HIROSHIMA MATH. J.
19 (1989), 567–573

A simple proof that certain capacities decrease under contraction

Bent FUGLEDE

(Received January 4, 1989)

It is well known that the Newton capacity, cap, decreases under contraction, in the following sense:

$$\operatorname{cap}\varphi(E) \leq \operatorname{cap}E$$

for any compact (or analytic) set $E \subseteqq \mathbf{R}^n$, $n \geq 3$, and for any contraction $\varphi: E \to \mathbf{R}^n$:

$$(1) \qquad d(\varphi(x), \varphi(y)) \leq d(x, y), \qquad \forall x, y \in E,$$

d denoting the Euclidean distance. Similarly for the logarithmic capacity in the plane. As observed by Landkof [11, p.406 below] this is an easy consequence of the characterization of the logarithmic capacity as the transfinite diameter in the sense of Fekete [6], cf. also Pólya and Szegö [12] for the Newton kernel in dimension $n > 2$. The fact that polar sets (= sets of logarithmic or Newton capacity 0) are preserved under contraction was noted by Brelot, cf. [4, p.50].

Replacing the Newton kernel $|x - y|^{2-n}$ by the Riesz kernels $|x - y|^{\alpha-n}$ of order α on $\mathbf{R}^n$, $0 < \alpha < n$, there are two notions of capacity, the energy capacity and the "ordinary" capacity (they differ for $\alpha > 2$), and correspondingly two notions of transfinite diameter, both generalizing the classical transfinite diameter in $\mathbf{R}^n$ associated with the Newton kernel. These two generalized transfinite diameters with respect to the Riesz kernels were studied by Pólya and Szegö [12] who showed that they are equal if and only if $\alpha \leq 2$. The identification of one of these two notions of transfinite diameter with the energy capacity (with respect to the same kernel) is due to Choquet [5], even for a general symmetric lower semicontinuous kernel G (infinite on the diagonal) on a locally compact space, cf. §1 below. This allows then for a simple proof that energy capacity decreases under contraction in the case of Riesz kernels. We bring a natural, short, direct proof of this (simpler than that of Landkof [11, Ch.II, §3]) applicable also in the general framework of Choquet [5] (see Theorem 1 below). In the same way we obtain a similar result for the ordinary capacity (Theorem 2). Probably this latter result could equally well be obtained by an extension of Fekete's result, now concerning the other notion of transfinite diameter, in the same spirit as in Choquet [5]. –The two results coalesce for symmetric kernels satisfying the maximum principle, valid in particular for the Riesz kernels of orders $\alpha \leq 2$ as established by Frostman [7], thus in particular for the Newton kernel.

The question whether similar results hold for L^p-capacities, $1 < p < \infty$ with respect to suitable kernels on $\mathbf{R}^n$ (cf. [10]) is studied by Aikawa [1]. For $p = 2$ and Riesz kernels of order $\alpha < n/2$ the answer is affirmative, being equivalent to Landkof's result on the energy capacity applied to the Riesz kernel of order 2α.[1)]

In the sequel X, X' denote locally compact (Hausdorff) spaces, and G, G' denote *kernels* on X, X' respectively, i.e., lower semicontinuous functions:

$$G: X \times X \longrightarrow [0, +\infty], \quad G': X' \times X' \longrightarrow [0, +\infty].$$

A continuous map $\varphi: X \to X'$ is called a *contraction with respect to* G, G' (briefly: a G, G'-contraction) if

$$G'(\varphi(x), \varphi(y)) \geq G(x, y), \qquad \text{for all } x, y \in X. \tag{2}$$

For example, if $X' = X$ is endowed with a metric d, if $G = G'$ is a decreasing function k of the distance:

$$G(x, y) = k(d(x, y)),$$

(with k: $[0, \operatorname{diam} X] \to [0, +\infty]$ l.s.c., ≥ 0, and decreasing), then every contraction $\varphi: X \to X$ (as in (1), now with $x, y \in X$) is a G-contraction (i.e., a G, G-contraction). This situation covers e.g. the Riesz kernels.

When G is a kernel on X, the *potential* $G\mu$ of a Radon measure $\mu \geq 0$ on X is the l.s.c. function $X \to [0, +\infty]$ defined by

$$G\mu(x) = \int G(x, y)\, d\mu(y).$$

The *energy* of μ is

$$\int G\mu d\mu = \iint G(x, y)\, d\mu(x)\, d\mu(y).$$

For any set $A \subset X$, $\mathscr{M}^+(A)$ denotes the set of Radon measures ≥ 0 on X with support contained in A.

LEMMA. *Let* φ: $E \to E'$ *be a continuous surjection between compact spaces* E, E'. *For any measure* $\mu' \in \mathscr{M}^+(E')$ *there exists a measure* $\mu \in \mathscr{M}^+(E)$ *such that* $\mu' = \varphi(\mu)$, *that is,*

$$\int f'\, d\mu' = \int (f' \circ \varphi)\, d\mu \tag{3}$$

1) For arbitrary p the answer is affirmative in dimension 1 for a class of convolution kernels comprising the Riesz kernels on $\mathbf{R}$, see [1], but hardly in higher dimensions, not even for the Newton kernel, I believe.

for all continuous functions $f'\colon E' \to \mathbf{R}$ (*hence also for all l.s.c. functions* $f'\colon E' \to]-\infty, +\infty]$).

PROOF. This lemma is well known. The following elegant proof was indicated to me by J.P. Reus Christensen. Let $\mathscr{K}(E)$ and $\mathscr{K}(E')$ denote the vector spaces of continuous functions $E \to \mathbf{R}$ and $E' \to \mathbf{R}$, respectively. The functional $p\colon \mathscr{K}(E) \to \mathbf{R}$ defined by

$$p(f) = \mu'(1) \max f(E)$$

is subadditive and positive homogeneous. On the vector subspace

$$\{f' \circ \varphi \mid f' \in \mathscr{K}(E')\}$$

of $\mathscr{K}(E)$ a linear form λ is defined by

$$\lambda(f' \circ \varphi) = \mu'(f') \tag{4}$$

since φ is surjective. It satisfies

$$\lambda(f' \circ \varphi) \le \mu'(1) \max f'(E') = \mu'(1) \max (f' \circ \varphi)(E) = p(f' \circ \varphi).$$

By the Hahn-Banach theorem [3, p.27 f.] λ extends to a linear form μ on $\mathscr{K}(E)$ satisfying

$$\mu(f) \le p(f) \qquad \text{for all } f \in \mathscr{K}(E).$$

In particular, $\mu(f) \le 0$ if $f \le 0$, whence $\mu(f) = -\mu(-f) \ge 0$ for $f \ge 0$, showing that μ is a positive Radon measure on E. And (3) follows from (4) since μ extends λ. □

1. The energy capacity.

In this section G denotes any kernel on a locally compact space X.

DEFINITION. The energy capacity (with respect to G) of a compact set $E \subset X$ is

$$e_G(E) = \sup\{\mu(E)^2 \mid \mu \in \mathscr{M}^+(E), \int G\mu d\mu \le 1\}. \tag{5}$$

It can be shown that this supremum is attained if it is finite (in particular if $G(x, x) > 0$ for all $x \in X$, cf. [8, Theorem 2.3 and Lemma 2.5.1]).

THEOREM. *Let G and G' be kernels on locally compact spaces X and X', respectively. Let E denote a compact subset of X and $\varphi\colon E \to X'$ a G, G'-contraction. Then*

$$e_{G'}(\varphi(E)) \le e_G(E).$$

PROOF. Write $E' = \varphi(E)$. Let $\mu' \in \mathcal{M}^+(E')$ satisfy $\int G'\mu' d\mu' \leq 1$, and choose $\mu \in \mathcal{M}^+(E)$ so that (3) in the above lemma holds. Then

$$G'\mu'(x') = \int G'(x', y')\, d\mu'(y') = \int G'(x', \varphi(y))\, d\mu(y), \tag{6}$$

and hence

$$1 \geq \int G'\mu' d\mu' = \int (G'\mu')(\varphi(x))\, d\mu(x) = \int \left(\int G'(\varphi(x), \varphi(y))\, d\mu(y) \right) d\mu(x).$$

Invoking (2) we obtain

$$1 \geq \iint G(x, y)\, d\mu(x)\, d\mu(y),$$

showing that $\mu(E) \leq \sqrt{e_G(E)}$, and consequently, from (3) and (5)

$$\int_{E'} 1\, d\mu' = \int_E (1 \circ \varphi)\, d\mu = \mu(E) \leq \sqrt{e_G(E)},$$

from which the result follows by allowing μ' to vary. □

2. The ordinary capacity.

In this section G denotes any kernel on a locally compact space X.

DEFINITION. The ordinary capacity (with respect to G) of a compact set $E \subset X$ is

$$c_G(E) = \sup\{\mu(E) \mid \mu \in \mathcal{M}^+(E),\ G\mu \leq 1 \text{ on } X\}. \tag{7}$$

It can be shown that this supremum is attained if it is finite, in particular if there is no $y \in X$ such that $G(x, y) = 0$ for all $x \in X$, cf. [8, Theorem 2.3] as to the former statement.

THEOREM. *Let G and G' be kernels on locally compact spaces X and X', respectively. Let E denote a compact subset of X, and $\varphi: X \to X'$ a G, G'-contraction. Then*

$$c_{G'}(\varphi(E)) \leq c_G(E).$$

PROOF. Write $E' = \varphi(E)$. Let $\mu' \in \mathcal{M}^+(E')$ satisfy $G'\mu' \leq 1$ on X', and choose $\mu \in \mathcal{M}^+(E)$ so that (3) in the above lemma holds. For any $x \in X$ we have from (2), noting that (6) holds again,

$$G\mu(x) = \int G(x, y)\,d\mu(y) \le \int G'(\varphi(x), \varphi(y))\,d\mu(y)$$
$$= \int G'(\varphi(x), y')\,d\mu'(y') = (G'\mu')(\varphi(x)) \le 1,$$

showing that $\mu'(E') = \mu(E) \le c_G(E)$, whence the desired conclusion by allowing μ' to vary. □

REMARK. For a kernel G on X, $G\mu \le 1$ implies $\int G\mu\,d\mu \le \mu(1)$, and it follows easily that $c_G(E) \le e_G(E)$, with equality if G satisfies Frostman's maximum principle, cf. [8, p.153]. The Newton kernel on $\mathbf{R}^n$ satisfies the maximum principle, and so do the Riesz kernels for $\alpha \le 2$.

3. Inner and outer capacity.

Let G be a kernel on X, and let cap_G stand for either c_G or e_G. The *inner* and *outer* capacities of an arbitrary set $E \subset X$ are defined by

$$\underline{\mathrm{cap}}_G E = \sup\{\mathrm{cap}_G K \mid K \text{ compact}, K \subset E\},$$

$$\overline{\mathrm{cap}}_G E = \inf\{\underline{\mathrm{cap}}_G \omega \mid \omega \text{ open}, \omega \supset E\},$$

respectively. Clearly, $\underline{\mathrm{cap}}_G E \le \overline{\mathrm{cap}}_G E$. In case of equality, E is said to be *capacitable*, and we may omit the bars. Compact sets and open sets are always capacitable. (More about capacitability see below.) The relation

$$\underline{\mathrm{cap}}_G \bigcup K_n = \sup_n \mathrm{cap}_G K_n \tag{8}$$

holds for any increasing sequence of compact sets [8, Lemma 2.3.3].

Consider now two kernels G, G' on X, X', respectively, and a G, G'-contraction $\varphi: X \to X'$. Then

$$\underline{\mathrm{cap}}_{G'} \varphi(E) \le \underline{\mathrm{cap}}_G E \tag{9}$$

for any K_σ-set $E = \bigcup_n K_n$, where (K_n) denotes an increasing sequence of compact subsets of E. This follows immediately from Theorems 1 and 2, respectively, in view of (8), noting that $\varphi(E) = \bigcup_n \varphi(K_n)$ with the $\varphi(K_n)$ compact and increasing.

We proceed to prove that

$$\overline{\mathrm{cap}}_{G'} \varphi(E) \le \overline{\mathrm{cap}}_G E \tag{10}$$

for *every* set $E \subset X$ under the following hypotheses: The spaces X and X' are metrizable and σ-compact (i.e., K_σ-sets in themselves). In the case of the *energy capacity* we suppose that G' is (symmetric, positive semidefinite, and) *consistent* in the sense of [8, p.167]. In the case of the *ordinary capacity* we suppose that G' satisfies the *continuity principle* and "vanishes at infinity" (if X' is non-compact) in the usual sense[2].

Under these hypotheses, every open subset of X or X' is a K_σ-set, and every K_σ-set of X' is capacitable, cf. [8, Lemma 4.1.2], resp. [9, p.80]. To establish (10), consider any open (hence K_σ) set $\omega \subset X$ such that $E \subset \omega$, and apply (9) to ω in place of E, noting that $\varphi(\omega)$ is a K_σ in X' and hence capacitable:

$$\overline{\text{cap}}_{G'}\,\varphi(E) \leq \overline{\text{cap}}_{G'}\,\varphi(\omega) = \underline{\text{cap}}_{G'}\,\varphi(\omega) \leq \underline{\text{cap}}_{G}\,\omega,$$

whence (10) by allowing ω to vary.

The above hypotheses on the kernel G' on X' actually imply, in the case of the *energy capacity*, that every analytic subset of X' is capacitable, see [8, Theorem 4.5]. –In the case of the *ordinary capacity*, capacitability of analytic subsets of X' will follow if we add to the above hypotheses on X' and G' the following further assumptions concerning G' (omitting sometimes the dash for the sake of simplicity of notation): The adjoint kernal $(x, y) \mapsto G(y, x)$ likewise satisfies the continuity principle, and vanishes at infinity (if X' is non-compact), and finally $(x, y) \mapsto G(x, y)$ should be finite and continuous on $X' \times X'$ off the diagonal (i.e., for $x \neq y$), see [9, Théorème 8.2].

All these hypotheses—in the case of the energy capacity as well as in the case of the ordinary capacity—are satisfied, e.g., by the Riesz kernels $G(x, y) = |x - y|^{\alpha - n}$ on $\mathbf{R}^n (0 < \alpha < n)$, cf. [8, p.205], and earlier Aronszajn and Smith [2], in the case of the energy capacity; and Frostman [7, p.26] in the case of the ordinary capacity.

References

[1] H. Aikawa: Tangential boundary behavior of Green potentials and contractive properties of L^p-capacities. Tokyo J. Math. **9** (1986), 223–245.

[2] N. Aronszajn, K. T. Smith: Functional spaces and functional completion. Ann. Institut Fourier Grenoble **6** (1955–56), 125–185.

[3] S. Banach: Théorie des opérations linéaires. Warszawa 1932.

[4] M. Brelot: Eléments de la théorie classique du potentiel. Centre de documentation universitaire. 3e édition, Paris 1965.

[5] G. Choquet: Diamètre transfini et comparaison de diverses capacités. Séminaire de Théorie du Potentiel, Paris, 3e année, n°4 (1960), 7pp.

2) G' vanishes at infinity if $G'(x', y') \to 0$ uniformly with respect to y' on any compact subset of X' as x' tends to the Alexandroff point adjoined to X' (non-compact).

[6] M. Fekete: Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math. Z. **17** (1923), 228–249.

[7] O. Frostman: Potentiel d'équilibre et capacité des ensembles. Comm. Sém. Math. Univ. Lund 3, 1935.

[8] B. Fuglede: On the theory of potentials in locally compact spaces. Acta Math. **103** (1960), 139–215.

[9] B. Fuglede: Le théorème du minimax et la théorie fine du potentiel. Ann. Inst. Fourier Grenoble **15**. 1 (1965), 65–88.

[10] B. Fuglede: Applications du théorème minimax à l'étude de diverses capacités. C. R. Acad. Sc. Paris **266** (1968), 921–923.

[11] N. S. Landkof: Foundations of Modern Potential Theory. Springer-Verlag, Berlin-Heidelberg-New York 1972.

[12] G. Pólya, G. Szegö: Über den transfiniten Durchmesser ... J. reine u. angew. Math. **165** (1931), 4–49.

Mathematics Institute
University of Copenhagen

(*Universitetsparken* 5
DK-2100 *Copenhagen* ∅, *Denmark*)