\mathrm{Max}\{X_j\}\le x)$ up to $O(n^{-k/2})$ and the order of the error terms in (4.11) and (4.12) is $O(n^{-k/2})$.

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