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The injective hull of homotopy types with respect to generalized homology functors

Dedicated to Professor Masahiro Sugawara on his sixtieth birthday

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A. K. Bousfield [1], [2] introduced the notion of the localization of spaces and spectra with respect to homology. In this note, we introduce the notion of the injective hull of spaces and spectra with respect to homology. We also prove that the class $A(Ho^s)$ of Bousfield equivalence classes of spectra [2] becomes a set i.e., has a cardinality.

1. Statement of results

$\mathscr{C}$, $\mathscr{S}$, $\tilde{\mathscr{C}}$, $\tilde{\mathscr{S}}$ denote the categories of CW-complexes, CW-spectra, and their homotopy categories respectively.

DEFINITION 1. Let $\mathscr{A}$, $\mathscr{B}$ be categories and $\mathscr{F}: \mathscr{A} \to \mathscr{B}$ a functor.

i) $A \in \mathrm{Ob}(\mathscr{A})$ is *$\mathscr{F}$-local* (resp. *$\mathscr{F}$-injective*) if for any B, $C \in \mathrm{Ob}(\mathscr{A})$ and any $f: B \to C$ with $\mathscr{F}(f)$: iso (resp. mono), $f^*: \mathscr{A}(C, A) \to \mathscr{A}(B, A)$ is an iso (resp. epi).

ii) A map $f: A \to B$ is an *$\mathscr{F}$-localization map* of A if B is $\mathscr{F}$-local and $\mathscr{F}(f)$ is an iso.

iii) A map $f: A \to B$ is an *$\mathscr{F}$-injective enveloping map* of A if f satisfies the following two conditions:

a) B is $\mathscr{F}$-injective and $\mathscr{F}(f)$ is a mono.

b) For any $C \in \mathrm{Ob}(\mathscr{A})$ and any $g: B \to C$, $\mathscr{F}(g)$ is monic if $\mathscr{F}(g \circ f)$ is monic.

iv) B is an *$\mathscr{F}$-injective hull* of A if there is an $\mathscr{F}$-injective enveloping map $f: A \to B$ of A.

Then we can prove the following:

THEOREM 1. *Let $h = (h_n | n \in \mathbf{Z}): \mathscr{D} \to \mathscr{G}r\mathscr{A}b$ ($\mathscr{D} \in \{\tilde{\mathscr{C}}, \tilde{\mathscr{S}}\}$) be a generalized homology functor which is representable by a spectrum where $\mathscr{G}r\mathscr{A}b$ is the category of $\mathbf{Z}$-graded abelian groups. Then it follows that*:

i) *Any object $A \in \mathrm{Ob}(\mathscr{D})$ has an h-injective enveloping map.*

ii) *Let $f: A \to B$ and $g: A \to C$ be h-injective enveloping maps of A. then there exists a map $k: B \to C$ such that $k \circ f = g$. Moreover, such a k is always an* *isomorphism in the category $\mathscr{D}$.*

DEFINITION 2. Let E, $F \in \mathrm{Ob}(\mathscr{S})$. Then E and F are *Bousfield equivalent* (resp. *B^*-equivalent*) if for any $X \in \mathrm{Ob}(\mathscr{S})$ (resp. for any $f \in \mathrm{Mor}(\mathscr{S})$) the following two conditions on X (resp. f) are equivalent;

a) $E_n(X) \cong 0$ (resp. $E_n(f)$ is monic) for any $n \in \mathbf{Z}$.

b) $F_n(X) \cong 0$ (resp. $F_n(f)$ is monic) for any $n \in \mathbf{Z}$.

Then we can prove

THEOREM 2. i) *The class of Bousfield equivalence classes of spectra becomes a set.* ii) *The class of B^*-equivalence classes of spectra becomes a set.*

Let E be a spectrum with finite stable cells i.e., (so-called) finite spectra and $R = (R_n = [E, E]_n | n \in \mathbf{Z})$ the graded ring of operations of the generalized homology functor $E_*(-)$. Then this functor $E_*(-)$ becomes a functor from $\tilde{\mathscr{C}}$ or $\tilde{\mathscr{S}}$ to the category $\mathscr{M}_R$ of R-modules.

Then we can prove

THEOREM 3. *Let A, B be spectra. Then:*

i) A is *$E_*(-)$-injective iff A is $E_*(-)$-local and $E_*(A)$ is injective over R.*

ii) *For any injective R-module I, there is an $E_*(-)$-injective spectruma X with $E_*(X) \cong I$.*

iii) *If B is $E_*(-)$-injective, then the map $E_*(-): [A, B]_* \to \mathrm{Hom}_R(E_*(A), E_*(B))$ is bijective.*

Let $f: A \to B$ be a map of spectra. Then:

iv) *f is an $E_*(-)$-injective enveloping map iff B is $E_*(-)$-injective and $E_*(f): E_*(A) \to E_*(B)$ is an injective enveloping map over R, i.e., $E_*(f)$ is monic, $E_*(B)$ is injective over R, and $E_*(f)(E_*(A))$ is an essential submodule of $E_*(B)$ (a submodule N of M is called* essential *if for any submodule N' of M with $N \cap N' = \{0\}$, N' vanishes i.e., $N' = \{0\}$).*

v) *Any injective enveloping map $\iota: E_*(A) \to I$ of R-module is realized by an $E_*(-)$-injective enveloping map from A.*

2. Proof of Theorem 1

We shall prove Theorem 1 only for the case of CW-complexes. The case of CW-spectra can be proved similarly. Moreover we can give a slightly more clear proof by using the additive and triangulable properties of the category $\tilde{\mathscr{S}}$. It is left for the reader. $\# X$ denotes the cardinality of a set X. Let $\alpha = \sum_{n \in \mathbf{Z}} \# \pi_n(E)$ where E is a spectrum representing the generalized homology functor h in Theorem 1. $\mathrm{c}(X)$ denotes the set of cells of a CW-complex X.

PROPOSITION 1. *Let A be a subcomplex of a CW-complex B and $\iota: A \to B$ the inclusion map, and suppose that $h(\iota)$ is* monic. *Then for any subcomplex C of*

B there is a subcomplex D of B with the properties:

i) $D \supset C$.

ii) *$h(j)$ is monic where $j: A \cap D \to D$ is the inclusion map.*

iii) $\# c(D) \leq \max(\# c(C), \alpha)$.

iv) *$h(k)$ is monic where $k: A \cup D \to B$ is the inclusion map.*

PROOF. a) Construction of D satisfying the properties i), ii), iii). Let $C_0 = C$, and suppose that C_n is defined. Let $a \in M = \{b \in \bigcup_m E_m(A \cap C_n) | b$ vanishes in $E_*(C_n)\}$ where $\cup$ means disjoint union, $R(a)$, $S(a)$, $\hat{a}$ be mathematical objects satisfying that $R(a)$ is a finite subcomplex of $A \cap C_n$, $\hat{a} \in E_*(R(a))$ is an element which coincides with a in $E_*(A \cap C)$, $S(a)$ is a finite subcomplex of A, containing $R(a)$, in which $\hat{a}$ vanishes. Such mathematical objects in fact exist since $h(\imath)$ is monic. Let $C_{n+1} = C_n \cup (\bigcup_{a \in M} S(a))$. Then any element $b \in E_*(C_n \cap A)$ which vanishes in $E_*(C_n)$ vanishes in $E_*(C_{n+1} \cap A)$, and $\# c(C_{n+1}) \leq \max(\alpha, \# c(C_n))$. Hence $D = \bigcup_n C_n$ is the required subcomplex.

b) Construction of D satisfying all the conditions of the proposition. We define C_n, D_n, inductively on n as follows. Let $C_0 = C$, and assume that C_n is defined. By the above construction, we give D_n satisfying the conditions i), ii), iii) of the proposition in which the letters C, D are changed to C_n, D_n respectively. Let $M = \{(a, b) \in E_m(A) \times E_m(D_n) | m \in \mathbf{Z},\ \imath_*(a) = l_*(b)$ where $l: D_n \subset B$ is the inclusion map$\}$. Then $\# M \leq \max(\alpha, \# c(D_n))$ since above a is uniquely determined by b if it exists because $h(\imath)$ is monic. For $(a, b) \in M$, let $R(a, b)$, $S(a, b)$, $T(a, b)$, $\hat{a}$, $\hat{b}$ be mathematical objects such that $R(a, b)$, $S(a, b)$ are finite subcomplexes of A and D_n respectively, $\hat{a} \in E_*(R(a, b))$, $\hat{b} \in E_*(S(a, b))$ are elements which coincide with a, b in $E_*(A)$, $E_*(D_n)$ respectively, $T(a, b)$ is a finite subcomplex of B containing $R(a, b) \cup S(a, b)$, and $\hat{a}$ and $\hat{b}$ become the same elements in $E_*(T(a, b))$. Such mathematical objects indeed exist. Let $C_{n+1} = D_n \cup (\bigcup_{(a,b) \in M} T(a, b))$. Then $\# c(C_{n+1}) \leq \max(\alpha, \# c(C))$. Considering the following Mayer-Veitoris exact sequence for A and D_n:

$$\longrightarrow E_m(A \cap D_n) \xrightarrow[\varphi]{} E_m(A) \oplus E_m(D_n) \xrightarrow[\lambda]{} E_m(A \cup D_n) \longrightarrow ,$$

we see that φ is monic and λ is epic since $h(j_n)$ is monic where $j_n: A \cap D_n \subset D_n$ is the inclusion map. Let $c \in E_m(A \cup D_n)$ be an element which vanishes in $E_m(B)$ and $(a, b) \in E_m(A) \times E_m(D_n)$ be an element with $\lambda(a \oplus b) = c$. Then a and b coincide with each other in $E_m(B)$. Hence by the construction of C_{n+1}, c vanishes in $E_m(A \cup C_{n+1})$, and therefore $D = \bigcup_n C_n$ satisfies the conditions i), ii), iv), and also satisfies iii) since $D = \bigcup_n D_n$.

DEFINITION 3. Let h be a generalized homology functor. Then a map $f: A \to B$ is a *versal h-mono* if $h(f)$ is monic and for any CW-complex C, C is h-injective iff $f^*: [B, C] \to [A, C]$ is epic.

PROPOSITION 2. *For any generalized homology functor $h = E_*(-)$, there* *exists a versal h-mono.*

PROOF. For a map $f: A \to B$, Dom(f) or Dom$f$, and Ran(f) or Ran$f$ denote the domain A and range B of f respectively. Let X be a set satisfying the properties that i) any $f \in X$ is an inclusion map $f: A \subset B$ of CW-complexes such that $h(f)$ is monic and $\# \mathrm{c}(B) \leq \alpha$ and ii) for any inclusion map $g: C \subset D$ such that $h(g)$ is monic and $\# \mathrm{c}(D) \leq \alpha$, there is an $f \in X$ which is isomorphic to g as a map. Such an X indeed exists. Let $g = \bigcup_{f \in X} f: P \to Q$ be the disjoint union where $P = \bigcup_{f \in X} \mathrm{Dom} f$ and $Q = \bigcup_{f \in X} \mathrm{Ran} f$. Then g is a versal h-mono. To prove this, it suffices since $h(g)$ is monic to show that any CW-complex R satisfying that $g^*: [Q, R] \to [P, R]$ is epic is h-injective. Let $\iota: A \to B$ be a map between CW-complexes with $h(\iota)$ is monic. We may assume that ι is an inclusion for the aim below. From here we work in the category $\mathscr{C}$ instead of $\tilde{\mathscr{C}}$. Let $f: A \to R$ be a map where R satisfies the above condition, and C be a finite subcomplex of B. Then there is a subcomplex D of B satisfying the conditions i), ii), iii), iv) of Proposition 1. Then map f extends to $A \cup D$ since there is a map $k \in X$ which is isomorphic to the inclusion $A \cap D \subset D$ (as maps in the category $\tilde{\mathscr{C}}$). Hence f extends to B by the transfinite induction and therefore R is h-injective.

The following proposition is obtained from the definitions.

PROPOSITION 3. i) *An h-injective complex is h-local.*

ii) *Let X and Y be h-local CW-complexes. Then any map $f: X \to Y$ with $h(f)$ iso is a homotopy equivalence.*

PROOF. i) Let X be an h-injective complex, $f: Y \to Z$ a map which induces an iso of $h(-)$, and $g: Y \to X$ a map. Then there is a map $k: Z \to X$ with $g \simeq k \circ f$. We assume that f is an inclusion map. Let $l: Z \to X$ be a map such that $g \simeq l \circ f$, $V = Y \times [0, 1] \cup Z \times \{0, 1\}$ in $W = Z \times [0, 1]$ and $m: V \to W$ an inclusion map, and let $p: V \to X$ be a map such that $p(x, 0) = k(x)$ and $p(x, 1) = l(x)$. Such a map indeed exists. Since $h(m)$ is monic, p extends to W i.e., $k \simeq l$. Therefore X is h-local. ii) is easy.

Let us fix a versal h-mono $p: P \to Q$ which is an inclusion map, and let β denote the smallest infinite cardinal greater than $\# \mathrm{c}(P)$.

PROPOSITION 4. *For any CW-complex X, there are a CW-complex Y and a map $f: X \to Y$ satisfying that $h(f)$ is monic and y is h-injective.*

PROOF. We define a tower of CW-complexes $(X_\xi | \xi\text{: ordinal})$ by the transfinite induction as follows. Let $X_0 = X$ and suppose that X_ξ is defind. Let $M_\xi = \mathrm{CMap}(P, X_\xi)$ where $\mathrm{CMap}(K, L)$ denotes the set of all cellular maps from K to L, and define $X_{\xi+1}$ as the push out in the following push out square diagram in the category $\mathscr{C}$:

$$\begin{array}{ccccc} & & X_\xi & & \\ & \overset{\varphi}{\nearrow} & & \overset{\subset}{\searrow} & \\ \bigcup_{m\in M_\xi} P & & & & X_{\xi+1}, \\ & \underset{\psi}{\searrow} & & \nearrow & \\ & & \bigcup_{m\in M_\xi} Q & & \end{array}$$

where φ is the composition of the disjont union $\bigcup_{m\in M_\xi} m$ and the codiagonal map $\bigcup_{m\in M_\xi} X_\xi \to X_\xi$ and ψ is the disjoint union $\bigcup_{m\in M_\xi} p$. For the limit ordinal ξ, define $X_\xi = \bigcup_{\zeta<\xi} X_\zeta$. Note that for any ordinals ξ, ζ with $\xi < \zeta$, the inclusion map $X_\xi \subset X_\zeta$ induces a mono of $h(-)$. Let κ be the smallest ordinal with cardinality β. Then X_κ is h-injective, because any map $k: P \to X_\kappa$ passes X_ξ for some $\xi < \kappa$, hence k extends to a map from Q which passes $X_{\xi+1}$, and therefore X_κ is h-injective since p is a versal h-mono.

Proof of i) of Theorem 1. Let X be a CW-complex. Then we define a tower of CW-complexes $(X_\xi | \xi\text{: ordinals})$ by the transfinite induction as follows. Let $X_0 = X$, and suppose that X_ξ is defined and let Y_ξ be a CW-complex containing X_ξ such that Y_ξ is h-injective and the inclusion map $X_\xi \subset Y_\xi$ induces a monomorphism of $h(-)$. Such a complex indeed exists by Proposition 4. Let $T_\xi = \{\mathrm{Ker}\, h(g) | Z$ is a complex, $g: Y_\xi \to Z$ is a map such that $h(g \circ f_\xi)$ is monic$\}$, where f_ξ is the inclusion map $X_0 \subset Y_\xi$, be a set of submodules of $h(Y_\xi)$. Then we see that T_ξ has a maximal element with respet to inclusion relation as follows. Let U be a linearly orderd subset of T_ξ, and for $M \in U$, let k_M and Z_M be mathematical objects such that Z_M is a complex and $k_M: Y_\xi \to Z_M$ is an inclusion map with $M = \mathrm{Ker}\, h(k_M)$ (note that then $h(k_M \circ f_\xi)$ is monic). We may assume that $Z_M \cap Z_{M'} = Y_\xi$ for any M, $M' \in U$ with $M \neq M'$. Let $Z = \bigcup_{M\in U} Z_M$, $k: Y_\xi \subset Z$ be the inclusion map. Then we can see that $\mathrm{Ker}\, h(k) = \bigcup_{M\in U} M \in T_\xi$ (and then $h(k \circ f_\xi)$ is monic). Thus T_ξ has a maximal element M_0 by Zorn's Lemma. Let k_0 and Z_0 be mathematical objects such that Z_0 is a complex, $k_0: Y_\xi \to Z_0$ is an inclusion map satisfying that $M_0 = \mathrm{Ker}\, h(k_0)$ (and then $h(k_0 \circ f_\xi)$ is a mono) where f_ξ and M_0 are as above. Such objects indeed exist. Define $X_{\xi+1}$ by $X_{\xi+1} = Z_0$, and for the limit ordinal ξ, define as $X_\xi = \bigcup_{\zeta<\xi} X_\zeta$. Let κ be the smallest ordinal with cardinality β, and $f: X \to X_\kappa$ be the inclusion map. Then $h(f)$ is monic, and X_κ is h-injective by the same reason as in the proof of Proposition 4. We show that f is an h-injective enveloping map. Let W be a complex, $g: X_\kappa \to W$ a map with $h(g \circ f)$ mono and $a \in \mathrm{Ker}\, h(g)$. Then a is represented by some $a' \in h(Y_\xi)$ for some $\xi < \kappa$. Hence by considering $g|_{X_{\xi+1}}$, we can show that $a = 0$. This completes the proof of i) of Theorem 1.

Proof of ii). Let $f: A \to B$ and $g: A \to C$ be h-injective enveloping maps. Then there is a map $k: B \to C$ such that $g = k \circ f$ (in $\tilde{\mathscr{C}}$) since C is h-injective and $h(f)$ is monic. Then $h(k)$ is monic since f is an h-injective enveloping map. Then there is a map $l: C \to B$ with $l \circ k = \mathrm{id}_B$ since B is h-injective. Then $h(l)$ is monic since $l \circ g = l \circ k \circ f = f$ and g is an h-injective enveloping map. Hence $h(l)$ is an iso and therefore l is a homotopy equivalence by Proposition 3, and thus k is also a homotopy equivalence.

3. Proof of Theorem 2

Let U, V, d, T be mathematical objects satisfying the following conditions:

i) U is a set of finite spectra.

ii) For any finite spectrum A, there is a unique element $B \in U$ such that $A \simeq B$.

iii) V is a set of maps between finite spectra.

iv) For any $A \in U$, any finite spectrum B and any map $f: A \to B$, there is a unique $g \in V$ such that there is an iso $h: B \to \mathrm{Ran}\, g$ with $g = h \circ f$ (in the category $\tilde{\mathscr{S}}$).

v) $d: V \to U$ is a map and $d(f) = \mathrm{Dom} f$ for any $f \in V$.

vi) T is the set of all sections of the bundle over U whose fiber of $A \in U$ is $\mathfrak{P}(\mathfrak{P}(d^{-1}(A)))$ where $\mathfrak{P}(X)$ denotes the set of all subsets of X, and topologies are discrete.

Such mathematical objects in fact exist, and let us fix one 4-tuple of such objects U, V, d, T.

DEFINITION 4. Let E be a spectrum, $A \in U$, and $f: A \to E$ a map. Then the *elementary type* $t(f)$ is the set $\{g \in V \mid d(g) = A$, and f extends to $\mathrm{Ran}\, g$ in the category $\tilde{\mathscr{S}}\} \in \mathfrak{P}(d^{-1}(A))$, and the *elementary type* $t(E)$ of E is the element $s \in T$ defined by $s(A) = \{t(f) \mid f: A \to E\}$. Spectra E and F are *elementarily equivalent* if $t(E) = t(F)$.

i) of Theorem 2 is directly obtained from ii) and ii) is a corollary of the next proposition since the class of elementary equivalence classes of spectra becomes a set.

PROPOSITION 5. *Let E and F be elementarily equivalent spectra and $f: X \to Y$ be a map between spectra. Then $F_*(f)$ is monic if $E_*(f)$ is monic.*

PROOF. Let $a \in F_n(X)$ and suppose $f_*(a) = 0$. Let A, B be finite spectra, $p: A \to X$, $q: B \to Y$, $r: A \to B$ be maps and $\hat{a} \in F_n(A)$ such that $f \circ p = q \circ r$ *in* $\tilde{\mathscr{S}}$, $p_*(\hat{a}) = a$ and $r_*(\hat{a}) = 0$. Such objects indeed exist. Let $g: \Sigma^n(A') \to C$ be a homotopy cofiber of $\Sigma^n(r'): \Sigma^n(B') \to \Sigma^n(A')$ where K', l' are (one of) the Spanier-Whitehead dual of a spectrum K and map l, and Σ is the suspension functor. We can assume that $g \in V$. Let $h: \Sigma^n(A') \to F$ be a map corresponding to $\hat{a}$. Since $t(E) = t(F)$, there is a map $k: \Sigma^n(A') \to E$ with $t(h) = t(k)$ and let $b \in E_n(A)$ be the corresponding element to k. Then $r_*(\hat{a}) = 0$ implies that $h \circ \Sigma^n(r') = 0$ i.e., h extends to C in the category $\tilde{\mathscr{S}}$, hence k extends to C since

$t(h)=t(k)$ and this implies $r_*(b)=0$. Then from the assumption of the proposition, we can construct a finite spectrum D, maps $m: A \to D$ and $n: D \to X$ such that $n \circ m = p$ and $m_*(b)=0$. Then we can show $a=0$ by the similar argument to the above. Therefore $F_*(f)$ is monic.

4. Proof of Theorem 3

Let E be a finite spectrum, E' its Spanier-Whitehead dual, $h = E_*(-)$, and $R = [E, E]_*$. Let us fix these expressions. Note that h and $[E', -]_*$ are canonically naturally equivalent as $\mathscr{M}_R$-valued functors.

PROPOSITION 6. *Let I be an injective R-module. Then there is a spectrum U such that $[-, U]_*$ and $\mathrm{Hom}_R(h(-), I)$ are naturally equivalent to each other as functors from $\mathscr{S}$ to $\mathscr{G}r\mathscr{A}b$. Such a U is unique up to homotopy type, and is h-local. Moreover $h(U) \cong I$ as R-modules.*

PROOF. Since I is R-injective, $\mathrm{Hom}_R(h(-), I)$ becomes a generalized cohomology functor and then there are a spectrum U and a natural equivalence $\varphi(-): \mathrm{Hom}_R(h(-), I) \to [-, U]_*$ by the representation theorem. Uniqueness up to homotopy of U is also implied from the representation theorem. The localness of U follows from the fact that a spectrum V is h-local iff for any h-acyclic spectrum X (i.e., $h(X)=0$), $[X, V]=0$. Finally by putting $- = E'$, we obtain a required isomophism $\varphi(E')$ of modules.

PROPOSITION 7. *A spectrum U is $E_*(-)$-injective iff the map $r_X: [X, U]_* \to \mathrm{Hom}_R(E_*(X), E_*(U))$ defined by $r_X(f) = E_*(f)$ is monomorphic for all spectrum X.*

PROOF. Assume that U is $E_*(-)$-injective and let $f: X \to U$ be a map with $E_*(f)=0$ and $g: U \to Y$ be a homotopy cofiber of f. Then id_U extends to Y because U is $E_*(-)$-injective and $E_*(g)$ is monic. Thus $f = \mathrm{id}_U \circ f = 0$. Conversely suppose that r_X is monic for all X and let $f: Y \to Z$ be a map with $E_*(f)$ mono, $g: W \to Y$ be a homotopy fiber of f and $k: Y \to U$ be a map. Then from the assumption $k \circ g = 0$ since $E_*(g)=0$. This implies that k extends to Z, and therefore U is $E_*(-)$-injective.

PROPOSITION 8. *Let I be an injective R-module and U be a spectrum representing the generalized cohomology functor $\mathrm{Hom}_R(E_*(-), I)$. Then:*

i) *The map $r_X: [X, U] \to \mathrm{Hom}_R(E_*(X), E_*(U))$ defined by $r_X(f) = E_*(f)$ is bijective for all X,*

ii) *U is $E_*(-)$-injective.*

PROOF. i) Let $\varphi_X: \mathrm{Hom}_R(E_*(X), E_*(U)) \to [X, U]$ be a natural equivalence obtained from the condition on U and $f: U \to U$ be a map realizing the natural transformation $\varphi_X \circ r_X$. Then $E_*(f)$ is an iso since $r_{E'}$ is a bijection. Then f is a homotopy equivalence from the spectrum version of Proposition 3 since U is $E_*(-)$-local from Proposition 6, thus r_X is a bijection for all X. ii) follows directly from Proposition 7.

PROPOSITION 9. *Let A be a spectrum and $f: E_*(A) \to I$ be an injective enveloping map of R-modules. Then f is realized by an $E_*(-)$-injective enveloping map of spectra from A.*

PROOF. From Proposition 8, f is realized by a map $g: A \to U$ from A to an $E_*(-)$-injective spectrum U. Let $k: U \to V$ be a map between spectra with $E_*(k \circ g)$ mono. Then $E_*(k)$ is monic since $E_*(g) = f$ is an injective enveloping map over R. Thus g is an $E_*(-)$-injective enveloping map.

PROPOSITION 10.

i) *If a spectrum U is $E_*(-)$-injective, then $E_*(U)$ is an injective R-module.*

ii) *If a spectrum U is $E_*(-)$-local and $E_*(U)$ is an injective R-module, then U is $E_*(-)$-injective.*

PROOF. i) From Proposition 9, there is an $E_*(-)$-injective enveloping map $f: U \to V$ realizing an injective enveloping map $g = E_*(f): E_*(U) \to E_*(V)$ of R-modules. Since U is $E_*(-)$-injective, there is a map $k: V \to U$ with $k \circ f = \mathrm{id}_U$. Then $E_*(U)$ becomes a direct summand of the injective R-module $E_*(V)$, and therefore $E_*(U)$ is also injective over R. ii) Also let $f: U \to V$ be an $E_*(-)$-injective enveloping map realizing an injective enveloping map $g = E_*(f): E_*(U) \to E_*(V)$ of R-modules, then f is a homotopy equivalence because g is an iso and U and V are $E_*(-)$-local. Thus U is $E_*(-)$-injective.

The following proposition is easily obtained from Proposition 9 and Theorem 1.

PROPOSITION 11. *Let $f: A \to U$ be an $E_*(-)$-injective enveloping map. Then $g = E_*(f)$ is an injective enveloping map of R-modules.*

Theorem 3 is proved by Propositions 6, 7,..., 11. Finally we can also get the following:

PROPOSITION 12. *Let U be an $E_*(-)$-injective spectrum and $E_*(U) = M \oplus N$ be a direct sum decomposition by R-submodules of $E_*(U)$. Then there is a corresponding direct sum decomposition of U in the category $\mathscr{S}$.*

The proof of this propositon is not difficult. As a corollary we obtain that if U is $E_*(-)$-injective, $E_*(U)$ is indecomposable as an R-module iff U is indecomposable in the category $\mathscr{S}$.

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