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Integral representations of Beppo Levi functions and the existence of limits at infinity

Dedicated to Professor Hisao Mizumoto on the occasion of his 60th brthday

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1. Introduction

Our main aim in this paper is to study the behavior at infinity of Beppo Levi functions $u \in BL_m(L^p_{loc}(R^n))$ such that

(1) $$\sum_{|\lambda|=m} \int |D^\lambda u(x)|^p \omega(|x|)dx < \infty,$$

where m is a positive integer, $1 < p < \infty$, $D^\lambda = (\partial/\partial x)^\lambda$ and ω is a positive monotone function on the interval $[0, \infty)$; for the definition and properties of Beppo Levi functions, see Deny-Lions [1]. For this purpose we need an integral representation of u as a generalization of [7; Theorem 1], where the case $\omega(r) \equiv 1$ was discussed.

We recall the following integral representation of $\varphi \in C_0^\infty(R^n)$ (see Wallin [8; p.71]):

(2) $$\varphi(x) = \sum_{|\lambda|=m} a_\lambda \int D^\lambda k_m(x-y) D^\lambda \varphi(y)dy,$$

where $\{a_\lambda\}$ are constants independent of φ, k_m denotes the Riesz kernel of order $2m$, which is defined by

$$k_m(x) = \begin{cases} |x|^{2m-n} & \text{if } 2m < n \text{ or if } 2m > n \text{ and } n \text{ is odd,} \\ -|x|^{2m-n} \log |x| & \text{if } 2m \geqq n \text{ and } n \text{ is even.} \end{cases}$$

If φ does not have compact support, then the integrals of (2) may fail to be absolutely convergent at any x. This requires us to modify the kernel functions $D^\lambda k_m$, in such a way that all the integrals, which will appear in the representations, are absolutely convergent at almost every x. To do so, we introduce the following kernel functions $K_{m,\lambda,\ell}$ (cf. Hayman-Kennedy [2], Mizuta [6]):

$$K_{m,\lambda,\ell}(x, y) = \begin{cases} D^\lambda k_m(x - y) - \sum_{|\mu| \leqq \ell} (x^\mu/\mu!)(D^{\lambda+\mu} k_m)(-y) & \text{if } |y| \geqq 1, \\ D^\lambda k_m(x-y) & \text{if } |y| < 1. \end{cases}$$

Our aim is to find an integer ℓ such that the functions $\int K_{m,\lambda,\ell}(x, y) D^\lambda u(y)dy$ are absolutely convergent at almost every x and the equality

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y) D^\lambda u(y) dy + P(x)$$

holds for almost every $x \in \boldsymbol{R}^n$, where P is a polynomial which is polyharmonic of order m in $\boldsymbol{R}^n$ (see Theorems 1 and 1′).

By using the above integral representation, we can give extensions of the results in the papers [5], [6] and [7] about the existence of radial limits.

2. Preliminary lemmas

Let k_m be the Riesz kernel of order $2m$, which is defined as above. Then, for a multiindex λ with length $|\lambda|$, we see that $D^\lambda k_m(x)$ is of the form $(\sum b_\mu x^\mu) h(|x|) + (\sum c_\nu x^\nu)|x|^{2m-n-2|\lambda|}$, where $b_\mu (|\mu| = 2m - n - |\lambda|)$, $c_\nu (|\nu| = |\lambda|)$ are constants and

$$h(r) = \begin{cases} \log r & \text{in case } m \geqq n \text{ and } n \text{ is even,} \\ 1 & \text{otherwise;} \end{cases}$$

in case $2m - n < |\lambda|$, $\sum b_\mu x^\mu$ is understood to be zero.

We first state some elementary facts concerning the properties of $K_{m,\lambda,\ell}$ (cf. [6; Lemmas 1 and 4], [7; Lemma 1]).

LEMMA 1. (i) *The function* $K_{m,\lambda,\ell}(\cdot\, y)$ *is polyharmonic of order m in* $R^n - \{y\}$, *that is,* $\Delta^m K_{m,\lambda,\ell}(\cdot, y) = 0$ *on* $R^n - \{y\}$.

(ii) *If* $2m - |\lambda| - n - \ell \leqq 0$, *then*

$$K_{m,\lambda,\ell}(rx, ry) = r^{2m-n-|\lambda|} K_{m,\lambda,\ell}(x, y) \quad \textit{for } r > 0,$$

whenever $|y| \geqq \max\{r^{-1}, 1\}$.

LEMMA 2. *If* $\ell \geqq \max\{-1, 2m-n-|\lambda|\}$, *then there exists a positive constant M such that*

$$|K_{m,\lambda,\ell}(x, y)| \leqq M|x|^{\ell+1}|y|^{2m-n-|\lambda|-\ell-1}$$

whenever $|y| \geqq 2|x|$ *and* $|y| \geqq 1$.

REMARK. If $\ell \leqq -1$ or $y \in B(0, 1)$, then

$$|K_{m,\lambda,\ell}(x, y)| = |D^\lambda k_m(x-y)| \leqq M|x - y|^{2m-n-|\lambda|}[|h(|x-y|)| + 1]$$

for any x, where $B(x, r)$ denotes the open ball with center at x and radius $r > 0$, and M is a positive constant independent of x and y.

LEMMA 3. *If* $\ell \geqq \max\{0, 2m-n-|\lambda|\}$, *then there exists a positive constant M such that*

$$|K_{m,\lambda,\ell}(x, y)| \leqq M|x|^{\ell}|y|^{2m-n-|\lambda|-\ell}h(4|x|/|y|)$$
$$\text{whenever } 1 \leqq |y| < 2|x| \text{ and } |x-y| \geqq |x|/2$$

and

$$|K_{m,\lambda,\ell}(x, y)| \leqq M[|x|^{2m-n-|\lambda|} + |x-y|^{2m-n-|\lambda|}h(|x|/|x-y|)]$$
$$\text{whenever } 1 \leqq |y| < 2|x| \text{ and } |x-y| < |x|/2.$$

PROOF. For a function $K(x, y)$, we write $K^{(\ell)}(x, y) = K(x, y) - \sum_{|\mu| \leq \ell}(x^{\mu}/\mu!)$ $[(\partial/\partial x)^{\mu}K](0, y)$. We know that $(D^{\lambda}k_m)(x-y)$ is of the form

$$\left(\sum_{|\mu|=2m-n-|\lambda|} b_{\mu}(x-y)^{\mu}\right) h(|x-y|/|y|)$$
$$+ \left(\sum_{|\mu|=2m-n-|\lambda|} b_{\mu}(x-y)^{\mu}\right) h(|y|) + \left(\sum_{|\mu|=|\lambda|} c_{\mu}(x-y)^{\mu}\right)|x-y|^{2m-n-2|\lambda|}$$
$$= K_1(x, y) + K_2(x, y) + K_3(x, y).$$

Since $K_2^{(\ell)}(x, y) \equiv 0$, $K_{m,\lambda,\ell}(x, y) = K_1^{(\ell)}(x, y) + K_3^{(\ell)}(x, y)$ for $|y| \geqq 1$, from which we can derive the desired result.

For simplicity, we set $\Omega(x) = \omega(|x|)$ for a positive monotone function ω on the interval $[0, \infty)$. Further, fixing m and p, we let ℓ_{ω} be the smallest integer ℓ satisfying

$$\int_1^{\infty} r^{p'(m-n/p-\ell-1)}\omega(r)^{-p'/p}r^{-1}dr < \infty,$$

if it exists, where $1/p + 1/p' = 1$; and for $\ell \geqq \max\{\ell_{\omega}, m-n\}$, let

$$\omega_{\ell}(r) = \left(\int_r^{\infty} s^{p'(m-n/p-\ell-1)}\omega(s)^{-p'/p}s^{-1}ds\right)^{1/p'}.$$

REMARK. If ω is a positive monotone function on the interval $[0, \infty)$ for which there exists $A > 0$ such that

$$(\omega 1) \qquad A^{-1}\omega(r) \leqq \omega(2r) \leqq A\omega(r) \qquad \text{for } r > 0,$$

then ℓ_{ω} exists and $\ell_{\omega} \leqq m-n/p + \alpha/p$, where $\alpha = \log_2 A$. In case $\omega(r) = r^{-\delta}$ for $r > 1$, we note that $\ell_{\omega} \leqq m-n/p + \delta/p < \ell_{\omega} + 1$.

Throughout this paper, let ω be a positive monotone function on $[0, \infty)$ satisfying condition $(\omega 1)$.

LEMMA 4. *If $\ell \geqq \max\{-1, \ell_{\omega}, m-n\}$ and f is a nonnegative measurable function on R^n satisfying $\int_{R^n} f(y)^p\Omega(y)dy < \infty$, then*

$$\int_{R^n - B(0,2|x|)} |K_{m,\lambda,\ell}(x, y)|f(y)dy \leqq M|x|^{\ell+1}\Omega_{\ell}(x)F(x)$$

whenever $|\lambda| = m$ *and* $x \in R^n - B(0, 2)$, *where* M *is a positive constant independent of* x, $\Omega_\ell(x) = \omega_\ell(|x|)$ *and*

$$F(x) = \left(\int_{R^n - B(0,2|x|)} f(y)^p \Omega(y) dy \right)^{1/p}.$$

PROOF. By Lemma 2 we have

$$\int_{R^n - B(0,2|x|)} |K_{m,\lambda,\ell}(x, y)| f(y) dy \leq M|x|^{\ell+1} \int_{R^n - B(0,2|x|)} |y|^{m-n-\ell-1} f(y) dy.$$

By Hölder's inequality, we see that the right hand side is dominated by

$$M_1 |x|^{\ell+1} \left(\int_{R^n - B(0,2|x|)} (|y|^{m-n-\ell-1} \Omega(y)^{-1/p})^{p'} dy \right)^{1/p'} F(x) \leq M_2 |x|^{\ell+1} \omega_\ell(|x|) F(x)$$

with positive constants M_1 and M_2. Thus the lemma is proved.

LEMMA 2′. *If* $2m - n - |\lambda| > \ell \geqq -1$, *then*

$$|K_{m,\lambda,\ell}(x, y)| \leqq M|x|^{\ell+1} |y|^{2m-n-|\lambda|-\ell-1} h(2|y|)$$

whenever $|y| \geqq 2|x|$ *and* $|y| \geqq 1$, *where* M *is a positive constant independent of* x *and* y.

LEMMA 3′. *If* $2m - n - |\lambda| > \ell \geqq -1$, *then*

$$|K_{m,\lambda,\ell}(x, y)| \leqq M|x|^{2m-n-|\lambda|} h(4|x|) \quad \textit{whenever } 1 \leqq |y| \leqq 2|x|,$$

where M *is a positive constant independent of* x *and* y.

Let ℓ'_ω be the smallest integer ℓ satisfying

$$\int_1^\infty r^{p'(m-n/p-\ell-1)} h(r)^{p'} \omega(r)^{-p'/p} r^{-1} dr < \infty.$$

We note that $\ell'_\omega = \ell_\omega$ or $\ell_\omega + 1$. If $\ell'_\omega \leqq \ell < m - n$, then we set

$$\omega_\ell(r) = \left(\int_r^\infty s^{p'(m-n/p-\ell-1)} h(s)^{p'} \omega(s)^{-p'/p} s^{-1} ds \right)^{1/p'}$$

(compare it with that defined for $\ell \geqq \max\{\ell_\omega, m-n\}$).

REMARK. If $\omega(r) = r^{-\delta}$ on the interval $(1, \infty)$, then $\ell_\omega = \ell'_\omega$ and, for $\ell_\omega \leqq \ell < m - n$, we have

$$\omega_\ell(r) \leqq M r^{m-n/p-\ell-1+\delta/p} \log r,$$

where M is a positive constant independent of $r > 2$.

LEMMA 4′. *If* $|\lambda| = m$, $\max\{-1, \ell_\omega'\} \leqq \ell < m-n$ *and* f *is a nonnegative measurable function on* R^n *satisfying* $\int_{R^n} f(y)^p \Omega(y) dy < \infty$, *then*

$$\int_{R^n - B(0,2|x|)} |K_{m,\lambda,\ell}(x, y)| f(y) dy \leqq M|x|^{\ell+1} \Omega_\ell(x) F(x)$$

for every $x \in R^n - B(0, 2)$, *where* M *is a positive constant independent of* x, $\Omega_\ell(x) = \omega_\ell(|x|)$ *and* F *is as in Lemma* 4.

3. L^p-estimates with weight

In this section we give L^p-estimates with weight of $D^\mu \int K_{m,\lambda,\ell}(x,y) f(y) dy$, $|\mu| = m$, for functions f satisfying $\int |f(y)|^p \Omega(y) dy < \infty$.

We begin with showing the following technical lemma.

LEMMA 5. *Let* f *be a nonnegative measurable function on* R^n *such that* $\int f(y)^p \Omega(y) dy < \infty$. *Let* ℓ *be an integer such that* $\ell \geqq \max\{-1, \ell_\omega, m-n\}$ *or* $\max\{-1, \ell_\omega'\} \leqq \ell < m-n$. *For* $R > 1$, *we write*

$$U_\ell f(x) = \int K_{m,\lambda,\ell}(x, y) f(y) dy$$

and

$$U_{\ell,R} f(x) = \int_{B(0,2R)} K_{m,\lambda,\ell}(x, y) f(y) dy.$$

Then $U_\ell f \in BL_m(L^p_{loc}(R^n))$ *and* $U_{\ell,R} f$ *tends to* $U_\ell f$ *in* $BL_m(L^p_{loc}(R^n))$ *as* $R \to \infty$.

PROOF. If we set $V_{\ell,R} f(x) = \int_{R^n - B(0,2R)} K_{m,\lambda,\ell}(x, y) f(y) dy$, then Lemmas 4 and 4′ imply that $V_{\ell,R} f(x)$ is absolutely convergent for every $x \in B(0, R)$. Further, since $(\partial/\partial x)^\mu K_{m,\lambda,\ell}(x, y) = K_{m,\lambda+\mu,\ell-|\mu|}(x, y)$, we see, in view of Lemmas 2 and 2′ (cf. the proof of Lemma 4), that $V_{\ell,R} f$ is infinitely differentiable and $(\partial/\partial x)^\mu V_{\ell,R}(x) = \int_{R^n - B(0,2R)} K_{m,\lambda+\mu,\ell-|\mu|}(x, y) f(y) dy$ on $B(0, R)$. On the other hand, by Lemma 3.3 in [4], we find that $U_{\ell,R} f \in BL_m(L^p_{loc}(R^n))$, because $U_{\ell,R}f(x)=\int_{B(0,2R)} D^{\lambda}k_m(x-y)f(y)dy$ + a polynomial. Consequently, $U_\ell f \in BL_m(L^p_{loc}(R^n))$. By Lemmas 2 and 2′ again, we see that $(\partial/\partial x)^\mu V_{\ell,R}(x)$ are all convergent to 0 locally uniformly as $R\to\infty$ on R^n, so that $U_{\ell,R}f(x)\to U_\ell f(x)$ in $BL_m(L^p_{loc}(R^n))$ as $R\to\infty$. Thus Lemma 5 is proved.

REMARK. We can also prove that $\int |K_{m,\lambda,\ell}(x, y)|f(y)dy \in L^p_{loc}(R^n)$, since $\int_{B(0,2R)} |D^\lambda k_m(x-y)|f(y)dy \in L^p_{loc}(R^n)$ and $\int_{R^n-B(0,2R)} |K_{m,\lambda,\ell}(x, y)|f(y)dy$ is bounded in $B(0, R)$.

PROPOSITION 1. *Let $\ell \geqq m$ and ω be a positive nonincreasing function on the interval $[0, \infty)$ satisfying $(\omega 1)$ and the following conditions:*

(i) *There exists a number α such that $\alpha > n+\ell-m$ and*

$$(\omega 2)\qquad \int_1^r s^{-\alpha p'+n}\omega(s)^{-p'/p}s^{-1}ds \leqq M_1 r^{-\alpha p'+n}\omega(r)^{-p'/p} \qquad \textit{for any } r>1.$$

(ii) *There exists a number β such that $\beta < n+\ell-m+1$ and*

$$(\omega 3)\qquad \int_r^\infty s^{-\beta p'+n}\omega(s)^{-p'/p}s^{-1}ds \leqq M_2 r^{-\beta p'+n}\omega(r)^{-p'/p} \qquad \textit{for any } r>0.$$

Here M_1 and M_2 are positive constants independent of r. If $|\lambda|=|\mu|=m$, then

$$\int \left| D^\mu \int K_{m,\lambda,\ell}(x, y)f(y)dy \right|^p \Omega(x)dx \leqq M\int f(y)^p\Omega(y)dy$$

for any nonnegative measurable function f on R^n, where M is a positive constant independent of f.

REMARK. If (ii) is fulfilled, then, since $-\beta p'+n > p'(m-n/p-\ell-1)$, we see that $\ell \geqq \ell'_\omega$ $(\geqq \ell_\omega)$.

PROOF OF PROPOSITION 1. By Lemma 5 we may assume that f vanishes outside a compact set in R^n. Then it follows from [4; Lemma 5.1] that $(\partial/\partial x)^\mu U_\ell f(x)$ is of the form

$$af(x)+\int D^{\mu+\lambda}k_m(x-y)f(y)dy$$
$$-\sum_{|\nu|\leqq \ell-m}(\nu!)^{-1}x^\nu\int_{R^n-B(0,1)} D^{\lambda+\mu+\nu}k_m(-y)f(y)dy$$

with a constant a. Here $\int D^{\mu+\lambda}k_m(x-y)f(y)dy$ is understood to be $\lim_{r\downarrow 0}\int_{R^n-B(x,r)} D^{\mu+\lambda}k_m(x-y)f(y)dy$, which exists almost everywhere on R^n and, since $f\in L^p(R^n)$, it belongs to $L^p(R^n)$ because of [4; Lemma 3.3]. For $x\in R^n$ and $|\mu|=m$, we set

$$u_1(x)=\int_{B(0,2|x|)} D^{\mu+\lambda}k_m(x-y)f(y)dy$$

$$-\sum_{|\nu|\leqq \ell-m}(\nu!)^{-1}x^\nu\int_{B(0,2|x|)-B(0,1)} D^{\lambda+\mu+\nu}k_m(-y)f(y)dy$$

and

$$u_2(x)=\int_{R^n-B(0,2|x|)} D^{\mu+\lambda}k_m(x-y)f(y)dy$$

$$-\sum_{|\nu|\leqq \ell-m}(\nu!)^{-1}x^\nu\int_{R^n-B(0,2|x|)-B(0,1)} D^{\lambda+\mu+\nu}k_m(-y)f(y)dy$$

$$=\int_{R^n-B(0,2|x|)} K_{m,\lambda+\mu,\ell-m}(x,y)f(y)dy.$$

If $x\in B(0,2^{j+1})-B(0,2^j)$, then

$$|u_1(x)|\leqq M_1\Big(\Big|\int_{B(0,2^{j+2})-B(0,2^{j-1})} D^{\mu+\lambda}k_m(x-y)f(y)dy\Big|$$

$$+\int_{A(x)}|D^{\mu+\lambda}k_m(x-y)|f(y)dy$$

$$+|x|^{\ell-m}\int_{B(0,2|x|)-B(0,1)}|y|^{m-n-\ell}f(y)dy\Big)$$

$$=M_1[u_{11}(x)+u_{12}(x)+u_{13}(x)]$$

with a positive constant M_1 independent of x, where $A(x)=B(0,2^{j-1})\cup[B(0,2^{j+2})-B(0,2|x|)]$. First we have by Lemma 3.3 in [4]

$$\int u_{11}(x)^p\Omega(x)dx\leqq \sum_j \omega(2^j)\int\Big|\int_{B(0,2^{j+2})-B(0,2^{j-1})} D^{\mu+\lambda}k_m(x-y)f(y)dy\Big|^p dx$$

$$\leqq M_2\sum_j \omega(2^j)\int_{B(0,2^{j+2})-B(0,2^{j-1})} f(y)^p dy$$

$$\leqq M_3\int f(y)^p\Omega(y)dy$$

with positive constants M_2 and M_3 independent of f. Next, since $|x-y|\geqq |x|/2$ for $y\in A(x)$, $u_{12}(x)\leqq M_4|x|^{-n}\int_{B(0,4|x|)} f(y)dy$ with a positive constant M_4 independent of x. Since $\Omega(x)\leqq A^2\Omega(y)$ whenever $y\in B(0,4|x|)$, letting $0<\delta<n/p'$, we have

$$\int u_{12}(x)^p \Omega(x)dx \leqq M_4^p \int |x|^{-np}\left(\int_{B(0,4|x|)} |y|^{-\delta p'}dy\right)^{p/p'}$$
$$\times \left(\int_{B(0,4|x|)} |y|^{\delta p} f(y)^p dy\right)\Omega(x)dx$$
$$\leqq M_5 \int\left(|x|^{-\delta p-n}\left(\int_{B(0,4|x|)} |y|^{\delta p} f(y)^p \Omega(y)dy\right)dx\right.$$
$$= M_5 \int |y|^{\delta p} f(y)^p \Omega(y)\left(\int_{R^n - B(0,|y|/4)} |x|^{-\delta p-n}dx\right)dy$$
$$\leqq M_6 \int f(y)^p \Omega(y)dy$$

with positive constants M_5 and M_6. Similarly, using (ω2), we see that

$$\int u_{13}(x)^p \Omega(x)dx \leqq \int |x|^{(\ell-m)p}\left(\int_{B(0,2|x|)-B(0,1)} |y|^{-\alpha p'}\Omega(y)^{-p'/p}dy\right)^{p/p'}$$
$$\times \left(\int_{B(0,2|x|)} |y|^{(\alpha-n-\ell+m)p} f(y)^p \Omega(y)dy\right)\Omega(x)dx$$
$$\leqq M_7 \int (|x|^{-\alpha p+np/p'+(\ell-m)p}\left(\int_{B(0,2|x|)} |y|^{(\alpha-n-\ell+m)p} f(y)^p \Omega(y)dy\right)dx$$
$$= M_7 \int |y|^{(\alpha-n-\ell+m)p} f(y)^p \Omega(y)\left(\int_{R^n-B(0,|y|/2)} |x|^{-\alpha p+np/p'+(\ell-m)p}dx\right)dy$$
$$\leqq M_8 \int f(y)^p \Omega(y)dy$$

with positive constants M_7 and M_8.

On the other hand, by Lemma 2 we obtain

$$|u_2(x)| \leqq M_9 \int_{R^n-B(0,2|x|)} |x|^{\ell-m+1}|y|^{-n-(\ell-m)-1} f(y)dy$$
$$+ M_9 \int_{B(0,1)-B(0,2|x|)} |y|^{-n} f(y)dy = M_9[u_{21}(x) + u_{22}(x)]$$

with a positive constant M_9. It follows from condition (ω3) that

$$\int |u_{21}(x)|^p \Omega(x)dx \leqq \int |x|^{(\ell-m+1)p}\left(\int_{R^n-B(0,2|x|)} |y|^{-\beta p'}\Omega(y)^{-p'/p}dy\right)^{p/p'}$$
$$\times \left(\int_{R^n-B(0,2|x|)} |y|^{(\beta-n-\ell+m-1)p} f(y)^p \Omega(y)dy\right)\Omega(x)dx$$
$$\leqq M_{10} \int\left(|x|^{-\beta p+np/p'+(\ell-m+1)p}\right.$$

$$\times \left(\int_{R^n - B(0,2|x|)} |y|^{(\beta - n - \ell + m - 1)p} f(y)^p \Omega(y) dy \right) dx$$

$$\leqq M_{11} \int |y|^{(\beta - n - \ell + m - 1)p} f(y)^p \Omega(y)$$

$$\times \left(\int_{B(0,|y|/2)} |x|^{-\beta p + np/p' + (\ell - m + 1)p} dx \right) dy$$

$$\leqq M_{12} \int f(y)^p \Omega(y) dy$$

with positive constants $M_{10} \sim M_{12}$. Letting $n/p' < \gamma < n$ and noting that $u_{22}(x) = 0$ for $x \in R^n - B(0, 1/2)$ and both $\Omega(x)$ and $\Omega(x)^{-1}$ are bounded on $B(0, 1)$, we establish

$$\int |u_{22}(x)|^p \Omega(x) dx \leqq \int_{B(0,1/2)} \left(\int_{R^n - B(0,2|x|)} |y|^{-\gamma p'} dy \right)^{p/p'}$$

$$\times \left(\int_{B(0,1) - B(0,2|x|)} |y|^{(\gamma - n)p} f(y)^p dy \right) \Omega(x) dx$$

$$\leqq M_{13} \int_{B(0,1/2)} |x|^{-\gamma p + np/p'}$$

$$\times \left(\int_{B(0,1) - B(0,2|x|)} |y|^{(\gamma - n)p} f(y)^p dy \right) dx$$

$$\leqq M_{13} \int_{B(0,1)} |y|^{(\gamma - n)p} f(y)^p \left(\int_{B(0,|y|/2)} |x|^{-\gamma p + np/p'} dx \right) dy$$

$$\leqq M_{14} \int_{B(0,1)} f(y)^p dy \leqq M_{15} \int f(y)^p \Omega(y) dy$$

with positive constants $M_{13} \sim M_{15}$. Thus Proposition 1 is proved.

PROPOSITION 2. *Let $\ell < m$ and ω be a positive nonincreasing function on* $[0, \infty)$ *satisfying* (ω1) *and* (ii) *in Proposition* 1. *Then the same conclusion as in Proposition* 1 *holds.*

The proof can be carried out in the same way as that of Proposition 1. In fact, in this case, $D^\mu \int K_{m,\lambda,\ell}(x, y) f(y) dy$ is of the form

$$af(x) + \int D^{\mu+\lambda} k_m(x - y) f(y) dy$$

with a constant a, and $\left| \int D^{\mu+\lambda} k_m(x - y) f(y) dy \right| \leqq M[u_{11}(x) + u_{12}(x) + v(x)]$, where u_{11} and u_{12} are as in the proof of Proposition 1 and $v(x) = \int_{R^n - B(0,2|x|)} |y|^{-n} f(y)dy$. Since u_{11} and u_{12} are evaluated in the proof of Proposition 1, we have only to treat the function v. By noting that β in (ω3) is smaller than n, we establish

$$\begin{aligned}\int v(x)^p \Omega(x)dx &\leqq \int\left(\int_{R^n - B(0,2|x|)} |y|^{-\beta p'} \Omega(y)^{-p'/p} dy\right)^{p/p'} \\ &\quad\times\left(\int_{R^n - B(0,2|x|)} |y|^{(\beta-n)p} f(y)^p \Omega(y)dy\right)\Omega(x)dx \\ &\leqq M_1 \int\left(|x|^{-\beta p + np/p'}\left(\int_{R^n - B(0,2|x|)} |y|^{(\beta-n)p} f(y)^p \Omega(y)dy\right)dx\right. \\ &\leqq M_2 \int |y|^{(\beta-n)p} f(y)^p \Omega(y)\left(\int_{B(0,|y|/2)} |x|^{-\beta p + np/p'} dx\right)dy \\ &\leqq M_3 \int f(y)^p \Omega(y)dy\end{aligned}$$

with positive constants M_1, M_2 and M_3.

REMARK. Let $\omega(r) = r^{-\delta}$ for $r > 1$, where $\delta \geqq 0$. If $-1 \leqq \ell < m - n/p + \delta/p < \ell + 1$, then ω satisfies conditions (ω1), (i) and (ii). If $\ell = m - n/p + \delta/p \geqq -1$, then ω satisfies (ω1) and (ii), but not (i).

In view of the proof of Proposition 1, we can establish the following variant of Proposition 1.

PROPOSITION 3. *Let ω be a positive nonincreasing function on the interval* $[0, \infty)$ *satisfying condition* (ω1) *together with* (ii) *in Proposition* 1. *If ω_ℓ^* is a positive nonincreasing function on* $[0, \infty)$ *such that*

$$\omega_\ell^*(r) = r^{(m-\ell-n/p)p}\left(\int_1^r s^{p'(m-n/p-\ell)}\omega(s)^{-p'/p} s^{-1} ds\right)^{-p/p'} \quad \textit{for } r > 2,$$

then

$$\int\left|D^\mu \int K_{m,\lambda,\ell}(x, y)f(y)dy\right|^p \Omega_\ell^*(x)dx \leqq M \int f(y)^p \Omega(y)dy$$

for $|\mu| = m$, where $\Omega_\ell^*(x) = \omega_\ell^*(|x|)$ and M is a positive constant independent of f.

REMARK. If ω satisfies condition (ω1), then we can find a positive constant M_1 such that $\omega_\ell^*(r) \leqq M_1\omega(r)$ for $r \geqq r_0 > 1$. If $\ell \geqq \ell_\omega$, then $\omega_\ell^*(r) \geqq M_2 r^{p(m-n/p-\ell-1)}$ for $r > 1$ with a positive constant M_2.

PROPOSITION 1′. *Let $\ell \geqq m$ and ω be a positive nondecreasing function on* *the interval* $[0, \infty)$ *satisfying* (ω1), (i), (ii) *in Proposition* 1 *and*

$$(\omega 4) \qquad \int_1^\infty r^{-np+n}\omega(r)r^{-1}dr < \infty.$$

If $|\lambda| = |\mu| = m$, *then*

$$\int \left| D^\mu \int K_{m,\lambda,\ell}(x, y)f(y)dy \right|^p \Omega(x)dx \leqq M \int f(y)^p \Omega(y)dy$$

for any nonnegative measurable function f *on* R^n, *where* M *is a positive constant independent of* f.

PROOF. Let f be a nonnegative measurable function on R^n such that $\int f(y)^p\Omega(y)dy < \infty$. As in the proof of Proposition 1, we may assume that f vanishes outside a compact set in R^n, and write $D^\mu \int K_{m,\lambda,\ell}(x, y)f(y)dy = af(x) + u_1(x) + u_2(x)$, where a is a positive constant, $x \in R^n$ and $|\mu| = m$. As in the proof of Proposition 1, $|u_1(x)| \leqq M_1[u_{11}(x) + u_{12}(x) + u_{13}(x)]$, and we can prove that

$$\int u_{11}(x)^p\Omega(x)dx \leqq M_2 \int f(y)^p\Omega(y)dy$$

and

$$\int u_{13}(x)^p\Omega(x)dx \leqq M_2 \int f(y)^p\Omega(y)dy$$

with a positive constant M_2 independent of f. Also, $|u_{12}(x)| \leqq M_3[u'_{12}(x) + u''_{12}(x)]$ with a positive constant M_3, where $u'_{12}(x) = |x|^{-n}\int_{B(0,4|x|)-B(0,1)} f(y)dy$ and $u''_{12}(x) = |x|^{-n}\int_{B(0,4|x|)\cap B(0,1)} f(y)dy$. We derive from ($\omega$2)

$$\begin{aligned}
\int u'_{12}(x)^p\Omega(x)dx &\leqq \int |x|^{-np}\left(\int_{B(0,4|x|)-B(0,1)} |y|^{-\alpha p'}\Omega(y)^{-p'/p}dy\right)^{p/p'} \\
&\quad \times \left(\int_{B(0,4|x|)} |y|^{\alpha p}f(y)^p\Omega(y)dy\right)\Omega(x)\, dx \\
&\leqq M_4 \int \left(|x|^{-\alpha p-n}\left(\int_{B(0,4|x|)} |y|^{\alpha p}f(y)^p\Omega(y)dy\right)dx\right. \\
&= M_4 \int |y|^{\alpha p}f(y)^p\Omega(y)\left(\int_{R^n - B(0,|y|/4)} |x|^{-\alpha p-n}dx\right)dy
\end{aligned}$$

$$\leqq M_5 \int f(y)^p \Omega(y) dy.$$

Moreover, letting $0 < \delta < n/p'$ and using (ω4), we find

$$\int u''_{12}(x)^p \Omega(x) dx \leqq \int_{B(0,1/4)} |x|^{-np} \left(\int_{B(0,4|x|)} |y|^{-\delta p'} dy \right)^{p/p'}$$
$$\times \left(\int_{B(0,4|x|)} |y|^{\delta p} f(y)^p dy \right) \Omega(x) dx$$
$$+ \int_{R^n - B(0,1/4)} |x|^{-np} \left(\int_{B(0,1)} f(y) dy \right)^p \Omega(x) dx$$
$$\leqq M_6 \int_{B(0,1/4)} |x|^{-\delta p - n} \left(\int_{B(0,4|x|)} |y|^{\delta p} f(y)^p dy \right) dx$$
$$+ M_6 \left(\int_{R^n - B(0,1/4)} |x|^{-np} \Omega(x) dx \right) \left(\int_{B(0,1)} f(y)^p dy \right)$$
$$\leqq M_6 \int_{B(0,1)} |y|^{\delta p} f(y)^p \left(\int_{R^n - B(0,|y|/4)} |x|^{-\delta p - n} dx \right) dy$$
$$+ M_7 \int_{B(0,1)} f(y)^p dy$$
$$\leqq M_8 \int_{B(0,1)} f(y)^p dy \leqq M_9 \int f(y)^p \Omega(y) dy$$

with positive constants $M_6 \sim M_9$.

Since the same evaluations as in the proof of Proposition 1 are true for u_2, we complete the proof of Proposition 1′.

PROPOSITION 2′. *Let* $-1 \leqq \ell < m$ *and* ω *be a positive nondecreasing function on* $[0, \infty)$ *satisfying* (ω1), (ω2) *with* $\alpha > 0$, (ω4) *and* (ii) *in Proposition* 1. *Then the same conclusion as in Proposition* 1 *holds.*

PROPOSITION 3′. *Let* ω *be a positive nondecreasing function on the interval* $[0,\infty)$ *satisfying conditions* (ω1), (ω2) *with* $\alpha > 0$, (ω4) *and* (ii) *in Proposition* 1. *Suppose* $\omega_\ell^*(r) = r^{(m-\ell-n/p)p} \left(\int_1^r s^{p'(m-n/p-\ell)} \omega(s)^{-p'/p} s^{-1} ds \right)^{-p/p'}$ *is nondecreasing on some interval* $[r_0, \infty)$; *and set* $\omega_\ell^*(r) = \omega_\ell^*(r_0)$ *for* $r < r_0$. *Then*

$$\int \left| D^\mu \int K_{m,\lambda,\ell}(x, y) f(y) dy \right|^p \Omega_\ell^*(x) dx \leqq M \int f(y)^p \Omega(y) dy$$

for $|\mu| = m$, *where* $\Omega_\ell^*(x) = \omega_\ell^*(|x|)$ *and* M *is a positive constant independent of* f.

4. Integral representation

Now we establish the integral representation of Beppo Levi functions as given in the Introduction.

THEOREM 1. *Let* ω *be a positive monotone function on the interval* $[0, \infty)$ *satisfying condition* (ω1), *and suppose further* $\ell_\omega \geqq m-n$. *If* u *is a function in* $BL_m(L^p_{loc}(R^n))$ *satisfying* (1), *then there exists a polynomial* P, *which is polyharmonic of order* m *in* R^n, *such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell_\omega}(x, y)D^\lambda u(y)dy + P(x) \qquad a.e. \quad on\ R^n.$$

REMARK. We recall that $\ell_\omega \leqq m-n/p+\alpha/p$ with $\alpha = \log_2 A$ (see the Remark given before Lemma 4). We shall show below that the degree of P is at most $\max\{m-1, \ell_\omega+1\}$.

PROOF OF THEOREM 1. For $\ell \geqq \max\{-1, \ell_\omega\}$, set $U_\ell(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy$. By Lemma 5 and its Remark, $U_\ell \in BL_m(L^p_{loc}(R^n))$ and, moreover,

$$\iint |K_{m,\lambda,\ell}(x, y)D^\lambda u(y)\varphi(x)|dydx < \infty$$

for any $\varphi \in C_0^\infty(R^n)$. By (2), there exists a number c_m such that $\Delta^m = c_m \sum_{|\lambda|=m} a_\lambda D^{2\lambda}$ (cf. [4; §4]). Hence we have by Fubini's theorem and the fact that $\Delta^m_x[K_{m,\lambda,\ell}(x, y) - D^\lambda k_m(x-y)] = 0$,

$$\begin{aligned}
\int U_\ell(x)\Delta^m\varphi(x)dx &= \int \sum_{|\lambda|=m} a_\lambda\left(\int K_{m,\lambda,\ell}(x, y)\Delta^m\varphi(x)dx\right)D^\lambda u(y)dy \\
&= \int \sum_{|\lambda|=m} a_\lambda\left(\int D^\lambda k_m(x-y)\Delta^m\varphi(x)dx\right)D^\lambda u(y)dy \\
&= \int \sum_{|\lambda|=m} a_\lambda\left((-1)^{|\lambda|}\int k_m(x-y)D^\lambda\Delta^m\varphi(x)dx\right)D^\lambda u(y)dy \\
&= \int \sum_{|\lambda|=m} a_\lambda[c_m(-1)^m D^\lambda\varphi(y)]D^\lambda u(y)dy \\
&= \int \Delta^m\varphi(y)u(y)dy.
\end{aligned}$$

Hence $\Delta^m(u - U_\ell) = 0$ in the sense of distributions. What remains is to show that $P_\ell \equiv u - U_\ell$ is a polynomial.

In view of Proposition 3 and the Remark after Proposition 3, we see that if ω is nonincreasing and satisfies (ω1) and (ii) with $\ell = \ell^* \equiv \max\{-1, \ell_\omega\}$, then the function P_{ℓ^*} satisfies

$$\int [|D^\mu P_{\ell^*}(x)|(|x|+1)^{m-n/p-\ell^*-1}]^p dx < \infty \qquad \text{for } |\mu| = m.$$

By noting that $\Delta^m P_{\ell^*} = 0$ on R^n and considering the Fourier transform, we find that P_{ℓ^*} is a polynomial of degree at most $\max\{m-1, \ell^*\}$ (cf. [4; Lemma 4.1]). If $\ell \geqq \max\{-1, \ell_\omega\}$, then

$$(3) \qquad P_\ell = P_{\ell^*} - \sum_{|\lambda|=m} a_\lambda \int [K_{m,\lambda,\ell}(\cdot, y) - K_{m,\lambda,\ell^*}(\cdot, y)] D^\lambda u(y) dy,$$

so that P_ℓ is a polynomial of degree at most $\max\{m-1, \ell\}$. In case ω is nonincreasing and satisfies (ω1) only, we see from the definition of ℓ_ω that $\omega(r) \geqq M r^{p(m-n/p-\ell_\omega-1)}$ for $r > 1$ and $m - n/p - \ell_\omega - 1 < 0$. If we let $\omega^\sim(r) = (r+1)^{p(m-n/p-\ell_\omega-1)}$, then u satisfies (1) with ω replaced by $\omega^\sim$. Since $\omega^\sim$ satisfies condition (ii) with $\ell = \ell^\sim \equiv \max\{-1, \ell_\omega + 1\}$, from the above considerations we find that for $\ell \geqq \max\{-1, \ell_\omega\}$, P_ℓ is a polynomial of degree at most $\max\{m-1, \ell^\sim, \ell\}$; this implies that the degree of P_{ℓ_ω} is at most $\max\{m-1, \ell_\omega + 1\}$ and the degree of P_ℓ, $\ell \geqq \ell_\omega + 1$, is at most $\max\{m-1, \ell\}$.

If ω is nondecreasing, then $u \in BL_m(L^p(R^n))$; i.e., (1) holds with $\omega(r) \equiv 1$. Hence, by the above discussion, it follows that $P_{\ell^\#}$, where $\ell^\#$ is the integer such that $\ell^\# \leqq m - n/p < \ell^\# + 1$, is a polynomial of degree at most $m-1$. By (3), P_ℓ for $\ell \geqq \max\{-1, \ell_\omega\}$ is a polynomial of degree at most $\max\{m-1, \ell\}$. Thus the proof of Theorem 1 is completed.

The case $\ell_\omega < m - n$ can be derived along the same lines as in the proof of Theorem 1, by using Lemmas 2′, 3′ and 4′ instead of Lemmas 2, 3 and 4.

THEOREM 1′. *Let ω be a positive monotone function on the interval* $[0, \infty)$ *satisfying condition* (ω1). *If $\ell_\omega < m - n$ and u is a function in $BL_m(L^p_{loc}(R^n))$ satisfying* (1), *then there exists a polynomial P such that*

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell'_\omega}(x, y) D^\lambda u(y) dy + P(x) \qquad a.e. \ \ on \ R^n.$$

OUTLINE OF THE PROOF. We shall deal only with the case when ω is nonincreasing. For $\ell \geqq \max\{-1, \ell'_\omega\}$, we set $U_\ell(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y) D^\lambda u(y) dy$ and $P_\ell = u - U_\ell$. If $\ell \geqq m - n$, then the proof of Theorem 1 implies that P_ℓ is a polynomial. If $\ell < m - n$, then from Lemmas 5 it follows that U_ℓ belongs to $BL_m(L^p_{loc}(R^n))$, and

$$\iint |K_{m,\lambda,\ell}(x, y)D^\lambda u(y)\varphi(x)|dydx < \infty$$

for any $\varphi \in C_0^\infty(R^n)$. Therefore, as in the proof of Theorem 1, we see that $\Delta^m(u - U_\ell) = 0$ in the sense of distributions. To show that P_ℓ is a polynomial, we first note that $\omega(r) \geqq Mr^{p(m-n/p-\ell'_\omega-1)}h(r)^p$ for $r > 2$ and $m - n/p - \ell'_\omega - 1 < 0$. Thus u satisfies (1) with ω replaced by $\omega^\sim(r) = (r+1)^{p(m-n/p-\ell^\sim)}$, where $\ell^\sim = \max\{-1, \ell'_\omega + 1\}$. Moreover $\ell^\sim < m$ and condition (ii) in Proposition 1 is satisfied with $\ell = \ell^\sim$. Consequently we can apply Proposition 2 to obtain

$$\int |D^\mu P_{\ell^\sim}(x)|^p \omega^\sim(|x|)dx < \infty \qquad \text{for } |\mu| = m.$$

Thus $P_{\ell^\sim}$ is a polynomial, and then for $\ell \geqq \ell'_\omega$, $P_\ell = P_{\ell^\sim} - \sum_{|\lambda|=m} a_\lambda \int [K_{m,\lambda,\ell}(x, y) - K_{m,\lambda,\ell^\sim}(x, y)]D^\lambda u(y)dy$ is a polynomial.

5. Behavior at infinity of Beppo Levi functions

For sets E and $G \subset R^n$, we define $C_{m,p}(E; G) = \inf \|f\|_p^p$, where the infimum is thaken over all nonnegative measurable functions f such that $f = 0$ outside G and $\int_G |x-y|^{m-n}f(y)dy \geqq 1$ for every $x \in E$; for the properties of the capacity $C_{m,p}$, we refer to the paper of Meyers [3]. We say that a function u is (m, p)-quasi continuous on R^n if for any $\varepsilon > 0$ and any bounded open set $G \subset R^n$, there exists an open set $B \subset G$ such that $C_{m,p}(B; G) < \varepsilon$ and u is continuous as a function on $G - B$; for details, we refer the reader to [4].

Let u be an (m, p)-quasi continuous function on R^n satisfying condition (1); here ω is assumed to satisfy condition (ω1). Then Theorems 1 and 2 imply the existence of an integer ℓ and a polynomial P_ℓ of degree at most $\max\{m-1, \ell+1\}$ such that

$$u(x) = \sum_{|\lambda|=m} a_\lambda \int K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy + P_\ell(x) \qquad \text{a.e. on } R^n. \tag{4}$$

If we write

$$U_\lambda(x) = \int K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy = \int_{B(0,2R)} K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy$$
$$+ \int_{R^n - B(0,2R)} K_{m,\lambda,\ell}(x, y)D^\lambda u(y)dy = U_{\lambda,R}(x) + V_{\lambda,R}(x)$$

for $R > 0$, then we see that $U_{\lambda,R}$ is (m, p)-quasi continuous on R^n and $V_{\lambda,R}$ is continuous on $B(0, R)$, on account of [4; Lemma 3.3]. Hence U_λ is (m, p)-quasi continuous on R^n, so that equality (4) holds for any $x \in R^n - E_0$, where E_0 is a set satisfying $C_{m,p}(E_0 \cap B(0, r); B(0, 2r)) = 0$ for any $r > 0$. We first study the behavior at infinity of the functions U_λ. More generally, we deal with the function $U(x) = \int K_{m,\lambda,\ell}(x, y)f(y)dy$, where ℓ is an integer such that $\ell \geqq -1$ and f is a nonnegative measurable function on R^n such that $\int f(y)^p \omega(|y|)dy < \infty$. For $x \in R^n - B(0, 2)$, write $U = v + w$, where

$$v(x) = \int_{B(0,2|x|)} K_{m,\lambda,\ell}(x, y)f(y)dy$$

and

$$w(x) = \int_{R^n - B(0,2|x|)} K_{m,\lambda,\ell}(x, y)f(y)dy.$$

By Lemmas 4 and 4′, we know that

$$|w(x)| \leqq M|x|^{\ell+1}\omega_\ell(|x|)F(x) \tag{5}$$

with a positive constant M independent of x.

In case $\ell \geqq \max\{0, m-n\}$, by use of Lemma 3, we find a positive constant M such that

$$|v(x)| \leqq M\{v'(x) + v''(x) + v'''(x)\},$$

where

$$v'(x) = \int_{B(0,1)} |x-y|^{m-n}[|h(|x-y|)| + 1]f(y)dy,$$

$$v''(x) = |x|^\ell \int_{B(0,2|x|)-B(0,1)} |y|^{m-n-\ell}h(4|x|/|y|)f(y)dy$$

and

$$v'''(x) = \int_{B(x,|x|/2)} |x-y|^{m-n}h(|x|/|x-y|)f(y)dy.$$

Then we first note that $v'(x) = O(|x|^{m-n}h(|x|))$ as $|x| \to \infty$.

As to v'', by Hölder's inequality we obtain

$$v''(x) \leqq M|x|^\ell \Omega'_\ell(x)G(x) \tag{6}$$

for any $x \in R^n - B(0, 2)$, where $\Omega'_\ell(x) = \omega'_\ell(|x|)$ with

$$\omega'_\ell(r) = \left(\int_1^r s^{p'(m-n/p-\ell)}h(2r/s)^{p'}\omega(s)^{-p'/p}s^{-1}ds\right)^{1/p'}$$

and $G(x) = \left(\int_{B(0,2|x|)} f(y)^p \Omega(y) dy \right)^{1/p}$.

REMARK. Let $\omega(r) = r^{-\delta}$ for $r > 1$. If $\ell < m - n/p + \delta/p$, then $\omega'_\ell(r) = M_1 r^{m-n/p-\ell+\delta/p}$; if $\ell = m - n/p + \delta/n$, then $\omega'_\ell(r) \leqq M_2 h(r)(\log r)^{1/p'}$ for $r > 2$, where M_1 and M_2 are positive constants.

Finally we treat the function v'''.

LEMMA 6. *Let f be a nonnegative measurable function on R^n such that $\int f(y)^p \Omega(y) dy < \infty$, and let $\varphi(r)$ be a positive function on the interval $(0, \infty)$ for which there exists $M > 0$ such that $\varphi(r) \leqq M \varphi(s)$ whenever $0 < r \leqq s \leqq 2r$. If $mp \leqq n$, then there exists a set $E \subset R^n$ having the following properties:*

(i) $\lim_{|x| \to \infty, x \in R^n - E} \varphi(|x|)^{-1} \omega(|x|)^{1/p} v'''(x) = 0$.

(ii) $\sum_{j=1}^\infty \varphi(2^j)^p C_{m,p}(E_j; G_j) < \infty$,

where $E_j = E \cap B_j$ and $G_j = B_{j-1} \cup B_j \cup B_{j+1}$ with $B_j = B(0, 2^j) - B(0, 2^{j-1})$. If $mp > n$, then

$$v'''(x) \leqq M'|x|^{m-n/p} \omega(|x|)^{-1/p} G(x) \leqq M''|x|^\ell \Omega'_\ell(x) G(x)$$

for any $x \in R^n - B(0, 2)$, where M' and M'' are positive constants independent of x and f.

PROOF. The case $mp > n$ can be derived readily from Hölder's inequality. In case $mp \leqq n$, we choose a sequence $\{a_j\}$ of positive numbers such that $\lim_{j \to \infty} a_j = \infty$ and $\sum_{j=1}^\infty a_j \int_{G_j} f(y)^p \Omega(y) dy < \infty$. For each positive integer j, we define

$$E_j = \{x \in B_j;\ v'''(x) \geqq \varphi(2^j) \omega(2^j)^{-1/p} a_j^{-1/p}\}.$$

If $x \in B_j$, then $v'''(x) \leqq \int_{G_j} |x - y|^{m-n} f(y) dy$. Hence it follows from the definition of $C_{m,p}$ that

$$C_{m,p}(E_j) \leqq \varphi(2^j)^{-p} \omega(2^j) a_j \int_{G_j} f(y)^p dy \leqq A^2 \varphi(2^j)^{-p} a_j \int_{G_j} f(y)^p \Omega(y) dy.$$

This implies that $E = \bigcup_{j=1}^\infty E_j$ satisfies (ii). It is easy to see that (i) is fulfilled with this set E. Thus the lemma is proved.

In case $\ell = -1 \geqq m - n$, $|K_{m,\lambda,\ell}(x, y)| = |D^\lambda k_m(x - y)| \leqq M_1 |x - y|^{m-n}$, so that

(7) $|v(x)| \leqq M_2\left(|x|^{m-n}\int_{B(0,2|x|)} f(y)dy + v'''(x)\right) \leqq M_3|x|^{\ell}\Omega'_{\ell}(x)G(x) + M_2 v'''(x),$

where $M_1 \sim M_3$ are positive constants independent of $x \in R^n - B(0, 2)$.

In case $\ell < m - n$, by using Lemma 3′, we find a positive constant M_1 such that

$$|v(x)| \leqq M_1|x|^{m-n}h(|x|)\int_{B(0,2|x|)} f(y)dy.$$

Hence Hölder's inequality gives

(8) $$|v(x)| \leqq M_2|x|^{\ell}\Omega'_{\ell}(x)G(x),$$

where M_2 is a positive constant independent of x, $\Omega'(x) = \omega'_{\ell}(|x|)$ with

$$\omega'_{\ell}(r) = r^{m-n-\ell}h(r)\left(\int_1^r \omega(s)^{-p'/p}s^{n-1}ds\right)^{1/p'}$$

and $G(x) = \left(\int_{B(0,2|x|)} f(y)^p\Omega(y)dy\right)^{1/p'}$.

We now define $A_{\ell}(r) = r^{\ell+1}\omega_{\ell}(r) + r^{\ell}\omega'_{\ell}(r)$ for an integer ℓ such that $\ell \geqq \max\{-1, \ell_{\omega}, m-n\}$ or $\max\{-1, \ell'_{\omega}\} \leqq \ell < m-n$. Then condition ($\omega$1) implies that $A_{\ell}(r) \geqq Mr^{m-n/p}\omega(r)^{-1/p}$ for $r > 1$, where M is a positive constant independent of r. If $\ell \geqq \max\{-1, m-n\}$, then $\liminf_{r\to\infty} h(r)^{-1}\omega'_{\ell}(r) \geqq \left(\int_1^{\infty} s^{p'(m-n/p-\ell)}\omega(s)^{-p'/p}s^{-1}ds\right)^{1/p'} \equiv a_{\ell} > 0$, so that

$$\limsup_{r\to\infty} A_{\ell}(r)^{-1}[r^{\ell}h(r)] \leqq a_{\ell}^{-1} < \infty.$$

Further we set $b_{\ell} = \limsup_{r\to\infty} A_{\ell}(r)^{-1}[r^{m-n}h(r)]$. If $\ell \geqq m-n$, then $b_{\ell} < \infty$ by the above, and if $\ell < m-n$, then $A_{\ell}(r) \geqq r^{m-n}h(r)\left(\int_1^r \omega(t)^{-p'/p}t^{n-1}dt\right)^{1/p'}$, so that b_{ℓ} is finite, too.

THEOREM 2. *Let ω be a positive monotone function on $[0, \infty)$ satisfying condition (ω1), and ℓ be given as above. If f is a nonnegative measurable function on R^n satisfying $\int f(y)^p\Omega(y)dy < \infty$, then there exists a set $E \subset R^n$ such that*

(i) $\limsup_{|x|\to\infty, x\in R^n - E} A_{\ell}(|x|)^{-1}|u(x)| < \infty$;

(ii) $\sum_{j=1}^{\infty} A_{\ell}(2^j)^p\omega(2^j)C_{m,p}(E_j; G_j) < \infty$,

where $u(x) = \int K_{m,\lambda,\ell}(x, y)f(y)dy$, $E_j = E \cap B_j$ and $G_j = B_{j-1} \cup B_j \cup B_{j+1}$ with B_j

$= B(0, 2^j) - B(0, 2^{j-1})$; *in case* $mp > n$, *E can be taken as the empty set.*

PROOF. By (5), (6), (7) and (8), we see that

(9) $$|u(x)| \leqq M_1 A_\ell(|x|)[F(x) + G(x)] + M_1 |x|^{m-n}[|h(|x|)| + 1] \int_{B(0,1)} f(y)dy + M_1 v'''(x)$$

for any $x \in R^n - B(0, 2)$, where M_1 is a positive constant independent of x. In case $mp \leqq n$, applying Lemma 6 with $\varphi(r) = A_\ell(r)\omega(r)^{1/p}$, we see that v''' fulfills (i) in Lemma 6 with an appropriate set E satisfying (ii), so that

(10) $$\limsup_{|x| \to \infty, x \in R^n - E} A_\ell(|x|)^{-1} |u(x)| \leqq M_1 \limsup_{|x| \to \infty} G(x) + M_1 b_\ell \int_{B(0,1)} f(y)dy < \infty;$$

in case $mp > n$, this remains true if we take E as the empty set by the second half of Lemma 6. Thus the proof of Theorem 2 is completed.

REMARK. If $a_\ell = \infty$ (this holds when $\ell = \ell_\omega$) and $b_\ell = 0$, then $\lim_{|x| \to \infty, x \in R^n - E} A_\ell(|x|)^{-1} u(x) = 0$ in the above theorem.

In order to prove this, we write

$$u(x) = \int_{B(0,2R)} K_{m,\lambda,\ell}(x, y) f(y)dy + \int_{R^n - B(0,2R)} K_{m,\lambda,\ell}(x, y) f(y)dy = U_{\ell,R} f(x) + V_{\ell,R} f(x)$$

for $R > 1$ as before. Then, by our assumptions, $\lim_{|x| \to \infty} A_\ell(|x|)^{-1} |U_{\ell,R} f(x)| = 0$. Next, noting that M_1 in (9) is determined to be independent of f, we find from the arguments in the proof of Theorem 2 that

$$\limsup_{|x| \to \infty, x \in R^n - E} A_\ell(|x|)^{-1} |V_{\ell,R} f(x)| \leqq M_1 \left(\int_{R^n - B(0,2R)} f(y)^p \Omega(y) dy \right)^{1/p}$$

with the same E as above. This proves the required assertion.

COROLLARY 1. *Let* ω *be a positive monotone function on* $[0, \infty)$ *satisfying condition* (ω1), *and* ℓ *be as above. If u is an* (m, p)*-quasi continuous function belonging to* $BL_m(L^p_{loc}(R^n))$ *and satisfying condition* (1), *then there exist a polynomial P and a set* $E \subset R^n$ *such that*

(i) $\limsup_{|x| \to \infty, x \in R^n - E} A_\ell(|x|)^{-1} |u(x) - P(x)| < \infty$;

(ii) $\sum_{j=1}^{\infty} A_\ell(2^j)^p \omega(2^j) C_{m,p}(E_j; G_j) < \infty$;

in case $mp > n$, *E can be taken as the empty set.*

PROOF. First we can find a polynomial P_ℓ and a set E_0 such that equality (4) holds for any $x \in R^n - E_0$ and $C_{m,p}(E_0 \cap B(0, r); B(0, 2r)) = 0$ for any $r > 0$. Clearly, $C_{m,p}(E_{0j}; G_j) = 0$, so that E_0 satisfies condition (ii). Therefore the Corollary follows readily from Theorem 2.

LEMMA 7. *If $\omega(r) = r^{-\delta}$ for $r > 1$, then $\ell_\omega \leqq m - n/p + \delta/p < \ell_\omega + 1$ and $\ell'_\omega = \ell_\omega$; moreover for $\ell = \max\{-1, \ell_\omega\}$,*

$$A_\ell(r) \sim r^{m-n/p+\delta/p} \qquad \text{in case } m - n/p + \delta/p > \ell \geqq m - n,$$

$$A_\ell(r) \sim r^\ell h(r)(\log r)^{1/p'} \qquad \text{in case } \ell = m - n/p + \delta/p \geqq m - n,$$

$$A_\ell(r) \sim r^\ell \qquad \text{in case } m - n/p + \delta/p < \ell \text{ and } m - n \leqq \ell$$

and

$$A_\ell(r) \sim r^{m-n}h(r) \qquad \text{in case } \ell < m - n,$$

where $\varphi(r) \sim \psi(r)$ means that $0 < \lim_{r\to\infty} \varphi(r)/\psi(r) < \infty$.

With the aid of Lemma 7, Corollary 1 and the Remark after Theorem 2 give the following result.

COROLLARY 2. *If u is an (m, p)-quasi continuous function in $BL_m(L^p_{loc}(R^n))$ satisfying (1) with $\omega(r) = r^{-\delta}$, then there exist a set E and a polynomial P of degree at most $\max\{m-1, \ell\}$, where $\ell = \max\{-1, \ell_\omega\}$, such that*

$$\lim_{|x|\to\infty, x\in R^n - E} |x|^{-(m-n/p+\delta/p)}[u(x) - P(x)] = 0$$
$$\text{in case } m - n/p + \delta/p > \ell \geqq m - n,$$

$$\lim_{|x|\to\infty, x\in R^n - E} |x|^{-\ell}[h(|x|)]^{-1}(\log|x|)^{-1/p'}[u(x) - P(x)] = 0$$
$$\text{in case } m - n/p + \delta/p = \ell \geqq m - n,$$

$$\limsup_{|x|\to\infty, x\in R^n - E} |x|^{-\ell}|u(x) - P(x)| < \infty$$
$$\text{in case } m - n/p + \delta/p < \ell \text{ and } m - n \leqq \ell,$$

$$\limsup_{|x|\to\infty, x\in R^n - E} [|x|^{m-n}h(|x|)]^{-1}|u(x) - P(x)| < \infty$$
$$\text{in case } \ell < m - n$$

and

$\sum_j \varphi(2^j)^p C_{m,p}(E_j; B_j) < \infty$ with $\varphi(r) = A_\ell(r)\omega(r)^{1/p}$ ($\geqq M r^{m-n/p}$); in case $mp > n$, E can be taken as the empty set.

REMARK. This corollary gives the radial limit theorem [7; Theorem 3], where the case $\omega(r) \equiv 1$ is treated.

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