HIROSHIMA MATH. J.
19 (1989), 251–258

Moduli space of 1-instantons on a quaternionic projective space HP^n

Hideo DOI and Takayuki OKAI

(Received May 20, 1988)

Introduction

The moduli space of 1-instantons on $S^4 = HP^1$ is isomorphic to $Sp(2)\backslash SL(2, H)$ ([2], [3], [9]). The main purpose of this paper is to generalize this basic fact to the case of HP^n. More precisely, we consider self-dual connections, i.e. solutions to a first order equation which is a reduction of the Yang-Mills equation given by physicists [4], [20].

At present a general theory for self-dual connections on quaternionic Kähler manifolds is developed by M. Mamone Capria & S. M. Salamon [12] and T. Nitta [15]. Thus it would be worthwhile to study self-dual connections concretely. In this point of view E. Corrigan, P. Goddard & A. Kent [5] have provided an interesting family of self-dual connections on HP^n, as a generalization of the ADHM construction. They have also counted the number of parameters of this family. For 1-instantons (see § 1), from the table of H. T. Laquer [11], we know that this number coincides with the nullity of the second variation of the Yang-Mills functional at the canonical connection for the symmetric space $Sp(n+1)/Sp(1) \times Sp(n)$. However, even in this case, the completeness of the ADHM construction is a problem [5]. In Theorem 1.1, we will give an affirmative answer to this, using a result in algebraic geometry due to H. Spindler [19]. In Theorem 1.2, we will give a compactification of the moduli space of 1-instantons. In Theorem 1.3, we will examine the convergence of the Yang-Mills action densities.

1. Notation and the results

We begin with a review of quaternionic geometry (for details, see [12], [14, 15], [16, 17, 18]). Let M^{4n} be a *quaternionic Kähler manifold*. By definition its holonomy group is contained in $Sp(n) \cdot Sp(1) \subset SO(4n)$. Note that the natural representation of $GL(n, H) \times Sp(1)$ on $\Lambda^2(C^{2n} \otimes C^2)$ is decomposed to $\Lambda^2 C^{2n} \otimes S^2 C^2 + S^2 C^{2n} \otimes \Lambda^2 C^2$. Accordingly, we have a decomposition $\Lambda^2 T^*M \otimes C = \Lambda_2 + \Lambda_0$. Let E be a complex unitary vector bundle over M with a unitary connection D. We assume that its curvature form $F(D)$ is a section of $\Lambda_0 \otimes \mathrm{u}(E)$. Then D is said to be *self-dual*. Note that D becomes a Yang-Mills connection. If a transformation $g: M \to M$ preserves the $GL(n, H) \cdot$

$Sp(1)$-structure of M, then g^*D is also self-dual. Let Z be the twistor space of M and let $p: Z \to M$ be the canonical projection. We note that Z has a complex structure, and that $F(p^*D)$ is a $(1, 1)$-form. Hence the pull-back connection p^*D defines a unique holomorphic vector bundle structure on p^*E. Moreover, if the scalar curvature of M is positive, then Z has a Kähler metric and p^*D turns out to be an Einstein-Hermitian connection. In particular, D attains the minimum of the Yang-Mills functional. Also, it should be remarked that the Atiyah-Ward correspondence is established by T. Nitta [15].

Clearly the symmetric space $HP^n = Sp(n+1)/Sp(1) \times Sp(n)$ is a quaternionic Kähler manifold. We set $E = Sp(n+1) \times_\nu H^n$ for the projection ν: $Sp(1) \times Sp(n) \to Sp(n)$, and we call self-dual connections on E *1-instantons*. Let ∇ be the unique invariant connection on the homogeneous vector bundle E. Then ∇ is self-dual and called *the standard 1-instanton*. The action of $GL(n+1, H)$ on HP^n preserves the $GL(n, H)\cdot Sp(1)$-structure. Thus we have a self-dual connection $g^*\nabla$ on g^*E for $g \in GL(n+1, H)$. Using an $Sp(n)$-bundle equivalence $\gamma_g: E \to g^*E$, we obtain a 1-instanton $\gamma_g^* g^* \nabla = \nabla \cdot g$, which is unique up to $Sp(n)$-gauge transformations on E. Now we can state the main result.

THEOREM 1.1 *Let $\mathscr{M}_n$ denote the moduli space of 1-instantons on HP^n. Then $\mathscr{M}_n$ is identified with $Sp(n+1)\backslash SL(n+1, H)$ via the correspondence $g \mapsto \nabla \cdot g$ for $g \in SL(n+1, H)$.*

Let $M(m, H)$ denote the set of $m \times m$ quaternionic matrices. We set $\mathscr{P}_{n+1} = \{A \in M(n+1, H);\ {}^\dagger A = A,\ A > 0\}$ and $\hat{\mathscr{P}}_{n+1} = \{B \in M(n+1, H);\ {}^\dagger B = B,\ B \geq 0\}$, where † denotes the Hermitian conjugation. Then $Sp(n+1)\backslash SL(n+1, H) \to \mathscr{P}_{n+1}/R_+^\times$, $g \mapsto {}^\dagger g \cdot g$, is an isomorphism. Therefore we may identify $\mathscr{M}_n$ with $\mathscr{P}_{n+1}/R_+^\times$ and we will usually use the notation D_A instead of $\nabla \cdot A^{1/2}$ for $A \in \mathscr{P}_{n+1}$.

Let $\{D_i\}$ be a sequence of 1-instantons. Proposition 3.3 will provide the following situation: There exist a subsequence $\{j\} \subset \{i\}$, a linear subvariety S in HP^n, gauge transformations $\{\gamma_j\}$ on E, and a self-dual connection D_∞ on $E|HP^n\backslash S$ such that $\gamma_j^* D_j$ converges to D_∞ in C_{loc}^∞ on $HP^n\backslash S$. For the above $\{j\}$, we remark that if an exceptional set S is minimal, then S is unique. So, Proposition 3.3 would imply

THEOREM 1.2. *Let $\hat{\mathscr{M}}_n = \{(D_\infty, S);\ D_\infty$ is a limit of 1-instantons, S is the minimal exceptional set$\}/\sim$, where $(D_\infty, S) \sim (D'_\infty, S')$ means that $S = S'$ and D_∞ is gauge equivalent to D'_∞. Then we have an identification*

$$\hat{\mathscr{M}}_n = (\hat{\mathscr{P}}_{n+1}\backslash\{0\})/R_+^\times .$$

Thus we have a natural compactification $\hat{\mathscr{M}}_n$ of $\mathscr{M}_n$ in view of H. Nakajima's work [13].

In [7], S. K. Donaldson introduces the rough compactification of the moduli space of (anti) self-dual connections on 4-manifolds, using the convergence of the Yang-Mills action densities $|F|^2$ as measure. We also investigate the behavior of $|F|^2$ when the connections converge to $\hat{\mathscr{M}}_n \setminus \mathscr{M}_n$.

We give first some definitions. For $X = (X_{ij}) \in M(m, H)$, let $\operatorname{tr} X = \sum X_{ii}$. Then $\operatorname{tr}(gAg^{-1}) = \operatorname{tr} A$ for $g \in Sp(n+1)$ and $A \in \hat{\mathscr{P}}_{n+1}$. For $l \times m$ quaternionic matrices X and Y, we define an inner product $(X, Y) = \operatorname{tr}({}^{\dagger}XY)$. Let $S^{4n+3} = \{z \in H^{n+1}; (z, z) = 1\}$ and equip $HP^n = S^{4n+3}/Sp(1)$ with the standard metric induced from that of S^{4n+3}. For $A \in \hat{\mathscr{P}}_{n+1}$ and $z \in H^{n+1}$, we set

$$\begin{aligned}\Phi(A)(z) = 8(Az, z)^{-4}(z, z)^2\{&3(A^2z, z)^2 + (\operatorname{tr} A^2 + 2(\operatorname{tr} A)^2)(Az, z)^2 \\ &- 4 \operatorname{tr} A(A^2z, z)(Az, z) - 2(A^3z, z)(Az, z)\}\end{aligned}$$

and we consider $\Phi(A)$ as a rational function on HP^n. Let F_A denote the curvature of D_A for $A \in \mathscr{P}_{n+1}$. We shall prove that $|F_A|^2 = \Phi(A)$ in Proposition 3.1. Now we can state the following

THEOREM 1.3. *Let* $A \in \mathscr{P}_{n+1}$, $B \in \hat{\mathscr{P}}_{n+1} \setminus \{0\}$ *and let* S_B *denote the linear subvariety* $\{z \in HP^n; Bz = 0\}$. *We assume that* A *approaches* B.

(i) *If* $\operatorname{rank} B \geq 2$, *then* $\lim_{A \to B} |F_A|^2 = \Phi(B)$ *in* $L^1(HP^n)$.

(ii) *If* $\operatorname{rank} B = 1$, *then for any continuous function* ϕ *on* HP^n, *we have* $\lim_{A \to B} \int_{HP^n} \phi |F_A|^2 = 4\pi^2 \int_{S_B} \phi$, *where the integrals stand for those with respect to the canonical Riemannian volume elements.*

2. The moduli space of 1-instantons

In this section, we give a proof of Theorem 1.1, following the program of R. Hartshorne [9]. Hereafter we denote H^{n+1} *by* V when it is regarded as a right C-vector space. Let $p: P(V) = (V \setminus \{0\})/C^{\times} \to HP^n = (H^{n+1} \setminus \{0\})/H^{\times}$ be the natural projection. We note that $P(V)$ is the twistor space of HP^n. Therefore, as mentioned in § 1, a 1-instanton D gives a holomorphic vector bundle N_D, which is C^{∞}-isomorphic to p^*E. We know that $c_2(N_D) = 1$ and $N_D|p^{-1}(\text{point})$ is holomorphically isomorphic to the trivial bundle $CP^1 \times C^{2n}$. Due to H. Spindler [19, p. 20, Cor.] it follows that N_D is a null correlation bundle.

Let N be a null correlation bundle on $P(V)$. Then, by definition, there exists a resolution

$$0 \to \mathcal{O}(-1) \to \Omega(1) \to N \to 0,$$

where Ω denotes the holomorphic contangent bundle of $P(V)$. We know that $\operatorname{Hom}(\mathcal{O}(-1), \Omega(1)) = \{\varphi \in \operatorname{Hom}(V, V^{\vee}); \varphi^{\vee} = -\varphi\}$, where $V^{\vee}$ is the dual vector space of V and $\varphi^{\vee}$ is the transposed mapping of φ. Let

$\mathscr{A}^c = \{\varphi \in \mathrm{Hom}(V, V^\vee);\ \varphi^\vee = -\varphi,\ \varphi \text{ is bijective}\}$ and let $\mathscr{N}^c$ denote the moduli space of null correlation bundles on $P(V)$. Then $\mathscr{N}^c$ is naturally identified with $\mathscr{A}^c/C^\times$ [19, Satz 4.2, a)].

For a complex manifold X, we denote by X^- the manifold with the opposite complex structure. The right action of $j \in H$ on V defines an isomorphism $j_R : P(V) \to P(V^-)$. We define an action of j on $\mathscr{N}^c$ by $j \cdot N = j_R^* N^-$ for $N \in \mathscr{N}^c$. Clearly, we see that if $N \in \mathscr{N}^c$ is induced by a 1-instanton, then $j \cdot N = N$.

Let $e_0, \cdots, e_n$ be the standard basis of H^{n+1} and set $e_{n+1+i} = je_i$. Thus we have a basis $e_0, \cdots, e_{2n+1}$ of V, and by this, we identify $\mathrm{Hom}(V, V^\vee)$ with the space $M(2n+2, C)$ of $(2n+2) \times (2n+2)$ complex matrices. Let λ denote the standard embedding of $M(n+1, H)$ into $M(2n+2, C)$, and set $J = \lambda(j 1_{n+1})$. We define an action of j on $\mathscr{A}^c$ by $j \cdot \varphi = {}^t J \bar{\varphi} J$ for $\varphi \in \mathscr{A}^c$, where ${}^t J$ is the transposed matrix of J and $\bar{\varphi}$ is the usual complex conjugate of φ. Then the induced action of j on $\mathscr{A}^c/C^\times$ coincides with the action of j on $\mathscr{N}^c$ under the above identification.

Let $\mathscr{N} = \{N \in \mathscr{N}^c = \mathscr{A}^c/C^\times;\ j \cdot N = N\}$ and $\mathscr{A} = \{A \in M(n+1, H);\ {}^\dagger A = A,\ \det \lambda(A) \neq 0\}$. Then we have an isomorphism $\mathscr{A}/R^\times \to \mathscr{N}$ induced by $\lambda(j\ \ )$. We note that $\lambda(j {}^\dagger g A g) = {}^t\lambda(g)\lambda(jA)\lambda(g)$ for $g \in GL(n+1, H)$ and $A \in \mathscr{A}$. Clearly $\mathscr{N}$ is stable under the action of $GL(n+1, H)$ on $\mathscr{N}^c$ which is induced by the action on HP^n.

Lemma 2.1. (1) *$\mathscr{N}/GL(n+1, H)$ has a finite set of complete representatives $\{J_l = \lambda(j\,\mathrm{diag}(1_{n+1-l}, -1_l));\ 0 \le l \le (n+1)/2\}$.*

(2) *Let N_l be the null correlation bundle corresponding to J_l. If $0 < l \le (n+1)/2$, there exists a point $z \in HP^n$ such that $N_l | p^{-1}(z)$ is holomorphically non-trivial.*

Proof. (1) This is immediate if we consider in $\mathscr{A}/GL(n+1, H)$.

(2) Let N be a null correlation bundle corresponding to $\varphi \in \mathscr{A}^c$. Let $w_1, w_2 \in V$ be linearly independent and let $P(W)$ denote the projective line $(w_1 C + w_2 C\ \backslash\{0\})/C^\times \subset P(V)$. Then it is easy to see that $N|P(W)$ is holomorphically non-trivial if and only if ${}^t w_1 \varphi w_2 = 0$. When $w_1 = e_0 + e_{n+1-l}$ and $w_2 = w_1 j$, we have ${}^t w_1 J_l w_2 = 0$. □

Proof of Theorem 1.1. From Lemma 2.1, it follows that for $0 < l \le (n+1)/2$ and $g \in GL(n+1, H)$, $g^* N_l$ is not isomorphic to any null correlation bundle induced by a 1-instanton. On the other hand, N_0 is induced by the standard 1-instanton ∇. Let D be a 1-instanton and let N denote the null correlation bundle induced by D. Considering the action of $GL(n+1, H)$, we may assume that there exists a holomorphic isomorphism $\psi : N_0 \to N$. Then $\psi^* p^* D$ is an Einstein-Hermitian connection on N_0. From the uniqueness of

Einstein-Hermitian connections due to S. K. Donaldson [6, 8] (see also [10]), it follows that $\psi^*p^*D = p^*\nabla$ and ψ is an isometry. Furthermore, ψ is constant along the fibers of p because p^*D and $p^*\nabla$ are trivial on the fibers. Hence ψ defines a gauge transformation γ on E. Therefore, $\gamma^*D = \nabla$. Thus we know that $GL(n+1, H)$ acts transitively on $\mathscr{M}_n$. Clearly the isotropy subgroup of ∇ is $Sp(n+1)$. □

3. Limits of 1-instantons

In this section, we give proofs of Theorems 1.2 and 1.3. To begin with, we notice that $E = \{(z, v) \in HP^n \times H^{n+1};\ {}^{\dagger}zv = 0\}$. Let p_E denote the orthogonal projection $HP^n \times H^{n+1} \to E$. Then the standard 1-instanton ∇ is given by a covariant derivative $p_E \circ d$. Let π denote the projection $H^{n+1}\backslash\{0\} \to HP^n$ and let s be a mapping $H^n \to H^{n+1}\backslash\{0\}$ defined by $s(x) = e_0 + x$ with $x = \sum_{i=1}^n e_i x_i \in H^n$. Now we identify H^n with $\pi \circ s(H^n)$ and regard s as a local section of $HP^n \times H^{n+1}$. Then we have an expression of the curvature of ∇: $F_1 = |s|^{-2} p_E \cdot ds \wedge d^{\dagger}s \cdot p_E$ (see [1]).

Next, we shall prove that $|F_A|^2 = \Phi(A)$ for $A \in \mathscr{P}_{n+1}$ as mentioned in §1. Recall that $|F(\nabla \cdot g)|^2 = |g^*F_1|^2$ for any $g \in GL(n+1, H)$ and in particular, $|a^*F_1|^2 = |F_A|^2$ for $a \in \mathscr{P}_{n+1}$ with $A = a^2$.

PROPOSITION 3.1. $|a^*F_1|^2 = \Phi(a^2)$ *for* $a \in \mathscr{P}_{n+1}$.

PROOF. For $A \in \mathscr{P}_{n+1}$ and $g \in Sp(n+1)$, we know that $g^*\Phi(A) = \Phi({}^{\dagger}gAg)$ and $|F_{{}^{\dagger}gAg}|^2 = g^*|F_A|^2$. Therefore we may assume that $a = \operatorname{diag}(a_0, \cdots, a_n)$. If $g \in Sp(n+1)$ is diagonal, then $ga = ag$. Hence it is enough to show that $|a^*F_1|^2 = \Phi(a^2)$ at $y = \sum_{i=1}^n e_i y_i$ with $y_i \in R$. Let $f_i = (1_{n+1} - |s|^{-2} s \cdot {}^{\dagger}s)e_i$. Then at y,

$$F_1 = |s|^{-2} \sum_{i,j=1}^n f_i \cdot {}^{\dagger}f_j\, dx_i \wedge d\bar{x}_j\,.$$

Let $\theta_{ij} = dz_i \wedge d\bar{z}_j - y_i\, dz_0 \wedge d\bar{z}_j - y_j\, dz_i \wedge d\bar{z}_0 + y_i y_j\, dz_0 \wedge d\bar{z}_0$, where $z_0, \cdots, z_n$ are the standard coordinates of H^{n+1}. Let $(\ ,\ )_{HP^n}$ and $(\ ,\ )$ denote the standard metrics on HP^n and H^{n+1} respectively. Then we have that $(dx_i \wedge d\bar{x}_j, dx_k \wedge d\bar{x}_l)_{HP^n} = |s|^4(\theta_{ij}, \theta_{kl})$ at y because $\pi^*(dx_i \wedge d\bar{x}_j) = \theta_{ij}$. Let $Q_{ijkl} = a_i a_j a_k a_l (\delta_{ik}(a^2 s, s) - a_i a_k y_i y_k)(\delta_{jl}(a^2 s, s) - a_j a_l y_j y_l)$. Then we have at y,

$$|a^*F_1|^2 = (a^2 s, s)^{-4} |s|^4 \sum_{i,j,k,l=1}^n Q_{ijkl}(\theta_{ij}, \theta_{kl})\,.$$

Note that $(dz_i \wedge d\bar{z}_j, dz_i \wedge d\bar{z}_j) = 16$, $(dz_i \wedge d\bar{z}_j, dz_j \wedge d\bar{z}_i) = 8$ for $i \neq j$, $(dz_i \wedge d\bar{z}_i, dz_i \wedge d\bar{z}_i) = 24$, and the others are 0. Then a straightforward calculation shows that $|a^*F_1|^2(y) = \Phi(a^2)(s(y))$. □

Corollary 3.2. *Let A and B be as in Theorem* 1.3. *Then we have* $\lim_{A\to B}|F_A|^2(z) = \infty$ *for any* $z \in S_B$.

For $B \in \hat{\mathscr{P}}_{n+1}$, we set $M_B = HP^n \backslash S_B$, $K_B = M_B \times \operatorname{Ker} B$, $P_B = ((\operatorname{Ker} B)^\perp \backslash \{0\})/H^\times$, and $E_B = \{(z, v) \in P_B \times (\operatorname{Ker} B)^\perp; {}^\dagger zv = 0\}$. Let κ_B be the orthogonal projection $H^{n+1} \to \operatorname{Ker} B$ and let $\varepsilon_B = 1_{n+1} - \kappa_B$. Then ε_B induces a projection $\pi_B : M_B \to P_B$. By Theorem 1.1, $B|(\operatorname{Ker} B)^\perp$ defines a 1-instanton d_B on E_B. Let t_B denote the trivial connection on K_B. Clearly $\pi_B^* d_B + t_B$ is a self-dual connection on $\pi_B^* E_B + K_B$. From this connection, we obtain a self-dual connection D_B on $E|M_B$, because $E|M_B$ is isomorphic to $\pi_B^* E_B + K_B$.

Proposition 3.3. *Let A and B be as in Theorem* 1.3. *Then, after suitable gauge transformations on E, D_A approaches D_B on M_B.*

Proof. We assume, without loss of generality, that $B = \operatorname{diag}(\beta^2, 0)$, $A = \operatorname{diag}(\beta^2, \alpha^2)$ and that α converges to zero. Let us define an isometry $\tau : \pi_B^* E_B + K_B \to E|M_B$ by

$$\tau_z(v_1, v_2) = v_1 + (\kappa_B - |\varepsilon_B z|^{-2} \varepsilon_B z \cdot {}^\dagger(\kappa_B z)) h_z(v_2)\,,$$

where $z \in M_B$, $v_1 \in (\pi_B^* E_B)_z$, $v_2 \in (K_B)_z$ and $h_z = (1 + |\varepsilon_B z|^{-2} \kappa_B z \cdot {}^\dagger(\kappa_B z))^{-1/2} \in \operatorname{Hom}(\operatorname{Ker} B, \operatorname{Ker} B)$. Let $a = \operatorname{diag}(\beta, \alpha)$ and $\iota_\alpha = \operatorname{diag}(1, \alpha)$. We note that $a^*(\pi_B^* E_B + K_B) = \operatorname{diag}(\beta, 1)^*(\pi_B^* E_B + K_B)$. Hence it is enough to show that $\lim_{\alpha\to 0} a^* \tau^* \nabla = \pi_B^* \beta^* \nabla_B + t_B$, where ∇_B denotes the standard 1-instanton on E_B. Moreover, this is reduced to the case $\beta = 1 \in \operatorname{Hom}((\operatorname{Ker} B)^\perp, (\operatorname{Ker} B)^\perp)$.

Let σ be a section of $\pi_B^* E_B + K_B$. Setting $u_z = 1_{n+1} - |z|^{-2} z \cdot {}^\dagger z$ for $z \in M_B$, we have

$$(\iota_\alpha^* \tau^* \nabla)\sigma = \iota_\alpha^* \tau^{-1} \cdot \iota_\alpha^* u \cdot d(\iota_\alpha^* \tau \cdot \sigma)\,.$$

Also we see that $\lim_{\alpha\to 0} \iota_\alpha^* \tau = 1_{n+1}$ and $\lim_{\alpha\to 0} (\iota_\alpha^* u)_z = 1_{n+1} - |\varepsilon_B z|^{-2} \varepsilon_B z \cdot {}^\dagger(\varepsilon_B z)$. From this, it follows that $\lim_{\alpha\to 0} \iota_\alpha^* \tau^* \nabla = (1_{n+1} - |\varepsilon_B z|^{-2} \varepsilon_B z \cdot {}^\dagger(\varepsilon_B z)) \circ d = \pi_B^* \nabla_B + t_B$. □

Now the proof of Theorem 1.2 is completed as mentioned in § 1.

Proof of Theorem 1.3. We may assume that A and B are diagonal, and we use freely the notations in the proof of Proposition 3.1.

(i) From Lebesgue's dominated convergence theorem, it follows that $\Phi(A)$ converges to $\Phi(B)$ in $L^1(HP^n)$.

(ii) Note that $\lim_{A\to B}|F_A|^2(z) = 0$ for $z \in HP^n$ with $Bz \neq 0$. Thus we can assume that $B = \operatorname{diag}(0, 1, 0, \cdots, 0)$ and $a = A^{1/2} = \operatorname{diag}(a_0, 1, a_2, \cdots, a_n)$. Let $\rho = (a_0^2 + a_2^2 r_2^2 + \cdots + a_n^2 r_n^2)^{1/2}$ with $r_i = |x_i|$. Then $(As, s) = \rho^2 + r_1^2$ and $Q_{1111} = \rho^4$, where we substitute r_i for y_i. For $\varepsilon > 0$, $\omega_1 \in S^3$ and $\phi \in C^0(H^n)$ with compact support, we have

$$\lim_{A\to B}\int_0^{\varepsilon}(As, s)^{-4}Q_{1111}\phi(r_1\omega_1, x_2, \cdots, x_n)r_1^3\,dr_1$$
$$=\lim_{A\to B}\int_0^{\varepsilon/\rho}t^3(1+t^2)^{-4}\phi(\rho t\omega_1, x_2, \cdots, x_n)\,dt = \phi(0, x_2, \cdots, x_n)/12\,.$$

Note that $Q_{1212} = a_2^2\rho^2(a_0^2 + r_1^2 + a_3^2 r_3^2 + \cdots + a_n^2 r_n^2)$ and $(As, s)^4 \geq (2(\rho^2 r_1^2)^{1/2})^2 \cdot 2(r_1^2 \cdot a_2^2 r_2^2)^{1/2} \cdot (a_0^2 + r_1^2 + a_3^2 r_3^2 + \cdots + a_n^2 r_n^2)$. Hence $(As, s)^{-4}Q_{1212}r_1^3\cdots r_n^3 \leq a_2 r_2^2 r_3^3 \cdots r_n^3/8$. Similar arguments show that if $Q_{ijkl} \neq Q_{1111}$, then $\lim_{A\to B}(As, s)^{-4}Q_{ijkl}r_1^3\cdots r_n^3 = 0$. We notice that the Riemannian volume element on HP^n has an expression $(s, s)^{-2n-2}d^{4n}$. Here we denote the standard volume element on R^m by d^m. Now for $\phi \in C^0(H^n)$ with compact support, we have

$$\lim_{A\to B}\int_{H^n}\phi\cdot(As, s)^{-4}(s, s)^2\sum Q_{ijkl}(\theta_{ij}, \theta_{kl})(s, s)^{-2n-2}d^{4n}$$
$$=4\pi^2\int_{H^{n-1}}\phi(0, x_2, \cdots, x_n)\cdot(1 + r_2^2 + \cdots + r_n^2)^{-2n}d^{4n-4}\,.$$

This completes the proof. □

References

[1] M. F. Atiyah: Geometry of Yang-Mills Fields, Lezioni Fermiane Pisa, 1979.

[2] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin: Construction of Instantons, Phys. Letter **65A** (1978), 185–187.

[3] M. F. Atiyah, N. J. Hitchin and I. M. Singer: Self-duality in four-dimensional Riemannian geometry, Proc. Roy. Soc. London A **362** (1978), 425–461.

[4] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts: First-order equations for gauge fields in spaces of dimension greater than four, Nuclear Phys. **B214** (1983), 452–464.

[5] E. Corrigan, P. Goddard and A. Kent: Some comments on the ADHM construction in $4k$ dimensions, Comm. Math. Phys. **100** (1985), 1–13.

[6] S. K. Donaldson: Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles, Proc. London Math. Soc. (3) **50** (1985), 1–26.

[7] S. K. Donaldson: Connections, cohomology and the intersection forms of 4-manifolds, J. Differential Geom. **24** (1986), 275–341.

[8] S. K. Donaldson: Infinite determinants, stable bundles and curvature, Duke Math. J. **54** (1987), 231–247.

[9] R. Hartshorne: Stable vector bundles and instantons, Comm. Math. Phys. **59** (1978), 1–15.

[10] S. Kobayashi: Differential Geometry of Complex Vector Bundles, Iwanami, 1987.

[11] H. T. Laquer: Stability properties of the Yang-Mills functional near the canonical connection, Michigan Math. J. **31** (1984), 139–159.

[12] M. Mamone Capria and S. M. Salamon: Yang-Mills fields on quaternionic spaces, Nonlinearity **1** (1988), 517–530.

[13] H. Nakajima: Compactness of the moduli space of the Yang-Mills connections in higher dimensions, J. Math. Soc. Japan **40** (1988), 383–392.

[14] T. Nitta: Connections for vector bundles over quaternionic Kähler manifolds, Proc. Japan Acad. **63** Ser. A (1987), 23–25.

[15] T. Nitta: Vector bundles over quaternionic Kähler manifolds, Tôhoku Math. J. **40** (1988), 425–440.

[16] S. M. Salamon: Quaternionic Kähler manifolds, Invent. Math. **67** (1982), 143–171.

[17] S. M. Salamon: Quaternionic structures and twistor spaces, in "Global Riemannian Geometry", T. J. Willmore and N. Hitchin ed., Ellis Horwood Limited, 1984.

[18] S. M. Salamon: Differential geometry of quaternionic manifolds, Ann. Sci. École. Norm. Sup. 4^e série t. **19** (1986), 31–55.

[19] H. Spindler: Holomorphe Vectorbündel auf P_n mit $c_1 = 0$ und $c_2 = 1$, Manuscripta Math. **42** (1983), 171–198.

[20] R. S. Ward: Completely solvable gauge-field equations in dimension greater than four, Nuclear Phys. **B236** (1984), 381–396.

Department of Mathematics,
Faculty of Science,
Hiroshima University