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A remark on random fractals

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The second author of the present paper proved in [2] that the Hausdorff measure of a random fractal set is a non-random constant using a result of generalized random ergodic theorems. In this paper we remark that this constantness of the Hausdorff measure is proved directly (i.e., not using any result of generalized random ergodic theorems).

Let X be a compact subset of $\boldsymbol{R}^d$ with non-empty interior $\mathring{X}$. Consider a map $S: X \to X$ such that $\rho(Sx, Sy) = r\rho(x, y) (0 < r = r(S) < 1)$ for any $x, y \in X$, where $\rho(\cdot, \cdot)$ denotes the Euclidean metric. We call such a map S a contraction similarity of X with the contraction ratio r. For $0 < \delta < 1$, let $\mathscr{S}_\delta(X)$ be the set of all contraction similarities of X such that $r(S) > \delta$. We associate $\mathscr{S}_\delta(X)$ with the pointwise convergence topology and denote by $\mathscr{E}$ the Borel field of $\mathscr{S}_\delta(X)$.

For a fixed integer $N \geq 2$ we define $\mathscr{S} = (\mathscr{S}_\delta(X)^N)^D$ which plays a role of our fundamental space, where $D = \bigcup_{m \geq 0} C_m$, $C_m = \{1, 2, \cdots, N\}^m$, $m \geq 1$, and $C_0 = \{\varnothing\}$. Given a probability measure μ on $(\mathscr{S}_\delta(X)^N, \mathscr{E}^N)$, we define the product probability measure μ^D on $(\mathscr{S}, \mathscr{B} = (\mathscr{E}^N)^D)$.

Now we introduce a fractal set. For $\sigma = (\sigma_1, \cdots, \sigma_m)$, $\tau = (\tau_1, \cdots, \tau_r) \in D$, we define

$$\sigma | n = (\sigma_1, \cdots, \sigma_n), \qquad n \leq m,$$

and

$$\sigma * \tau = (\sigma_1, \cdots, \sigma_m, \tau_1, \cdots, \tau_r).$$

Each $s \in \mathscr{S}$ can be denoted by $s = (\underline{S}_\sigma)_{\sigma \in D}$,

$$\underline{S}_\sigma = (S_{\sigma * 1}(s), S_{\sigma * 2}(s), \cdots, S_{\sigma * N}(s)) \in \mathscr{S}_\delta(X)^N$$
$$\underline{S}_\varnothing = (S_1(s), S_2(s), \cdots, S_N(s)) \in \mathscr{S}_\delta(X)^N.$$

Then we define a set $K(s)$ for $s \in \mathscr{S}$,

$$K(s) = \bigcap_{m \geq 1} \bigcup_{\sigma \in C_m} S_{\sigma|1}(s) \circ S_{\sigma|2}(s) \circ \cdots \circ S_{\sigma|m}(s)(X)$$

which is called a *fractal set*. It is easily seen that $K(s)$ is a non-empty compact set. We take $s \in \mathscr{S}$ randomly according to μ^D. Thus we get a random fractal set $K(s)$.

For a set $K \subset \boldsymbol{R}^d$ and $0 \leq \lambda < \infty$, the Hausdorff measure and the Hausdorff dimension are defined as follows. Let

$$\mathscr{H}_\delta^\lambda(K) = \inf\{\textstyle\sum_{i=1}^\infty |V_i|^\lambda\colon K \subset \bigcup_{i=1}^\infty V_i,\ V_i \text{ is a closed set, } 0 < |V_i| \le \delta\}$$

where $|V|$ is the diameter of a set V, and let

$$\mathscr{H}^\lambda(K) = \lim_{\delta\to 0} \mathscr{H}_\delta^\lambda(K).$$

Then $\mathscr{H}^\lambda$ becomes an outer measure and so it defines a measure on $\boldsymbol{R}^d$, which is called the λ-dimensional *Hausdorff measure* and is denoted by the same letter $\mathscr{H}^\lambda$. It is easy to see that $\mathscr{H}^\lambda(S(K)) = r(S)^\lambda \mathscr{H}^\lambda(K)$ for a contraction similarity S. It is known that there is a number $0 \le \dim_{\mathrm{H}}(K) \le d$, called the *Hausdorff dimension* of K, such that $\mathscr{H}^\lambda(K) = \infty$ if $\lambda < \dim_{\mathrm{H}}(K)$ and $\mathscr{H}^\lambda(K) = 0$ if $\lambda > \dim_{\mathrm{H}}(K)$.

The following theorem was proved by Graf ([1]).

THEOREM 1. *Suppose that, for μ-a.e. $(S_1, S_2, \cdots, S_N) \in \mathscr{S}_\delta(X)^N$, $S_i(\mathring{X}) \cap S_j(\mathring{X}) = \varnothing$ holds for $i, j \in \{1, 2, \cdots, N\}$ with $i \ne j$. Let $\alpha > 0$ be the number such that*

$$\int \textstyle\sum_{i=1}^N r(S_i)^\alpha d\mu(S_1, S_2, \cdots, S_N) = 1.$$

Then

$$E(\mathscr{H}^\alpha(K(s))) = \int_S \mathscr{H}^\alpha(K(s))\, d\mu^D(s) < \infty$$

and

$$\dim_{\mathrm{H}}(K(s)) = \alpha \qquad \text{for } \mu^D\text{-a.e.} \qquad s \in \mathscr{S}.$$

Furthermore the following statements are equivalent:

(1) $\sum_{i=1}^N r(S_i)^\alpha = 1$ *for* μ*-a.e.* $(S_1, S_2, \cdots, S_N) \in \mathscr{S}_\delta(X)^N$,

(2) $\mathscr{H}^\alpha(K(s)) > 0$ *for* μ^D*-a.e.* $s \in \mathscr{S}$,

(3) $\mu^D(\{s\colon \mathscr{H}^\alpha(K(s)) > 0\}) > 0$.

Tsujii ([2]) proved following Theorem 2 using a generalized random ergodic theorem.

THEOREM 2. *Under the same assumption as in Theorem 1, there exists $0 \le \beta < \infty$ such that*

$$\mathscr{H}^\alpha(K(s)) = \beta \qquad \text{for } \mu^D\text{-a.e.} \qquad s \in \mathscr{S}.$$

We now prove this theorem without using random ergodic theorems. Let $\varphi_i\colon \mathscr{S} \to \mathscr{S}$ $(1 \le i \le N)$ be defined by $\underline{S}_\sigma(\varphi_i(s)) = \underline{S}_{i*\sigma}(s)$ for all $\sigma \in D$. Then φ_i is a measure-preserving transformation of $(\mathscr{S}, \mu^D)$ onto itself, i.e.,

$$\int f(\varphi_i(s))\, d\mu^D(s) = \int f(s)\, d\mu^D(s), \qquad \text{for all } f \in L^1(\mu^D).$$

It is easy to see that

$$\begin{aligned} K(s) &= \textstyle\bigcup_{i=1}^{N} S_i(s)\left(\bigcap_{m>0} \bigcup_{\sigma\in C_m} S_{(i*\sigma)|2}(s)\circ\cdots\circ S_{(i*\sigma)|m+1}(s)(X)\right) \\ &= \textstyle\bigcup_{i=1}^{N} S_i(s)K(\varphi_i(s)). \end{aligned}$$

Therefore we have

$$\mathscr{H}^{\alpha}(K(s)) \le \textstyle\sum_{i=1}^{N} \mathscr{H}^{\alpha}(S_i(s)K(\varphi_i(s))) = \sum_{i=1}^{N} r(S_i(s))^{\alpha}\mathscr{H}^{\alpha}(K(\varphi_i(s)))$$

and

$$\begin{aligned} E(\mathscr{H}^{\alpha}(K(s))) &\le \textstyle\sum_{i=1}^{N} E(r(S_i(s))^{\alpha}\mathscr{H}^{\alpha}(K(\varphi_i(s)))) \\ &= \textstyle\sum_{i=1}^{N} E(r(S_i(s))^{\alpha})E(\mathscr{H}^{\alpha}(K(s))) = E(\mathscr{H}^{\alpha}(K(s))) < \infty, \end{aligned}$$

because $r(S_i(s))$ is $\mathscr{B}_0^0$-measurable and $\mathscr{H}^{\alpha}(K(\varphi_i(s)))$ is $\mathscr{B}_1^{\infty}$-measurable and hence they are independent, where $\mathscr{B}_n^m$ is the Borel field generated by $\{S_\sigma(s)\colon \sigma\in\bigcup_{k=n}^{m} C_k\}$. Thus we have

(4) $$\mathscr{H}^{\alpha}(K(s)) = \textstyle\sum_{i=1}^{N} r(S_i(s))^{\alpha}\mathscr{H}^{\alpha}(K(\varphi_i(s))), \qquad \mu^D\text{-a.e. } s\in\mathscr{S}.$$

Let us define an operator U on $L^1(\mathscr{S}, \mu^D)$ by

$$Uf(s) = \textstyle\sum_{i=1}^{N} r(S_i(s))^{\alpha} f(\varphi_i(s)).$$

Then (4) implies that $\mathscr{H}^{\alpha}(K(s))$ is U-invariant, that is $U\mathscr{H}^{\alpha}(K(s)) = \mathscr{H}^{\alpha}(K(s))$, μ^D-a.e. $s\in\mathscr{S}$.

Now we will prove that if

(5) $$\textstyle\sum_{i=1}^{N} r(S_i(s))^{\alpha} = 1, \qquad \mu^D\text{-a.e. } s\in\mathscr{S},$$

then any U-invariant function f is a constant function. Indeed putting $r_i(s) = r(S_i(s))^{\alpha}$ we have

$$\begin{aligned} Uf(s) &= \textstyle\sum_{i=1}^{N} r_i(s) f(\varphi_i(s)), \\ U^2 f(s) &= \textstyle\sum_{i,j=1}^{N} r_i(s) r_j(\varphi_i(s)) f(\varphi_j\varphi_i(s)), \\ U^n f(s) &= \textstyle\sum_{i_1,\dots,i_n=1}^{N} r_{i_1}(s) r_{i_2}(\varphi_{i_1}(s))\cdots r_{i_n}(\varphi_{i_{n-1}}\cdots\varphi_{i_1}(s)) f(\varphi_{i_n}\cdots\varphi_{i_1}(s)). \end{aligned}$$

Here $r_{i_1}(s)$, $r_{i_2}(\varphi_{i_1}(s)),\cdots, r_{i_n}(\varphi_{i_{n-1}}\cdots\varphi_{i_1}(s))$ are $\mathscr{B}_0^{n-1}$-measurable and $f(\varphi_{i_n}\cdots\varphi_{i_1}(s))$ is $\mathscr{B}_n^{\infty}$-measurable where $\mathscr{B}_0^{n-1}$ and $\mathscr{B}_n^{\infty}$ are independent. Thus we have

$$\begin{aligned} E(f|\mathscr{B}_0^{n-1}) &= \textstyle\sum_{i_1,\dots,i_n=1}^{N} r_{i_1}(s) r_{i_2}(\varphi_{i_1}(s))\cdots r_{i_n}(\varphi_{i_{n-1}}\cdots\varphi_{i_1}(s))\, E(f) \\ &= E(f), \qquad \mu^D\text{-a.e.} \end{aligned}$$

The left hand side converges to $E(f|\mathscr{B}_0^{\infty}) = E(f|\mathscr{B}) = f$ μ^D-a.e. as $n\to\infty$. Therefore $f = E(f) =$ constant.

If (5) does not hold, then the condition (1) in Theorem 1 fails and so $\mu^D(\mathscr{H}^{\alpha}(K(s)) > 0) = 0$. Namely $\mathscr{H}^{\alpha}(K(s)) = 0$ μ^D-a.e. If (5) holds, $\mathscr{H}^{\alpha}(K(s))$

$= \beta\ \mu^D$-a.e. for some constant β, because $\mathscr{H}^\alpha(K(s))$ is U-invariant. This completes the proof.

References

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[2] Y. Tsujii, Generalized random ergodic theorems and Hausdorff-measures of random fractals, *Hiroshima Math. J.*, **19** (1989), 363–377.

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