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The orders of the canonical elements in $\tilde{K}(L^n(2^r))$ and an application

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§1. Introduction

Let k be a positive integer. For the standard sphere S^{2n+1} in complex $(n+1)$-space C^{n+1}, let T_k: $S^{2n+1} \to S^{2n+1}$ be the fixed point free transformation of period k defined by

$$T_k(z_0, z_1, \cdots, z_n) = (\lambda(k)z_0, \lambda(k)z_1, \cdots, \lambda(k)z_n),$$

where $\lambda(k) = e^{2\pi i/k}$, $\sum_{j=0}^{n}|z_j|^2 = 1$, $z_j \in C$ $(0 \leq j \leq n)$. Then T_k generates the cyclic group Z_k of order k. The orbit space S^{2n+1}/Z_k is the standard lens space mod k, and is denoted by $L^n(k)$.

Let η be the canonical complex line bundle over $L^n(2^r)$, and define the canonical elements in the reduced K-ring $\tilde{K}(L^n(2^r))$ of $L^n(2^r)$ by

$$\sigma(s) = \eta^{2^s} - 1 \ (0 \leq s \leq r), \ \sigma = \sigma(0).$$

The purpose of this paper is to determine the order of $\sigma(s)^i \in \tilde{K}(L^n(2^r))$.

THEOREM 1.1. *Let s and i be integers such that $0 \leq s \leq r$ and $1 \leq i \leq [(n-1)/2^s] + 1$. Then the element $\sigma(s)^i$ in $\tilde{K}(L^n(2^r))$ is of order $2^{r-s+1+[(n-1)/2^s]-i}$.* (Here $[x]$ is the integer part of a real number x.)

For $s = 0$, the result is obtained by T. Kawaguchi and M. Sugawara [2, Theorem 1.1(i)].

Let k be a positive integer. Then a fixed point free transformation $\bar{T}_2$: $L^n(k) \to L^n(k)$ of period 2 is induced by T_{2k}: $S^{2n+1} \to S^{2n+1}$, since $(T_{2k})^2 = T_k$.

As an application of Theorem 1.1, we prove

THEOREM 1.2. *Let q and t be odd integers. If there is a Z_2-equivariant map from $(L^n(2^r q), \bar{T}_2)$ to $(L^m(2^s t), \bar{T}_2)$, then $[(n-1)/2^r] \leq [(m-1)/2^s]$.*

In case $s = 0$ and $t = 1$, this result follows also from the mod 2 part of a theorem due to H. J. Munkholm and M. Nakaoka [9, Theorem 4] and K. Shibata [10, Theorem 7.4] (cf. J. W. Vick [11, Corollary (3.3)]), since $[(n-1)/2^r] \leq m-1$ if and only if $n \leq m \cdot 2^r$.

§2. Some relations between $\sigma^i \in \tilde{K}(L^n(2^r))$

First we recall the basic facts about the structure of the K-ring of the lens space according to [2].

Let k be any integer with $k > 1$. The canonical complex line bundle η over $L^n(k) = S^{2n+1}/Z_k$ is induced by the natural projection of $L^n(k)$ on the complex projective n-space $CP^n = S^{2n+1}/S^1$ from the canonical complex line bundle μ over CP^n. Put $\sigma = \eta - 1 \in \tilde{K}(L^n(k))$. Since $(\mu - 1)^{n+1} = 0$, we have

(2.1)([2, (2.8)]) $$\sigma^{n+1} = 0.$$

Now, let $k = 2^r$. The first Chern classes classify the complex line bundles, and so $\sigma(r) = \eta^{2^r} - 1 = (1 + \sigma)^{2^r} - 1 = 0$ ([2, (2.7)]). Hence we obtain that, in $\tilde{K}(L^n(2^r))$,

(2.2) $$\sum_{j=1}^{2^r} \binom{2^r}{j} \sigma^j = 0.$$

Combining (2.2) with (2.1) repeatedly, we obtain the following relations.

LEMMA 2.3. *For the element* $\sigma = \eta - 1 \in \tilde{K}(L^n(2^r))$,

(i) $2^{r+i}\sigma^{n-i} = 0 \qquad (i \leq n-1)$,

(ii) $2^{r+i}\sigma^{n-1-i} \pm 2^{r-1+i}\sigma^{n-i} = 0 \qquad (0 \leq i \leq n-2)$,

$2^{r-1}\sigma^n = 2^r\sigma^{n-1} = \cdots = 2^{r-1+i}\sigma^{n-i} = \cdots = 2^{r+n-2}\sigma \quad (0 \leq i \leq n-1)$,

(iii) $2^{r+i}\sigma^{n-2-i} + 2^{r-1+i}\sigma^{n-1-i} = 0 \quad (1 \leq i \leq n-3)$,

$2^r\sigma^{n-2} = -2^{r+1}\sigma^{n-3} = \cdots = (-1)^i 2^{r+i}\sigma^{n-2-i} = \cdots$

$= (-1)^{n+1}2^{r+n-3}\sigma \qquad (0 \leq i \leq n-3)$,

$2^{r-1}\sigma^{n-1} = 2^r\sigma^{n-2} \qquad$ *if* $r \geq 2$,

$\sigma^{n-1} = -2\sigma^{n-2} \qquad$ *if* $r = 1$,

(iv) $2^{r+i}\sigma^{n-3-i} + 2^{r-1+i}\sigma^{n-2-i} = 0 \qquad (2 \leq i \leq n-4)$,

$2^{r+1}\sigma^{n-4} = -2^{r+2}\sigma^{n-5} = \cdots = (-1)^{i-1}2^{r+i}\sigma^{n-3-i} = \cdots$

$= (-1)^{n+1}2^{r+n-4}\sigma \qquad (1 \leq i \leq n-4)$,

$2^r\sigma^{n-3} = \pm 2^{r+n-2}\sigma + (-1)^n 2^{r+n-4}\sigma = (-1)^{n+1}3 \cdot 2^{r+n-4}\sigma \quad$ *if* $r \geq 2$,

$2\sigma^{n-3} = -2^2\sigma^{n-4} = (-1)^n 2^{n-3}\sigma \qquad$ *if* $r = 1$.

This lemma is a generalization of Lemmas 4.1 and 4.3 in [4] for $r = 2$, Lemma (3.5) in [5] for $r = 3$, and Lemma A1 in [3] for $r = 4$.

PROOF. Though (i) is proved in [2, (3.3)], we give a proof for completeness. The equality $2^{r+i}\sigma^{n-i} = 0$ holds for any negative integer i by

(2.1). Assume that $2^{r+j}\sigma^{n-j}=0$ holds for any integer j with $j<i<n-1$. Multiply the equality (2.2) by $2^i\sigma^{n-1-i}$. Then we have

$$\sum_{j=1}^{2^r}\binom{2^r}{j}2^i\sigma^{n-1-i+j}=0.$$

Setting $j=q\cdot 2^t$, where q is odd and $0\le t\le r$, we obtain

$$\binom{2^r}{j}2^i\sigma^{n-1-i+j}=\frac{1}{q}\binom{2^r-1}{j-1}2^{r-t+i}\sigma^{n-1-i+j}=0 \quad \text{for } j>1$$

by the inductive assumption, and hence $2^{r+i}\sigma^{n-i}=0$ as desired.

To prove (ii), multiply (2.2) by $2^i\sigma^{n-2-i}$, where $0\le i\le n-2$. Then we have

$$\sum_{j=1}^{2r}\binom{2^r}{j}2^i\sigma^{n-2-i+j}=0.$$

Setting $j=q\cdot 2^t$, where q is odd and $0\le t\le r$, we obtain, by (i),

$$\begin{aligned}\binom{2^r}{j}2^i\sigma^{n-2-i+j}&=\frac{1}{q}\binom{2^r-1}{j-1}2^{r-t+i}\sigma^{n-2-i+j}=0 && \text{for } j>2,\\ &=\pm\, 2^{r-1+i}\sigma^{n-i} && \text{for } j=2.\end{aligned}$$

Thus we obtain (ii).

To prove (iii), multiply (2.2) by $2^i\sigma^{n-3-i}$, where $0\le i\le n-3$. Then we have, by (i), (ii) and (2.1),

$$\begin{aligned}\binom{2^r}{2}2^i\sigma^{n-1-i}&=-\,2^{r-1+i}\sigma^{n-1-i} && \text{for } r\ge 2,\\ &=2^i\sigma^{n-1-i} && \text{for } r=1,\\ \binom{2^r}{4}2^i\sigma^{n+1-i}&=\pm\, 2^{r+i}\sigma^{n-1-i} && \text{for } r\ge 2,\ 0<i\le n-3,\\ \binom{2^r}{j}2^i\sigma^{n-3-i+j}&=0 && \text{for } j=4,\ i=0,\end{aligned}$$

$$\text{for } j\ge 3,\ j\ne 4,\ 0\le i\le n-3, \text{ or for } j\ge 3,\ r=1.$$

Hence we obtain the desired equalities.

To prove (iv), multiply (2.2) by $2^i\sigma^{n-4-i}$, where $0\le i\le n-4$. Then we have, by (i) $\sim$ (iii),

$$\begin{aligned}\binom{2^r}{2}2^i\sigma^{n-2-i}&=-\,2^{r-1+i}\sigma^{n-2-i} && \text{for } r\ge 3,\\ &=\pm\, 2^n\sigma-2^{1+i}\sigma^{n-2-i} && \text{for } r=2,\end{aligned}$$

$$= 2^i\sigma^{n-2-i} \qquad \text{for } r = 1,$$

$$\binom{2^r}{3}2^i\sigma^{n-1-i} = \pm\, 2^{r+1+i}\sigma^{n-2-i} \qquad \text{for } r \geq 2,$$

$$\binom{2^r}{4}2^i\sigma^{n-i} = -\, 2^{r+i}\sigma^{n-2-i} \quad \text{for } r \geq 3,\ 2 \leq i \leq n-4,$$

$$= 2^{r+1}\sigma^{n-3} \qquad \text{for } r \geq 3,\ i = 1,$$

$$= 2^i\sigma^{n-i} \qquad \text{for } r = 2,$$

$$\binom{2^r}{j}2^i\sigma^{n-4-i+j} = 0 \qquad \text{for } j \geq 3 \text{ and } r = 1, \text{ or for } j \geq 5.$$

Hence we obtain the desired equalities. q.e.d.

On the other hand, the following is known.

LEMMA 2.4 ([2, (2.9)]). *The element* $\sigma^n \in \tilde{K}(L^n(k))$ *is of order* k.

Combining Lemma 2.3 (i), (ii) with Lemma 2.4, one has

THEOREM 2.5 (T. Kawaguchi and M. Sugawara [2, Theorem 1.1 (i)]). *For* $1 \leq i \leq n$, *the element* $\sigma^i \in \tilde{K}(L^n(2^r))$ *is of order* 2^{r+n-i}.

§3. The order of $\sigma(s)^i \in \tilde{K}(L^n(2^r))$

In the previous papers ([7, §3], [6, §2], [1, §2 ~ §3]) some relations in $\tilde{K}(L^n(2^r))$ have been studied. In this section we prove the relations in $\tilde{K}(L^n(2^r))$ which are necessary to determine the order of the canonical element $\sigma(s)^i = (\eta^{2^s} - 1)^i \in \tilde{K}(L^n(2^r))$. Though some of the relations in the lemmas below may have been already given in the previous papers (loc. cit.), we prove them for completeness.

It follows from the definition that

(3.1)([6, (2.2)]) $\qquad \sigma(s) = \sigma(s-1)^2 + 2\sigma(s-1) \quad (0 < s \leq r)$.

Since $(1 + \sigma(s))^{2^{r-s}} - 1 = \eta^{2^r} - 1 = 0$, we have

$$\sum_{j=1}^{2^{r-s}} \binom{2^{r-s}}{j}\sigma(s)^j = 0. \tag{3.2}$$

Let s be an integer with $0 \leq s \leq r$, and define integers l_s and k_s by the following:

$$n - 1 = l_s \cdot 2^s + k_s \qquad (0 \leq k_s < 2^s).$$

Then, clearly, $l_s = [(n-1)/2^s]$, and $l_{s-1} = 2l_s$ or $2l_s + 1$.

LEMMA 3.3. *For any positive integer i, $2^{r-s+1+l-i}\sigma(s)^i = 0$, where $l = l_s = [(n-1)/2^s]$.*

PROOF. If $s = 0$, the result follows from Lemma 2.3 (i). Assume that the result holds for $s-1$, i.e., assume that

$$2^{r-s+2+l'-i}\sigma(s-1)^i = 0 \qquad (i > 0,\ 0 < s \leq r),$$

where $l' = l_{s-1} = [(n-1)/2^{s-1}]$. By (3.1) and by the inductive assumption, we have

$$\begin{aligned} 2^{r-s+1+l-i}\sigma(s)^i &= 2^{r-s+1+l-i}(\sigma(s-1)^2 + 2\sigma(s-1))^i \\ &= \textstyle\sum_{j=0}^{i}\binom{i}{j}2^{r-s+1+l-i+j}\sigma(s-1)^{2i-j} = 0 \end{aligned}$$

for $i \geq l+1+\varepsilon$, where $\varepsilon = 0$ if $l' = 2l$, or $\varepsilon = 1$ if $l' = 2l+1$.

Multiply (3.2) by $2^{2+l-i}\sigma(s)^{i-2}$ $(i = l+1+\varepsilon)$. Then we have

$$2^{r-s+2+l-i}\sigma(s)^{i-1} = 0.$$

Thus the desired result is obtained by a downward induction on i. q.e.d.

LEMMA 3.4. *Let $l = l_s = [(n-1)/2^s]$, where $0 \leq s \leq r$. Then for $\sigma(s) = \eta^{2^s} - 1 \in \tilde{K}(L^n(2^r))$, the following relations hold.*

$$\begin{aligned}
&\text{(i)} && 2^{r-s+i}\sigma(s)^{l-i} \pm 2^{r-s-1+i}\sigma(s)^{l+1-i} = 0 \qquad (0 \leq i \leq l-1),\\
&&& 2^{r-s-1}\sigma(s)^{l+1} = 2^{r-s}\sigma(s)^l = \cdots = 2^{r-s+i}\sigma(s)^{l-i} = \cdots\\
&&& \qquad = 2^{r-s-1+l}\sigma(s) \qquad (-1 \leq i \leq l-1),\\
&\text{(ii)} && 2^{r-s-2}\sigma(s)^{l+2} = 0,\\
&\text{(iii)} && 2^{r-s+i}\sigma(s)^{l-1-i} + 2^{r-s-1+i}\sigma(s)^{l-i} = 0 \qquad (1 \leq i \leq l-2),\\
&&& 2^{r-s}\sigma(s)^{l-1} = -2^{r-s+1}\sigma(s)^{l-2} = \cdots = (-1)^{i-1}2^{r-s-1+i}\sigma(s)^{l-i}\\
&&& \qquad = \cdots = (-1)^l 2^{r-s-2+l}\sigma(s) \qquad (1 \leq i \leq l-1),\\
&&& 2^{r-s-1}\sigma(s)^l = 2^{r-s}\sigma(s)^{l-1} \qquad \textit{if } r-s \geq 2,\\
&&& \sigma(s)^l = -2\sigma(s)^{l-1} \qquad \textit{if } r-s = 1,\\
&\text{(iv)} && 2^{r-s+i}\sigma(s)^{l-2-i} + 2^{r-s-1+i}\sigma(s)^{l-1-i} = 0 \qquad (2 \leq i \leq l-3),\\
&&& 2^{r-s+1}\sigma(s)^{l-3} = -2^{r-s+2}\sigma(s)^{l-4} = \cdots = (-1)^i 2^{r-s-1+i}\sigma(s)^{l-1-i}\\
&&& \qquad = \cdots = (-1)^l 2^{r-s-3+l}\sigma(s) \qquad (2 \leq i \leq l-2),\\
&&& 2^{r-s}\sigma(s)^{l-2} = (-1)^l 3\cdot 2^{r-s-3+l}\sigma(s) \qquad \textit{if } r-s \geq 2,\\
&&& 2\sigma(s)^{l-2} = -2^2\sigma(s)^{l-3} = (-1)^l 2^{l-2}\sigma(s) \qquad \textit{if } r-s = 1.
\end{aligned}$$

PROOF. Since $\sigma(r) = 0$, we may assume that $r > s$.

(i) Multiply (3.2) by $2^i\sigma(s)^{l-1-i}$, where $0 \leq i \leq l-1$. Then we have

$$\sum_{j=1}^{2^{r-s}}\binom{2^{r-s}}{j}2^i\sigma(s)^{l-1-i+j}=0.$$

By Lemma 3.3, we obtain

$$\begin{aligned}\binom{2^{r-s}}{j}2^i\sigma(s)^{l-1-i+j} &= 0 && \text{for } j>2,\\ &= \pm 2^{r-s-1+i}\sigma(s)^{l+1-i} && \text{for } j=2.\end{aligned}$$

Thus we obtain (i).

(ii) By (3.1), we have

$$\begin{aligned}2^{r-s-2}\sigma(s)^{l+2} &= 2^{r-s-2}(\sigma(s-1)^2+2\sigma(s-1))^{l+2}\\ &= \sum_{j=0}^{l+2}\binom{l+2}{j}2^{r-s-2+j}\sigma(s-1)^{2l+4-j}.\end{aligned}$$

If $l'=l_{s-1}=2l$, by Lemma 3.3,

$$2^{r-s-2+j}\sigma(s-1)^{2l+4-j}=2^{r-s-2+j}\sigma(s-1)^{l'+4-j}=0.$$

If $l'=l_{s-1}=2l+1$, by (i),

$$2^{r-s-2+j}\sigma(s-1)^{2l+4-j}=2^{r-s-2+j}\sigma(s-1)^{l'+3-j}=\pm 2^{r-s+l'}\sigma(s-1),$$

and so, by Lemma 3.3,

$$\begin{aligned}2^{r-s-2}\sigma(s)^{l+2} &= \sum_{j=0}^{l+2}\binom{l+2}{j}2^{r-s+l'}\sigma(s-1)\\ &= 2^{l+2+r-s+l'}\sigma(s-1)=0.\end{aligned}$$

(iii) Multiply (3.2) by $2^i\sigma(s)^{l-2-i}$, where $0 \leq i \leq l-2$. Then we have, by Lemma 3.3, (i) and (ii),

$$\begin{aligned}\binom{2^{r-s}}{2}2^i\sigma(s)^{l-i} &= -2^{r-s-1+i}\sigma(s)^{l-i} && \text{for } r-s\geq 2,\\ &= 2^i\sigma(s)^{l-i} && \text{for } r-s=1,\\ \binom{2^{r-s}}{4}2^i\sigma(s)^{l+2-i} &= \pm 2^{r-s+i}\sigma(s)^{l-i} && \text{for } r-s\geq 2,\ 0<i\leq l-2,\\ \binom{2^{r-s}}{j}2^i\sigma(s)^{l-2-i+j} &= 0 && \text{for } j=4,\ i=0,\end{aligned}$$

for $j \geq 3$, $j \neq 4$, $0 \leq i \leq n-3$, or for $j \geq 3$, $r-s=1$.

Hence we obtain the desired equalities.

(iv) Multiply (3.2) by $2^i\sigma(s)^{l-3-i}$, where $0 \leq i \leq l-3$.

Then we have, by Lemma 3.3, (i) and (iii),

$$\begin{aligned}
\binom{2^{r-s}}{2}2^i\sigma(s)^{l-1-i} &= -2^{r-s-1+i}\sigma(s)^{l-1-i} && \text{for } r-s \geq 3,\\
&= \pm 2^{l+1}\sigma(s) - 2^{1+i}\sigma(s)^{l-1-i} && \text{for } r-s = 2,\\
&= 2^i\sigma(s)^{l-1-i} && \text{for } r-s = 1,\\
\binom{2^{r-s}}{3}2^i\sigma(s)^{l-i} &= \pm 2^{r-s+i+1}\sigma(s)^{l-1-i} && \text{for } r-s \geq 3,\\
&= \pm 2^{l+1}\sigma(s) && \text{for } r-s = 2,\\
\binom{2^{r-s}}{4}2^i\sigma(s)^{l+1-i} &= -2^{r-s+i}\sigma(s)^{l-1-i} && \text{for } r-s \geq 3,\ 2 \leq i \leq l-3,\\
&= 2^{r-s+1}\sigma(s)^{l-2} && \text{for } r-s \geq 3,\ i = 1,\\
&= 2^i\sigma(s)^{l+1-i} && \text{for } r-s = 2,
\end{aligned}$$

$$\binom{2^{r-s}}{j}2^i\sigma(s)^{l-3-i+j} = 0 \quad \text{for } j \geq 3 \text{ and } r-s=1, \text{ or for } j \geq 5.$$

Hence we obtain the desired equalities. q.e.d.

Now, we are ready to prove our main theorem.

PROOF OF THEOREM 1.1. If $s = 0$, the result is identical with Theorem 2.5. Assume that the result holds for $s-1$ $(1 \leq s \leq r)$. Put $l = l_s = [(n-1)/2^s]$ and $l' = l_{s-1} = [(n-1)/2^{s-1}]$. Then $l' = 2\ell$ or $l' = 2l+1$.

SUBLEMMA 3.5. *If $l' = 2l$, $2^{r-s+1}\sigma(s)^{l-1} = \pm 2^{r-s+l'}\sigma(s-1) \neq 0$.*

PROOF. By (3.1) and Lemma 3.4 (iv), we have

$$\begin{aligned}
&2^{r-s+1}\sigma(s)^{l-1} = 2^{r-s+1}(\sigma(s-1)^2 + 2\sigma(s-1))^{l-1}\\
&= \textstyle\sum_{j=0}^{l-1}\binom{l-1}{j}2^{r-s+1+j}\sigma(s-1)^{2l-2-j}\\
&= 2^{r-s+1}\sigma(s-1)^{l'-2} + \textstyle\sum_{j=1}^{l-1}\binom{l-1}{j}(-1)^{j+l'-1}2^{r-s+l'-2}\sigma(s-1).
\end{aligned}$$

Since $\sum_{j=1}^{l-1}\binom{l-1}{j}(-1)^j = (1-1)^{l-1} - 1 = -1$, we obtain, by Lemma 3.4 (iv),

$$2^{r-s+1}\sigma(s)^{l-1} = (-1)^{l'}2^{r-s+l'}\sigma(s-1).$$

By induction, $2^{r-s+l'}\sigma(s-1)$ is of order 2.

SUBLEMMA 3.6. *If* $l' = 2l + 1$, $2^{r-s+1}\sigma(s)^{l-1}\sigma(s-1) = \pm 2^{r-s+l'}\sigma(s-1) \neq 0$, *in particular*, $2^{r-s+1}\sigma(s)^{l-1} \neq 0$.

PROOF. As in the proof of the previous sublemma, we have

$$2^{r-s+1}\sigma(s)^{l-1}\sigma(s-1) = \textstyle\sum_{j=0}^{l-1}\binom{l-1}{j}2^{r-s+1+j}\sigma(s-1)^{l'-2-j}$$

$$= \pm 2^{r-s+l'}\sigma(s-1) \neq 0.$$

Combining Lemma 3.4 (i) with Sublemmas 3.5 and 3.6, we have

$$2^{r-s+l-i}\sigma(s)^i \neq 0 \quad (1 \leq i \leq l+1).$$

Therefore the result follows from Lemma 3.3. q.e.d.

§4. Proof of Theorem 1.2

Assume that there is a Z_2-equivariant map f from $(L^n(2^r q), \bar{T}_2)$ to $(L^m(2^s t), \bar{T}_2)$. Then f defines a continuous map $\bar{f}$ from $L^n(2^{r+1}q)$ to $L^m(2^{s+1}t)$ such that the following diagram is commutative:

$$\begin{array}{ccc} L^n(2^r q) & \xrightarrow{f} & L^m(2^s t) \\ \downarrow & & \downarrow \\ L^n(2^{r+1}q) & \xrightarrow{\bar{f}} & L^m(2^{s+1}t), \end{array}$$

where the vertical maps are the projections of the double coverings.

Let η/k denote the canonical complex line bundle η over $L^m(k)$ and let

$$\psi\colon \tilde{K}(L^m(2^{s+1})) \oplus \tilde{K}(L^m(t)) \longrightarrow \tilde{K}(L^m(2^{s+1}t))$$

be the isomorphism defined by

$$(\psi i_1)((\eta/2^{s+1}) - 1) = (\eta/2^{s+1}t)^t - 1,$$
$$(\psi i_2)((\eta/t) - 1) = (\eta/2^{s+1}t)^{2^{s+1}} - 1$$

(cf. [8, Theorem 2.2]), where

$$i_1\colon \tilde{K}(L^m(2^{s+1})) \longrightarrow \tilde{K}(L^m(2^{s+1})) \oplus \tilde{K}(L^m(t)),$$
$$i_2\colon \tilde{K}(L^m(t)) \longrightarrow \tilde{K}(L^m(2^{s+1})) \oplus \tilde{K}(L^m(t))$$

are the inclusions into the first or the second direct summand. Let $\pi\colon L^n(2^{r+1}) \to L^n(2^{r+1}q)$ be the projection of the q-fold covering.

Consider the composite homomorphism $\pi^*\bar{f}^*\psi i_1\colon \tilde{K}(L^m(2^{s+1})) \to \tilde{K}(L^n(2^{r+1}))$. Then we have

$$\begin{aligned}\pi^*\bar{f}^*\psi i_1((\eta/2^{s+1})^{2^s}-1) &= \pi^*\bar{f}^*((\eta/2^{s+1}t)^{2^st}-1)\\ &= \pi^*((\eta/2^{r+1}q)^{2^rq}-1) = (\eta/2^{r+1})^{2^r}-1\end{aligned}$$

(cf. [10, Lemma 7.1]).

For $\sigma(s) = (\eta/2^{s+1})^{2^s} - 1 \in \tilde{K}(L^m(2^{s+1}))$, we have $2^{1+[(m-1)/2^s]}\sigma(s) = 0$. Hence, for $\sigma(r) = (\eta/2^{r+1})^{2^r} - 1 \in \tilde{K}(L^n(2^{r+1}))$, we have $2^{1+[(m-1)/2^s]}\sigma(r) = 0$. But, by Theorem 1.1, we obtain $[(n-1)/2^r] \leq [(m-1)/2^s]$. q.e.d.

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