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Partial A-optimal balanced fractional 2^m factorial designs with $6 \leqq m \leqq 8$

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1. Introduction

A balanced array (B-array), which is a generalization of an orthogonal array, was first studied by Chakravarti [2] under the name of "partially B-array". A connection between a B-array and a balanced fractional factorial (BFF) design has been investigated so far by, e.g., Srivastava [31], Yamamoto, Shirakura and Kuwada [41], Kuwada [20] and Kuwada and Nishii [22]. The characteristic roots of the information matrix of a 2^m-BFF design of resolution V were obtained by Srivastava and Chopra [33]. By use of the triangular multidimentional partially balanced (TMDPB) association scheme and its algebra, Yamamoto, Shirakura and Kuwada [42] extended their results to a 2^m-BFF design of resolution $2\ell + 1$. The concept of the MDPB association scheme was introduced by Bose and Srivastava [1] and Srivastava [30] as a generalization of the ordinary association scheme.

A- and/or D-optimal 2^m-BFF designs of resolution V or VII were obtained by Srivastava and/or Chopra [4–9, 11–14, 34, 35] and Shirakura [24, 26]. More precise tables of Srivastava-Chopra optimal designs of resolution V have been presented by Nishii and Shirakura [23] for $4 \leqq m \leqq 6$, and Chopra, Kipngeno and Ghosh [10] for $7 \leqq m \leqq 10$. Some optimal fractional 2^m factorial (2^m-FF) designs were obtained by Cheng [3] and Kuwada [21]. Optimal 2^m-BFF designs of even resolution derived from B-arrays were obtained by Shirakura [25–27]. A necessary and sufficient condition for a B-array of strength 2ℓ to be a 2^m-BFF design of resolution 2ℓ was obtained by Shirakura [28]. Yamamoto and Hyodo [38, 39] introduced an extended concept of resolution, which includes the results due to Shirakura [25–28]. By utilizing the characterization of the information matrix, Yamamoto and Hyodo [38–40], Hyodo and Yamamoto [17–19] and Hyodo [15, 16] have shown that there are so many designs having various type resolution including both odd and even resolution as special cases.

Consider a two-symbol B-array of strength 6, m constraints, index set $\{\mu_0^{(6)}, \mu_1^{(6)},\dots, \mu_6^{(6)}\}$ and frequency set $\{z_0^{(m)}, z_1^{(m)},\dots, z_m^{(m)}\}$, where $z_j^{(m)}$ are the number of row vectors with weight j in the array. Such an array is traditionally denoted as a BA$(N, m, 2, 6)\{\mu_0^{(6)}, \mu_1^{(6)},\dots, \mu_6^{(6)}\}$, where N is the total number of assemblies. We, however, denote it here as

BA$(m, 6; z_0^{(m)}, z_1^{(m)},\ldots, z_m^{(m)})$ since the characterization of the information matrix can be explicitly expressed by $z_j^{(m)}$ (see [15, 16]). The indices $\mu_i^{(6)}$ are completely determined by $z_j^{(m)}$ as follows (cf. [15, 26, 32, 36, 37]):

$$\mu_i^{(6)} = \sum_{j=0}^{m} \binom{m-6}{j-i} \left\{ z_j^{(m)} \Big/ \binom{m}{j} \right\} \qquad \text{for } i = 0, 1,\ldots, 6.$$

Note that the usual boundary convention for the binomial coefficient $\binom{a}{b}$, i.e., $\binom{a}{b} = 0$ if and only if $b < 0$ or $0 \leqq a < b$, will be used throughout this paper.

In this paper, we shall consider a 2^m-BFF design derived from a BA$(m, 6; z_0^{(m)}, z_1^{(m)},\ldots, z_m^{(m)})$ such that the general mean and the main effects (or the main effects only) are estimable under the situation in which all four-factor and higher order interactions are assumed to be negligible. Such a design will be called a 2^m-BFF design having resolution $R^*(\{0, 1\}|P)$(or $R^*(\{1\}|P)$) as will be seen in Definition 3.3, where $P = \{0, 1, 2, 3\}$. For a given pair (N, m), there are so many 2^m-BFF designs having resolution $R^*(\{0, 1\}|P)$(or $R^*(\{1\}|P)$). We may note that these designs may be superior to resolution IV designs in the sense that the confounding of the three-factor interactions and the main effects can be always avoided even though the latter exist. A design considered here is explicitly described by some specified simple array (S-array) for the cases of $m = 6, 7, 8$ and $N < \sum_{i=0}^{3}\binom{m}{i}$($= \nu_3$, say) as will be seen in Proposition 3.3, where ν_3 is the total number of factorial effects up to the three-factor interactions (see [15–19, 38–40]). In Section 4, for the cases of $m = 6$, 7 and 8, partial A-optimal 2^m-BFF designs having resolution $R^*(\{0, 1\}|P)$ and $R^*(\{1\}|P)$) will be presented for each value of $N(< \nu_3)$. The covariance matrix of the estimates and the value of its trace are also given for such designs.

2. Preliminaries

Consider a 2^m-FF design with m factors $F_1,\ldots, F_m$, each at two levels 0 or 1, where $m \geqq 6$. Further consider the situation in which all four-factor and higher order interactions are assumed to be negligible. The $\nu_3 \times 1$ vector of factorial effects is denoted by

$$\begin{aligned}\underline{\theta}' &= (\theta_\phi; \theta_1,\ldots, \theta_m; \theta_{12},\ldots, \theta_{m-1m}; \theta_{123},\ldots, \theta_{m-2m-1m})\\ &= (\theta_\phi; \underline{\theta}_1'; \underline{\theta}_2'; \underline{\theta}_3'),\end{aligned}$$

where θ_ϕ, θ_{t_1} and, in general, $\theta_{t_1\ldots t_u}$ denote the general mean, the main effect of the factor F_{t_1} and the u-factor interaction of the factors $F_{t_1},\ldots, F_{t_u}$, respectively. Here A' and $\underline{\theta}_u$ denote, respectively, the transpose of a matrix A

and the $\binom{m}{u} \times 1$ vector of the u-factor interactions, especially $u = 0$ and $u = 1$ stand for the general mean, i.e., $\underline{\theta}_0 = \theta_\phi$, and the main effects, respectively. Let T be a (0, 1)-array of size $N \times m$ whose rows denote N assemblies of a design under consideration. The linear model based on T is then given by

$$\underline{y}_T = E_T \underline{\theta} + \underline{e}_T,$$

where $\underline{y}_T$, E_T and $\underline{e}_T$ denote a vector of N observations, the $N \times \nu_3$ design matrix whose elements are either 1 or -1, and an $N \times 1$ error vector with $E[\underline{e}_T] = \underline{0}_N$ and $\mathrm{Cov}[\underline{e}_T] = \sigma^2 I_N$, respectively. Here $\underline{0}_N$ and I_N are the $N \times 1$ vector with all zero and the identity matrix of order N, respectively. The normal equation for estimating $\underline{\theta}$ is given by

$$M_T \hat{\underline{\theta}} = E_T' \underline{y}_T,$$

where $M_T = E_T' E_T$ is the information matrix of order ν_3.

Among the four sets of factorial effects $\{\theta_\phi\}$, $\{\theta_{t_1}\}$, $\{\theta_{t_1 t_2}\}$ and $\{\theta_{t_1 t_2 t_3}\}$, a TMDPB association scheme is defined by introducing a natural relation of association such that $\theta_{t_1 \ldots t_u}$ and $\theta_{t_1' \ldots t_v'}$ are the a-th associates if and only if

$$|\{t_1, \ldots, t_u\} \cap \{t_1', \ldots, t_v'\}| = \min(u, v) - a,$$

where $|S|$ and $\min(u, v)$ denote the cardinality of a set S and the minimum of integers u and v, respectively.

It is known that a TMDPB association algebra $\boldsymbol{R}$ generated by the thirty ordered association matrices $D_a^{(u,v)} (0 \leqq a \leqq \min(u, v);\ u, v = 0, 1, 2, 3)$ is semi-simple and completely reducible. It is decomposed into the direct sum of the four two-sided ideals $\boldsymbol{R}_b$ generated by $(4\text{-}b)^2$ ideal bases $\{D_b^{(u,v)\#}: b \leqq u, v \leqq 3\}$ for $b = 0, 1, 2, 3$. The ideal $\boldsymbol{R}_b$ is isomorphic to the complete $(4-b) \times (4-b)$ matrix algebra with multiplicity $\binom{m}{b} - \binom{m}{b-1} (= \phi_b$, say). The details of the TMDPB association scheme and its algebra can be seen in Yamamoto, Shirakura and Kuwada [41, 42] and Shirakura [26]. It is known (see [15, 41, 42]) that the information matrix M_T of a 2^m-FF design T derived from a BA$(m, 6;\ z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$ belongs to the TMDPB association algebra $\boldsymbol{R}$ and is given by

$$\begin{aligned} M_T &= \textstyle\sum_{u=0}^{3} \sum_{v=0}^{3} \sum_{a=0}^{\min(u,v)} \gamma_{|u-v|+2a} D_a^{(u,v)} \\ &= \textstyle\sum_{b=0}^{3} \sum_{r=0}^{3-b} \sum_{s=0}^{3-b} \kappa_b^{r,s} D_b^{(b+r,b+s)\#} \in \boldsymbol{R}, \end{aligned} \tag{2.1}$$

where

$$\gamma_i = \textstyle\sum_{j=0}^{6} \sum_{q=0}^{i} (-1)^q \binom{i}{q} \binom{6-i}{j-i+q} \mu_j^{(6)}$$

$$= \sum_{j=0}^{m}\left\{\sum_{q=0}^{i}(-1)^q\binom{i}{q}\binom{m-i}{m-j-q}\right\}\left\{z_j^{(m)}\Big/\binom{m}{j}\right\} \qquad \text{for } i=0,1,\dots,6,$$

$$k_b^{r,s} = k_b^{s,r} = \sum_{a=0}^{b+r}\gamma_{s-r+2a}z_{ba}^{(b+r,b+s)} \qquad \text{for } 0 \leqq r \leqq s \leqq 3-b;\ b=0,1,2,3$$

and

$$z_{ba}^{(b+r,b+s)} = \sum_{c=0}^{a}(-1)^{a-c}\binom{r}{c}\binom{b+r-c}{b+r-a}\binom{m-2b-r+c}{c}$$
$$\cdot\left\{\binom{m-2b-r}{s-r}\binom{s}{r}\right\}^{1/2}\Big/\binom{s-r+c}{c} \qquad \text{for } r \leqq s.$$

Here the matrix $D_b^{(u,v)\#}$ of order v_3 is linearly linked with the ordered association matrices $D_a^{(u,v)}$ of the TMDPB association scheme as follows (see [26, 29, 42]):

$$D_a^{(u,v)} = \{D_a^{(v,u)}\}' = \sum_{b=0}^{u} z_{ba}^{(u,v)} D_b^{(u,v)\#} \qquad \text{for } 0 \leqq a \leqq u \leqq v \leqq 3$$

and

$$D_b^{(u,v)\#} = \{D_b^{(v,u)\#}\}' = \sum_{a=0}^{u} z_{(u,v)}^{ba} D_a^{(u,v)} \qquad \text{for } 0 \leqq b \leqq u \leqq v \leqq 3, \tag{2.2}$$

where

$$z_{(u,v)}^{ba} = \phi_b z_{ba}^{(u,v)}\Big/\left\{\binom{m}{u}\binom{u}{a}\binom{m-u}{v-u+a}\right\} \qquad \text{for } u \leqq v.$$

The matrices $D_b^{(u,v)\#}$ have the following properties (see [42]):

$$D_a^{(u,w)\#} D_b^{(s,v)\#} = \delta_{ws}\delta_{ab} D_b^{(u,v)\#}, \tag{2.3}$$

$$\sum_{b=0}^{u} D_b^{(u,u)\#} = D_0^{(u,u)}, \tag{2.4}$$

$$\sum_{u=0}^{3}\sum_{b=0}^{u} D_b^{(u,u)\#} = I_{v_3}$$

and

$$\operatorname{rank}[D_b^{(u,v)\#}] = \phi_b, \tag{2.5}$$

where δ_{ab} denotes Kronecker's delta. Each $(4-b)\times(4-b)$ symmetric matrix $K_b = [k_b^{r,s}]$ $(0 \leqq r, s \leqq 3-b;\ b=0,1,2,3)$ is called the irreducible matrix representation of M_T with respect to the ideal $\boldsymbol{R}_b$ with multiplicity ϕ_b and it can be expressed as follows (see [15, 19]):

$$K_b = \sum_{j=b}^{m-b}\left\{z_j^{(m)}\Big/\binom{m}{j}\right\}\underline{k}_{bj}\underline{k}_{bj}' \qquad \text{for } b=0,1,2,3,$$

where $\underline{k}_{bj}$ are given by

$$\underline{k}_{0j}' = \left\{\binom{m}{j}\right\}^{1/2}(1, (2j-m)/m^{1/2},\ \{(2j-m)^2-m\}/\{2m(m-1)\}^{1/2},$$

$$(2j-m)\{(2j-m)^2-3m+2\}/\{6m(m-1)(m-2)\}^{1/2}) \quad \text{for } 0 \leqq j \leqq m,$$

$$\underline{k}'_{1j} = 2\left\{\binom{m-2}{j-1}\right\}^{1/2}(1,(2j-m)/(m-2)^{1/2},$$

$$\{(2j-m)^2-m+2\}/\{2(m-2)(m-3)\}^{1/2}) \qquad \text{for } 1 \leqq j \leqq m-1,$$

$$\underline{k}'_{2j} = 4\left\{\binom{m-4}{j-2}\right\}^{1/2}(1,(2j-m)/(m-4)^{1/2}) \qquad \text{for } 2 \leqq j \leqq m-2$$

and

$$\underline{k}'_{3j} = 8\left\{\binom{m-6}{j-3}\right\}^{1/2} \qquad \text{for } 3 \leqq j \leqq m-3.$$

The matrices K_b have the following properties (see [15, 16, 19]):

PROPOSITION 2.1. (i) *rank*$[K_b] = \min(w(z_b^{(m)}, z_{b+1}^{(m)}, \ldots, z_{m-b}^{(m)}), 4-b)$ *for* $b = 0, 1, 2, 3$, *where* $w(\underline{x}')$ *denotes the number of nonzero elements of a row vector* $\underline{x}'$.

(ii) *If* $\text{rank}[K_b] = r$, *then the first* r *rows in* K_b *are always linearly independent.*

(iii) *There exist* (4-b) *linearly independent vectors in* $\underline{k}_{bb}, \underline{k}_{bb+1}, \ldots, \underline{k}_{bm-b}$, *which are contained in* K_b *as a column vector each.*

If T is an S-array with parameters $(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$, written SA$(m; \lambda_0, \lambda_1, \ldots, \lambda_m)$ for brevity, then it follows from $z_j^{(m)} = \binom{m}{j}\lambda_j$ $(j = 0, 1, \ldots, m)$ that

$$K_b = \textstyle\sum_{j=b}^{m-b} \lambda_j \underline{k}_{bj}\underline{k}'_{bj} \qquad \text{for } b = 0, 1, 2, 3.$$

3. 2^m-BFF designs having resolution $R^*(\{0, 1\}|P)$ and $R^*(\{1\}|P)$

For readers' convenience, we recall the definition of resolution here.

DEFINITION 3.1. Let $P = \{0, 1, 2, 3\}$ and $S \subset P$. Then a 2^m-FF design is said to be of resolution $R(S|P)$ if

(i) $D_0^{(s,s)}\underline{\theta}$, i.e., a vector of s-factor interactions $\underline{\theta}_s$, is estimable for every $s \in S$

and

(ii) $D_0^{(h,h)}\underline{\theta}$, i.e., a vector of h-factor interactions $\underline{\theta}_h$, is not estimable for every $h \in P$-S

under the situation in which all four-factor and higher order interactions are assumed to be negligible.

Note that resolution $R(\{0,1,2,3\}|P)$ and $R(\{0,1,2\}|P)$ (or $R(\{1,2\}|P)$) are, respectively, resolution VII and VI, where $P=\{0,1,2,3\}$.

DEFINITION 3.2. A 2^m-FF design of resolution $R(S|P)$ is said to be balanced and denoted by 2^m-BFF design of resolution $R(S|P)$ if the covariance matrix of the BLUE of $\sum_{s\in S} D_0^{(s,s)}\underline{\theta}$ is invariant under any permutation on m factors.

A 2^m-FF (or 2^m-BFF) design having resolution $R^*(\{0,1\}|P)$ (or $R^*(\{1\}|P)$) is defined as follows:

DEFINITION 3.3. If S is a set such that $P\supset S\supset Q$ for fixed P and Q, then a 2^m-FF (or 2^m-BFF) design of resolution $R(S|P)$ is called a 2^m-FF (or 2^m-BFF) design having resolution $R^*(Q|P)$, where $Q=\{0,1\}$ or $\{1\}$.

The following Propositions 3.1 and 3.2 are due to Hyodo [15] and Yamamoto and Hyodo [38], respectively.

PROPOSITION 3.1. *Let T be a 2^m-FF design derived from a $BA(m,6;z_0^{(m)},z_1^{(m)},\dots,z_m^{(m)})$. Then T is a 2^m-BFF design of resolution $R(S|P)$ if and only if T satisfies the following conditions:*

(i) $\operatorname{rank}[K_b^*]=\operatorname{rank}[K_b^*:\underline{f}_b^{(s)}]$ *for every* $b\in\{0,1,\dots,s\}\,(s\in S)$

and

(ii) $\operatorname{rank}[K_b^*]\neq\operatorname{rank}[K_b^*:\underline{f}_b^{(h)}]$ *for some* $b\in\{0,1,\dots,h\}\,(h\in P-S)$,

where $P=\{0,1,2,3\}$, $K_b^*=[z_b^{(m)}\underline{k}_{bb}, z_{b+1}^{(m)}\underline{k}_{bb+1},\dots,z_{m-b}^{(m)}\underline{k}_{bm-b}]$ *and* $\underline{f}_b^{(u)}$ *denotes the* $(4\text{-}b)\times 1$ *canonical basis vector whose* $(u-b+1)$*th element is unity.*

PROPOSITION 3.2. *Let T be a $BA(m,6;z_0^{(m)},z_1^{(m)},\dots,z_m^{(m)})$ and $P=\{0,1,2,3\}$.*

(I) *If T is a 2^m-BFF design having resolution $R^*(\{0,1\}|P)$, then the BLUE of a vector of estimable parametric functions $\sum_{u=0}^{1} D_0^{(u,u)}\underline{\theta}\,(=\underline{\Psi}_{01}$, say) and the covariance matrix of its estimate are, respectively, given by*

$$\hat{\underline{\Psi}}_{01}=X_{01}E_T'\underline{y}_T$$

and

$$\operatorname{Cov}[\hat{\underline{\Psi}}_{01}]=\sigma^2X_{01}M_TX_{01}'\in\boldsymbol{R}, \tag{3.1}$$

where $X_{01}(\in\boldsymbol{R})$ is a $v_3\times v_3$ matrix satisfying $X_{01}M_T=\sum_{u=0}^{1}D_0^{(u,u)}$.

(II) *If T is a 2^m-BFF design having resolution $R^*(\{1\}|P)$, then the BLUE of a vector of estimable parametric functions $D_0^{(1,1)}\underline{\theta}\,(=\underline{\Psi}_1$, say) and the covariance matrix of its estimate are, respectively, given by*

$$\hat{\underline{\Psi}}_1 = X_1 E_T' \underline{y}_T$$

and

$$\mathrm{Cov}[\hat{\underline{\Psi}}_1] = \sigma^2 X_1 M_T X_1' \in \boldsymbol{R}, \tag{3.2}$$

where $X_1 (\in \boldsymbol{R})$ *is a* $v_3 \times v_3$ *matrix satisfying* $X_1 M_T = D_0^{(1,1)}$.

It is known that a BA$(m, 6; z_0^{(m)}, z_1^{(m)}, \dots, z_m^{(m)})$ gives an SA$(m; \lambda_0, \lambda_1, \dots, \lambda_m)$ for the cases of $m = 6$ and 7. It has been shown in Hyodo [16] that a BA$(8, 6; z_0^{(8)}, z_1^{(8)}, \dots, z_8^{(8)})$ turns out to be an SA$(8; \lambda_0, \lambda_1, \dots, \lambda_8)$ provided the information matrix is singular. The following proposition is due to Hyodo [15, 16].

PROPOSITION 3.3. *Consider* 2^m*-BFF designs having resolution* $R^*(\{0, 1\}|P)$ *and* $R^*(\{1\}|P)$ *for the cases of* $m = 6, 7, 8$ *and* $N < v_3$, *where* $P = \{0, 1, 2, 3\}$. *Such designs are explicitly described by some specified* $SA(m; \lambda_0, \lambda_1, \dots, \lambda_m)$ *as will be seen in Tables* 3.1 *and* 3.2.

TABLE 3.1. 2^m-BFF designs having resolution $R^*(\{0, 1\}|P)$ with $6 \leqq m \leqq 8$

m	Resolution	Conditions on SA$(m; \lambda_0, \lambda_1, \dots, \lambda_m)$
6	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(6a) $\lambda_i > 0$ $(i=0,2,4,6)$, $\lambda_j = 0$ $(j=1,3,5)$;
		(6b) $\lambda_i > 0$ $(i=2,4,5)$, $\lambda_0 + \lambda_1 + \lambda_6 > 0$, $\lambda_3 = 0$;
		(6c) $\lambda_i > 0$ $(i=1,2,4)$, $\lambda_0 + \lambda_5 + \lambda_6 > 0$, $\lambda_3 = 0$;
		(6d) $\lambda_i > 0$ $(i=1,3,5)$, $\lambda_j = 0$ $(j=0,2,4,6)$; or
		(6e) $\lambda_i > 0$ $(i=1,3,5)$, $\lambda_0 + \lambda_6 > 0$, $\lambda_j = 0$ $(j=2,4)$
	$R(\{0, 1\}\|P)$	(6f) $\lambda_i > 0$ $(i=1,4,5)$, $\lambda_0 + \lambda_6 > 0$, $\lambda_j = 0$ $(j=2,3)$; or
		(6g) $\lambda_i > 0$ $(i=1,2,5)$, $\lambda_0 + \lambda_6 > 0$, $\lambda_j = 0$ $(j=3,4)$
7	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(7a) $\lambda_i > 0$ $(i=2,5,6)$, $\lambda_0 + \lambda_1 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=3,4)$; or
		(7b) $\lambda_i > 0$ $(i=1,2,5)$, $\lambda_0 + \lambda_6 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=3,4)$
	$R(\{0,1\}\|P)$	(7c) $\lambda_i > 0$ $(i=1,5,6)$, $\lambda_0 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=2,3,4)$;
		(7d) $\lambda_i > 0$ $(i=1,2,6)$, $\lambda_0 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=3,4,5)$;
		(7e) $\lambda_i > 0$ $(i=0,1,4,7)$, $\lambda_j = 0$ $(j=2,3,5,6)$;
		(7f) $\lambda_i > 0$ $(i=0,3,6,7)$, $\lambda_j = 0$ $(j=1,2,4,5)$;
		(7g) $\lambda_i > 0$ $(i=1,4,6)$, $\lambda_0 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=2,3,5)$; or
		(7h) $\lambda_i > 0$ $(i=1,3,6)$, $\lambda_0 + \lambda_7 > 0$, $\lambda_j = 0$ $(j=2,4,5)$

TABLE 3.1. (continued)

m	Resolution	Conditions on SA(m; λ_0, $\lambda_1,\dots,\lambda_m$)
8	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(8a) $\lambda_i>0$ ($i=2,6,7$), $\lambda_0+\lambda_1+\lambda_8>0$, $\lambda_j=0$ ($j=3,4,5$);
		(8b) $\lambda_i>0$ ($i=1,2,6$), $\lambda_0+\lambda_7+\lambda_8>0$, $\lambda_j=0$ ($j=3,4,5$); or
		(8c) $\lambda_i>0$ ($i=1,4,7$), $\lambda_0+\lambda_8>0$, $\lambda_j=0$ ($j=2,3,5,6$)
	$R(\{0,1\}\|P)$	(8d) $\lambda_i>0$ ($i=1,6,7$), $\lambda_0+\lambda_8>0$, $\lambda_j=0$ ($j=2,3,4,5$);
		(8e) $\lambda_i>0$ ($i=1,2,7$), $\lambda_0+\lambda_8>0$, $\lambda_j=0$ ($j=3,4,5,6$);
		(8f) $\lambda_i>0$ ($i=1,5,7$), $\lambda_0+\lambda_8>0$, $\lambda_j=0$ ($j=2,3,4,6$); or
		(8g) $\lambda_i>0$ ($i=1,3,7$), $\lambda_0+\lambda_8>0$, $\lambda_j=0$ ($j=2,4,5,6$)

TABLE 3.2. 2^m-BFF designs having resolution $R^*(\{1\}|P)$ with $6 \leqq m \leqq 8$

m	Resolution	Conditions on SA(m; λ_0, $\lambda_1,\dots,\lambda_m$)
6	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{1,2,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(6a)-(6e) in Table 3.1
	$R(\{1,3\}\|P)$	non-exist (see [15])
	$R(\{1,2\}\|P)$, i.e., VI	non-exist (see [15])
	$R(\{0,1\}\|P)$	(6f) and (6g) in Table 3.1
	$R(\{1\}\|P)$	(6h) $\lambda_i>0$ ($i=1,4,5$), $\lambda_j=0$ ($j=0,2,3,6$); or
		(6i) $\lambda_i>0$ ($i=1,2,5$), $\lambda_j=0$ ($j=0,3,4,6$)
7	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{1,2,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(7a) and (7b) in Table 3.1
	$R(\{1,3\}\|P)$	non-exist (see [15])
	$R(\{1,2\}\|P)$, i.e., VI	non-exist (see [15])
	$R(\{0,1\}\|P)$	(7c)-(7h) in Table 3.1
	$R(\{1\}\|P)$	non-exist (see [15])
8	$R(\{0,1,2,3\}\|P)$, i.e., VII	non-exist (since $N < v_3$)
	$R(\{0,1,3\}\|P)$	non-exist (see [15])
	$R(\{0,1,2\}\|P)$, i.e., VI	(8a)-(8c) in Table 3.1
	$R(\{1,3\}\|P)$	non-exist (see [15])
	$R(\{1,2\}\|P)$, i.e., VI	non-exist (see [15])
	$R(\{0,1\}\|P)$	(8d)-(8g) in Table 3.1
	$R(\{1\}\|P)$	non-exist (see [15])

4. PA-optimal 2^m-BFF designs having resolution $R^*(\{0,1\}|P)$ and $R^*(\{1\}|P)$ with $6 \leqq m \leqq 8$

We shall consider a 2^m-BFF design derived from a BA$(m, 6; z_0^{(m)}, z_1^{(m)}, \ldots, z_m^{(m)})$. For $P = \{0,1,2,3\}$, PA-optimal 2^m-BFF designs having resolution $R^*(\{0,1\}|P)$ and $R^*(\{1\}|P)$ are then defined as follows:

DEFINITION 4.1. A 2^m-BFF design having resolution $R^*(\{0,1\}|P)$ is said to be partial A-optimal, written PA-optimal 2^m-BFF design having resolution $R^*(\{0,1\}|P)$ for brevity, if $\mathrm{tr}(\mathrm{Cov}[\hat{\underline{\Psi}}_{01}]/\sigma^2)(= S_{01}$, say) is a minimum for a given pair (N, m), where $\mathrm{Cov}[\hat{\underline{\Psi}}_{01}]$ is given in (3.1) and $\mathrm{tr}(S)$ denotes the trace of a matrix S.

DEFINITION 4.2. A 2^m-BFF design having resolution $R^*(\{1\}|P)$ is said to be partial A-optimal, written PA-optimal 2^m-BFF design having resolution $R^*(\{1\}|P)$ for brevity, if $\mathrm{tr}(\mathrm{Cov}[\hat{\underline{\Psi}}_1]/\sigma^2)(= S_1$, say) is a minimum for a given pair (N, m), where $\mathrm{Cov}[\hat{\underline{\Psi}}_1]$ is given in (3.2).

Let $k_{i,j}^0$ and $k_{r,s}^1$ be, respectively, the $(i+1, j+1)$-element and $(r+1, s+1)$-element of

$$\text{(i)}\quad K_0^{-1} \text{ and } \begin{bmatrix} K_1^{0,0} & k_1^{0,1} \\ k_1^{1,0} & k_1^{1,1} \end{bmatrix}^{-1} \quad (= K_{(1)}^{-1}, \text{ say}) \tag{4.1}$$

for the series (6a), (7e) and (7f) in Proposition 3.3,

$$\text{(ii)}\quad \begin{bmatrix} k_0^{0,0} & k_0^{0,1} & k_0^{0,2} \\ & k_0^{1,1} & k_0^{1,2} \\ \text{sym.} & & k_0^{2,2} \end{bmatrix}^{-1} \quad (= K_{(0)}^{-1}, \text{ say}) \quad \text{and} \quad K_1^{-1} \tag{4.2}$$

for the series (6d), (6h) and (6i) in Proposition 3.3

and

(iii) K_0^{-1} and K_1^{-1} for the remaining series. (4.3)

Note that from Proposition 2.1, K_0 and $K_{(1)}$ in (4.1), $K_{(0)}$ and K_1 in (4.2), and K_0 and K_1 in (4.3) are nonsingular. Then we have the following:

THEOREM 4.1. (I) *If T is an array of Table* 3.1, *then* $\mathrm{Cov}[\hat{\underline{\Psi}}_{01}]$ *and* S_{01} *are, respectively, given by*

$$\mathrm{Cov}[\hat{\underline{\Psi}}_{01}] = \sigma^2 \textstyle\sum_{b=0}^{1} \sum_{r=0}^{1-b} \sum_{s=0}^{1-b} k_{r,s}^b D_b^{(b+r, b+s)\#} \in \boldsymbol{R} \tag{4.4}$$

and

$$S_{01} = (k_{0,0}^0 + k_{1,1}^0) + (m-1)k_{0,0}^1. \tag{4.5}$$

(II) *If T is an array of Table* 3.2, *then* $\mathrm{Cov}[\hat{\underline{\Psi}}_1]$ *and* S_1 *are, respectively, given by*

$$\mathrm{Cov}[\hat{\underline{\Psi}}_1] = \sigma^2\{k_{1,1}^0 D_0^{(1,1)\#} + k_{0,0}^1 D_1^{(1,1)\#}\} \in \boldsymbol{R} \tag{4.6}$$

and

$$S_1 = k_{1,1}^0 + (m-1)k_{0,0}^1. \tag{4.7}$$

Proof. (I) Consider T being an array of Table 3.1.

(i) If T is an array of the series (7e) and (7f), then using

$$X_{01} = \textstyle\sum_{r=0}^{1}\sum_{s=0}^{3} k_{r,s}^0\, D_0^{(r,s)\#} + \sum_{s=0}^{1} k_{0,s}^1\, D_1^{(1,s+1)\#} \in \boldsymbol{R},$$

it holds from (2.1), (2.3) and (2.4) that $X_{01}M_T = \sum_{u=0}^{1} D_0^{(u,u)}$. Furthermore substituting the above X_{01} into (3.1), we get (4.4) from (2.1), (2.2), (2.3) and (2.4).

(ii) For T being an array of the remaining series, let

$$X_{01} = \textstyle\sum_{b=0}^{1}\sum_{r=0}^{1-b}\sum_{s=0}^{3-b} k_{r,s}^b\, D_b^{(b+r,b+s)\#} \in \boldsymbol{R}.$$

Then from the argument similar to the above, we have (4.4). Applying (2.3) and (2.5) to (4.4), we have (4.5).

(II) Consider T being an array of Table 3.2.

(i) If T is an array of the series (6a), (7e) and (7f), then using

$$X_1 = \textstyle\sum_{s=0}^{3} k_{1,s}^0\, D_0^{(1,s)\#} + \sum_{s=0}^{1} k_{0,s}^1\, D_1^{(1,s+1)\#} \in \boldsymbol{R},$$

as computed in (I) we have (4.6).

(ii) If T is an array of the series (6d), (6h) and (6i), then using

$$X_1 = \textstyle\sum_{s=0}^{2} k_{1,s}^0\, D_0^{(1,s)\#} + \sum_{s=0}^{2} k_{0,s}^1\, D_1^{(1,1+s)\#} \in \boldsymbol{R},$$

we obtain (4.6).

(iii) If T is an array of the remaining series, then by use of

$$X_1 = \textstyle\sum_{s=0}^{3} k_{1,s}^0\, D_0^{(1,s)\#} + \sum_{s=0}^{2} k_{0,s}^1\, D_1^{(1,1+s)\#} \in \boldsymbol{R},$$

we can obtain (4.6). The formula (4.7) can be otained from (2.3), (2.5) and (4.6). This completes the proof.

Let $c_a^{(u,v)}$ be an element of $\mathrm{Cov}[\hat{\underline{\Psi}}_{01}]/\sigma^2$ ($= C_{01}$, say) or $\mathrm{Cov}[\hat{\underline{\Psi}}_1]/\sigma^2$ ($= C_1$, say) corresponding to the $\theta_{t_1\ldots t_u}$-th row and $\theta_{t'_1\ldots t'_v}$-th column, which are the a-th associates. Then the following theorem is immediately obtained from (2.2) and (4.4) (or (4.6)).

Theorem 4.2. (I) *If T is an array of Table* 3.1, *then the elements* $c_a^{(u,v)}$ $(0 \leqq a \leqq \min(u,v);\ u, v = 0, 1)$ *of* C_{01} *are given by*

$$c_0^{(0,0)} = k_{0,0}^0,$$

$$c_0^{(0,1)} = c_0^{(1,0)} = k_{0,1}^0/m^{1/2},$$

$$c_0^{(1,1)} = \{k_{1,1}^0 + (m-1)k_{0,0}^1\}/m$$

and

$$c_1^{(1,1)} = (k_{1,1}^0 - k_{0,0}^1)/m,$$

where $k_{r,s}^b$ $(0 \leqq b \leqq r \leqq s \leqq 1)$ *are given in* (I) *of Theorem* 4.1.

(II) *If* T *is an array of Table* 3.2, *then the elements* $c_a^{(1,1)}(a=0,1)$ *of* C_1 *are given by*

$$c_0^{(1,1)} = \{k_{1,1}^0 + (m-1)k_{0,0}^1\}/m$$

and

$$c_1^{(1,1)} = (k_{1,1}^0 - k_{0,0}^1)/m,$$

where $k_{1,1}^0$ *and* $k_{0,0}^1$ *are given in* (II) *of Theorem* 4.1.

We are interested in the estimation of the general mean and the main effects or the main effects only. By Theorems 4.1 and 4.2, PA-optimal 2^m-BFF designs having resolution $R^*(\{0,1\}|P)$ and $R^*(\{1\}|P)$ will be presented for $6 \leqq m \leqq 8$, where $P = \{0,1,2,3\}$. If $N \geqq \nu_3$, then there always exist a 2^m-BFF design of resolution VII. Thus we only consider the case of $N < \nu_3$. First, we shall consider 2^m-BFF designs having resolution $R^*(\{0,1\}|P)$, which satisfy (i) $m=6$, $28 \leqq N \leqq 41$, (ii) $m=7$, $36 \leqq N \leqq 63$ and (iii) $m=8$, $45 \leqq N \leqq 92$ as in Table 3.1. Note that the lower bounds of N for the existence of such designs can be obtained from the series (6f)(or (6g)) for $m=6$, (7c)(or (7d)) for $m=7$, and (8d)(or (8e)) for $m=8$. In Tables 4.1, 4.2 and 4.3, PA-optimal 2^m-BFF designs having resolution $R^*(\{0,1\}|P)$ for $m=6$, 7 and 8 are, respectively, given together with SA(m; $\lambda_0, \lambda_1, \ldots, \lambda_m$), resolution, S_{01} and $c_a^{(u,v)}(0 \leqq a \leqq \min(u,v)$; $u, v = 0, 1)$ for each N. Next we consider 2^m-BFF designs having resolution $R^*(\{1\}|P)$, which satisfy (i) $m=6$, $27 \leqq N \leqq 41$, (ii) $m=7$, $36 \leqq N \leqq 63$ and (iii) $m=8$, $45 \leqq N \leqq 92$ as in Table 3.2. We note that the lower bounds of N for the existence of such designs can be obtained from the series (6h)(or (6i)) for $m=6$, (7c)(or (7d)) for $m=7$, and (8d)(or (8e)) for $m=8$. In Tables 4.4, 4.5 and 4.6, PA-optimal 2^m-BFF designs having resolution $R^*(\{1\}|P)$ for $m=6$, 7 and 8 are, respectively, given together with SA(m; $\lambda_0, \lambda_1, \ldots, \lambda_m$), resolution, S_1 and $c_a^{(1,1)}(a=0,1)$ for each N. Note that for the designs in Tables 4.1 through 4.6, their complementary designs are also optimal and have the same resolution. In Tables 4.4, 4.5 and 4.6, the designs which are not PA-optimal designs having resolution $R^*(\{0,1\}|P)$ will be indicated by the asterisk $*$.

TABLE 4.1. PA-optimal 2^6-BFF designs having resolution $R^*(\{0,1\}|P)$ $(28 \leqq N \leqq 41)$

N	SA(6; $\lambda_0, \lambda_1, \ldots, \lambda_6$)	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
28	SA(6; 1, 1, 0, 0, 1, 1, 0)	$R(\{0,1\}|P)$	0.58333	0.08333	−0.01042
				0.08333	−0.01042
29	SA(6; 2, 1, 0, 0, 1, 1, 0)		0.57552	0.07552	−0.01042
				0.08333	−0.01042
30	SA(6; 2, 1, 0, 0, 1, 1, 1)		0.57224	0.07549	−0.01055
				0.08279	−0.01096
31	SA(6; 3, 1, 0, 0, 1, 1, 1)		0.56963	0.07292	−0.01042
				0.08279	−0.01096
32a	SA(6; 1, 0, 1, 0, 1, 0, 1)	$R(\{0,1,2\}|P)$	0.21875	0.03125	0.00000
				0.03125	0.00000
32b	SA(6; 0, 1, 0, 1, 0, 1, 0)				
33	SA(6; 1, 0, 1, 0, 1, 0, 2)		0.21533	0.03076	−0.00049
				0.03076	−0.00049
34	SA(6; 2, 0, 1, 0, 1, 0, 2)		0.21191	0.03027	0.00000
				0.03027	−0.00098
35	SA(6; 2, 0, 1, 0, 1, 0, 3)		0.21077	0.03011	−0.00016
				0.03011	−0.00114
36	SA(6; 3, 0, 1, 0, 1, 0, 3)		0.20964	0.02995	0.00000
				0.02995	−0.00130
37	SA(6; 3, 0, 1, 0, 1, 0, 4)		0.20907	0.02987	−0.00008
				0.02987	−0.00138
38	SA(6; 0, 1, 0, 1, 0, 2, 0)		0.19824	0.02832	−0.00195
				0.02832	−0.00098
39a	SA(6; 0, 1, 0, 1, 0, 2, 1)		0.19824	0.02832	−0.00195
				0.02832	−0.00098
39b	SA(6; 1, 1, 0, 1, 0, 2, 0)				
40	SA(6; 1, 1, 0, 1, 0, 2, 1)		0.19711	0.02811	−0.00177
				0.02817	−0.00113
41a	SA(6; 1, 1, 0, 1, 0, 2, 2)		0.19693	0.02808	−0.00174
				0.02814	−0.00116
41b	SA(6; 2, 1, 0, 1, 0, 2, 1)				

TABLE 4.2. PA-optimal 2^7-BFF designs having resolution $R^*(\{0,1\}|P)$ $(36 \leqq N \leqq 63)$

N	SA$(7;\lambda_0,\lambda_1,\ldots,\lambda_7)$	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
36	SA(7; 1, 1, 0, 0, 0, 1, 1, 0)	$R(\{0,1\}\|P)$	1.10500	0.13812	−0.01812
				0.13812	−0.01813
37	SA(7; 2, 1, 0, 0, 0, 1, 1, 0)		1.07750	0.11281	−0.01531
				0.13781	−0.01844
38	SA(7; 3, 1, 0, 0, 0, 1, 1, 0)		1.06833	0.10437	−0.01437
				0.13771	−0.01854
39	SA(7; 4, 1, 0, 0, 0, 1, 1, 0)		1.06375	0.10016	−0.01391
				0.13766	−0.01859
40	SA(7; 5, 1, 0, 0, 0, 1, 1, 0)		1.06100	0.09762	−0.01362
				0.13762	−0.01863
41	SA(7; 5, 1, 0, 0, 0, 1, 1, 1)		1.05870	0.09564	−0.01332
				0.13758	−0.01867
42	SA(7; 6, 1, 0, 0, 0, 1, 1, 1)		1.05666	0.09377	−0.01311
				0.13756	−0.01869
43	SA(7; 1, 1, 0, 0, 0, 1, 2, 0)		0.91250	0.13594	−0.01719
				0.11094	−0.01406
44	SA(7; 1, 1, 0, 0, 1, 0, 0, 1)		0.26389	0.03299	−0.00868
				0.03299	0.00868
45	SA(7; 2, 1, 0, 0, 1, 0, 0, 1)		0.24826	0.03103	−0.00673
				0.03103	0.00673
46	SA(7; 2, 1, 0, 0, 1, 0, 0, 2)		0.24132	0.03016	−0.00760
				0.03016	0.00586
47	SA(7; 3, 1, 0, 0, 1, 0, 0, 2)		0.23611	0.02951	−0.00694
				0.02951	0.00521
48	SA(7; 4, 1, 0, 0, 1, 0, 0, 2)		0.23351	0.02919	−0.00662
				0.02919	0.00488
49	SA(7; 4, 1, 0, 0, 1, 0, 0, 3)		0.23119	0.02890	−0.00691
				0.02890	0.00459
50	SA(7; 1, 1, 0, 0, 1, 0, 1, 0)		0.22271	0.03969	−0.00875
				0.02615	0.00184
51	SA(7; 1, 1, 0, 0, 1, 0, 1, 1)		0.21346	0.03270	−0.00725
				0.02582	0.00152
52	SA(7; 2, 2, 0, 0, 1, 0, 0, 1)		0.19965	0.02496	−0.00239
				0.02496	0.00412
53	SA(7; 2, 2, 0, 0, 1, 0, 0, 2)		0.19271	0.02409	−0.00326
				0.02409	0.00326

TABLE 4.2. (continued)

N	SA(7; $\lambda_0, \lambda_1, \ldots, \lambda_7$)	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
54	SA(7; 3, 2, 0, 0, 1, 0, 0, 2)		0.18750	0.02344	−0.00260
				0.02344	0.00260
55	SA(7; 4, 2, 0, 0, 1, 0, 0, 2)		0.18490	0.02311	−0.00228
				0.02311	0.00228
56	SA(7; 4, 2, 0, 0, 1, 0, 0, 3)		0.18258	0.02282	−0.00257
				0.02282	0.00199
57	SA(7; 5, 2, 0, 0, 1, 0, 0, 3)		0.18102	0.02263	−0.00237
				0.02263	0.00179
58	SA(7; 2, 2, 0, 0, 1, 0, 1, 0)		0.17406	0.02654	−0.00354
				0.02107	0.00024
59	SA(7; 3, 2, 0, 0, 1, 0, 1, 0)		0.17146	0.02508	−0.00305
				0.02091	0.00008
60	SA(7; 3, 2, 0, 0, 1, 0, 1, 1)		0.17013	0.02376	−0.00309
				0.02091	0.00008
61	SA(7; 4, 2, 0, 0, 1, 0, 1, 1)		0.16905	0.02329	−0.00289
				0.02082	−0.00001
62	SA(7; 5, 2, 0, 0, 1, 0, 1, 1)		0.16837	0.02299	−0.00276
				0.02077	−0.00006
63	SA(7; 4, 3, 0, 0, 1, 0, 0, 3)		0.16638	0.02080	−0.00112
				0.02080	0.00112

TABLE 4.3. PA-optimal 2^8-BFF designs having resolution $R^*(\{0,1\}|P)$ ($45 \leqq N \leqq 92$)

N	SA(8; $\lambda_0, \lambda_1, \ldots, \lambda_8$)	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
45	SA(8; 1, 1, 0, 0, 0, 0, 1, 1, 0)	$R(\{0,1\}|P)$	2.01000	0.22333	−0.02667
				0.22333	−0.02667
46	SA(8; 2, 1, 0, 0, 0, 0, 1, 1, 0)		1.94750	0.16778	−0.01972
				0.22247	−0.02753
47	SA(8; 3, 1, 0, 0, 0, 0, 1, 1, 0)		1.92667	0.14926	−0.01741
				0.22218	−0.02782
48	SA(8; 4, 1, 0, 0, 0, 0, 1, 1, 0)		1.91625	0.14000	−0.01625
				0.22203	−0.02797
49	SA(8; 5, 1, 0, 0, 0, 0, 1, 1, 0)		1.91000	0.13444	−0.01556
				0.22194	−0.02806

TABLE 4.3. (continued-1)

N	SA(8; $\lambda_0, \lambda_1, \ldots, \lambda_8$)	Resolution	S_{01}	$c_0^{(0,0)}$ / $c_0^{(1,1)}$	$c_0^{(0,1)}$ / $c_1^{(1,1)}$
50	SA(8; 5, 1, 0, 0, 0, 0, 1, 1, 1)		1.90494	0.12940	−0.01567
				0.22194	−0.02806
51	SA(8; 6, 1, 0, 0, 0, 0, 1, 1, 1)		1.90039	0.12531	−0.01518
				0.22189	−0.02811
52	SA(8; 7, 1, 0, 0, 0, 0, 1, 1, 1)		1.89714	0.12239	−0.01484
				0.22184	−0.02816
53	SA(8; 1, 1, 0, 0, 0, 0, 1, 2, 0)		1.51000	0.21639	−0.02580
				0.16170	−0.01799
54	SA(8; 2, 1, 0, 0, 0, 0, 1, 2, 0)		1.44750	0.16083	−0.01885
				0.16083	−0.01885
55	SA(8; 3, 1, 0, 0, 0, 0, 1, 2, 0)		1.42667	0.14231	−0.01654
				0.16054	−0.01914
56	SA(8; 4, 1, 0, 0, 0, 0, 1, 2, 0)		1.41625	0.13306	−0.01538
				0.16040	−0.01929
57	SA(8; 5, 1, 0, 0, 0, 0, 1, 2, 0)		1.41000	0.12750	−0.01469
				0.16031	−0.01938
58	SA(8; 6, 1, 0, 0, 0, 0, 1, 2, 0)		1.40583	0.12380	−0.01422
				0.16025	−0.01943
59	SA(8; 7, 1, 0, 0, 0, 0, 1, 2, 0)		1.40286	0.12115	−0.01389
				0.16021	−0.01947
60	SA(8; 8, 1, 0, 0, 0, 0, 1, 2, 0)		1.40063	0.11917	−0.01365
				0.16018	−0.01951
61	SA(8; 1, 1, 0, 0, 0, 0, 1, 3, 0)		1.34333	0.21407	−0.02551
				0.14116	−0.01509
62	SA(8; 2, 1, 0, 0, 0, 0, 1, 3, 0)		1.28083	0.15852	−0.01856
				0.14029	−0.01596
63	SA(8; 3, 1, 0, 0, 0, 0, 1, 3, 0)		1.26000	0.14000	−0.01625
				0.14000	−0.01625
64	SA(8; 4, 1, 0, 0, 0, 0, 1, 3, 0)		1.24958	0.13074	−0.01509
				0.13986	−0.01639
65	SA(8; 1, 0, 1, 0, 0, 0, 1, 1, 0)	$R(\{0, 1, 2\} \mid P)$	0.89702	0.04253	0.00045
				0.10681	−0.01428
66	SA(8; 1, 0, 1, 0, 0, 0, 1, 1, 1)		0.89087	0.03638	0.00047
				0.10681	−0.01428
67	SA(8; 2, 0, 1, 0, 0, 0, 1, 1, 1)		0.88740	0.03323	0.00012
				0.10677	−0.01432
68	SA(8; 2, 0, 1, 0, 0, 0, 1, 1, 2)		0.88529	0.03112	0.00011
				0.10677	−0.01432

TABLE 4.3. (continued-2)

N	SA(8; $\lambda_0, \lambda_1, \ldots, \lambda_8$)	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
69	SA(8; 3, 0, 1, 0, 0, 0, 1, 1, 2)		0.88406	0.03000	−0.00001
				0.10676	−0.01434
70	SA(8; 3, 0, 1, 0, 0, 0, 1, 1, 3)		0.88300	0.02893	−0.00001
				0.10676	−0.01434
71	SA(8; 4, 0, 1, 0, 0, 0, 1, 1, 3)		0.88236	0.02835	−0.00008
				0.10675	−0.01434
72	SA(8; 0, 1, 1, 0, 0, 0, 1, 1, 0)		0.59750	0.04500	0.00000
				0.06906	−0.00906
73	SA(8; 1, 1, 0, 0, 0, 1, 0, 1, 0)	$R(\{0,1\} \mid P)$	0.32000	0.08000	−0.01922
				0.03000	0.00266
74	SA(8; 2, 1, 0, 0, 0, 1, 0, 1, 0)		0.29000	0.06000	−0.01422
				0.02875	0.00141
75	SA(8; 3, 1, 0, 0, 0, 1, 0, 1, 0)		0.28000	0.05333	−0.01255
				0.02833	0.00099
76	SA(8; 3, 1, 0, 0, 0, 1, 0, 1, 1)		0.27493	0.04927	−0.01184
				0.02821	0.00086
77	SA(8; 4, 1, 0, 0, 0, 1, 0, 1, 1)		0.27107	0.04679	−0.01118
				0.02804	0.00069
78	SA(8; 5, 1, 0, 0, 0, 1, 0, 1, 1)		0.26869	0.04525	−0.01078
				0.02793	0.00059
79	SA(8; 6, 1, 0, 0, 0, 1, 0, 1, 1)		0.26708	0.04421	−0.01051
				0.02786	0.00051
80	SA(8; 6, 1, 0, 0, 0, 1, 0, 1, 2)		0.26588	0.04320	−0.01035
				0.02783	0.00049
81	SA(8; 1, 2, 0, 0, 0, 1, 0, 1, 0)		0.26141	0.06438	−0.01434
				0.02463	0.00168
82	SA(8; 2, 2, 0, 0, 0, 1, 0, 1, 0)		0.23141	0.04438	−0.00934
				0.02338	0.00043
83	SA(8; 3, 2, 0, 0, 0, 1, 0, 1, 0)		0.22141	0.03771	−0.00767
				0.02296	0.00001
84	SA(8; 4, 2, 0, 0, 0, 1, 0, 1, 0)		0.21641	0.03438	−0.00684
				0.02275	−0.00020
85	SA(8; 5, 2, 0, 0, 0, 1, 0, 1, 0)		0.21341	0.03238	−0.00634
				0.02263	−0.00032
86	SA(8; 6, 2, 0, 0, 0, 1, 0, 1, 0)		0.21141	0.03104	−0.00600
				0.02255	−0.00040
87	SA(8; 1, 1, 0, 0, 1, 0, 0, 1, 0)	$R(\{0,1,2\} \mid P)$	0.16574	0.01173	−0.00100
				0.01925	0.00710

TABLE 4.3. (continued-3)

N	SA(8; $\lambda_0, \lambda_1, \ldots, \lambda_8$)	Resolution	S_{01}	$c_0^{(0,0)}$ $c_0^{(1,1)}$	$c_0^{(0,1)}$ $c_1^{(1,1)}$
88	SA(8; 1, 1, 0, 0, 1, 0, 0, 1, 1)		0.14331	0.01136	0.00000
				0.01649	0.00434
89	SA(8; 2, 1, 0, 0, 1, 0, 0, 1, 1)		0.13848	0.01129	−0.00022
				0.01590	0.00375
90	SA(8; 2, 1, 0, 0, 1, 0, 0, 1, 2)		0.13530	0.01116	0.00000
				0.01552	0.00336
91	SA(8; 3, 1, 0, 0, 1, 0, 0, 1, 2)		0.13381	0.01111	−0.00010
				0.01534	0.00318
92	SA(8; 3, 1, 0, 0, 1, 0, 0, 1, 3)		0.13257	0.01104	0.00000
				0.01519	0.00304

TABLE 4.4. PA-optimal 2^6-BFF designs having resolution $R^*(\{1\}|P)$ $(27 \leqq N \leqq 41)$

N	SA(6; $\lambda_0, \lambda_1, \ldots, \lambda_6$)	Resolution	S_1	$c_0^{(1,1)}$	$c_1^{(1,1)}$
*27	SA(6; 0, 1, 0, 0, 1, 1, 0)	$R(\{1\}|P)$	0.49680	0.08280	−0.01095
*28a	SA(6; 0, 1, 0, 0, 1, 1, 1)	$R(\{0, 1\}|P)$	0.50000	0.08333	−0.01042
28b	SA(6; 0, 1, 1, 0, 0, 1, 1)				
*29	SA(6; 1, 1, 0, 0, 1, 1, 1)		0.49688	0.08281	−0.01094
*30	SA(6; 1, 1, 0, 0, 1, 1, 2)		0.49632	0.08272	−0.01103
*31	SA(6; 1, 1, 0, 0, 1, 1, 3)		0.49609	0.08268	−0.01107
32a	SA(6; 1, 0, 1, 0, 1, 0, 1)	$R(\{0, 1, 2\}|P)$	0.18750	0.03125	0.00000
32b	SA(6; 0, 1, 0, 1, 0, 1, 0)				
33	SA(6; 1, 0, 1, 0, 1, 0, 2)		0.18457	0.03076	−0.00049
34	SA(6; 2, 0, 1, 0, 1, 0, 2)		0.18164	0.03027	−0.00098
35	SA(6; 2, 0, 1, 0, 1, 0, 3)		0.18066	0.03011	−0.00114
36	SA(6; 3, 0, 1, 0, 1, 0, 3)		0.17969	0.02995	−0.00130
37	SA(6; 3, 0, 1, 0, 1, 0, 4)		0.17920	0.02987	−0.00138
38	SA(6; 0, 1, 0, 1, 0, 2, 0)		0.16992	0.02832	−0.00098
39a	SA(6; 0, 1, 0, 1, 0, 2, 1)		0.16992	0.02832	−0.00098
39b	SA(6; 0, 2, 0, 1, 0, 1, 1)				
40	SA(6; 1, 1, 0, 1, 0, 2, 1)		0.16900	0.02817	−0.00113
41a	SA(6; 1, 1, 0, 1, 0, 2, 2)		0.16885	0.02814	−0.00116
41b	SA(6; 2, 1, 0, 1, 0, 2, 1)				

TABLE 4.5. PA-optimal 2^7-BFF designs having resolution $R^*(\{1\}|P)$ $(36 \leqq N \leqq 63)$

N	SA(7; $\lambda_0, \lambda_1, \ldots, \lambda_7$)	Resolution	S_1	$c_0^{(1,1)}$	$c_1^{(1,1)}$
36	SA(7;1,1,0,0,0,1,1,0)	$R(\{0,1\}\|P)$	0.96687	0.13812	−0.01813
37	SA(7;2,1,0,0,0,1,1,0)		0.96469	0.13781	−0.01844
38	SA(7;3,1,0,0,0,1,1,0)		0.96396	0.13771	−0.01854
39	SA(7;4,1,0,0,0,1,1,0)		0.96359	0.13766	−0.01859
*40	SA(7;4,1,0,0,0,1,1,1)		0.96331	0.13762	−0.01863
41	SA(7;5,1,0,0,0,1,1,1)		0.96306	0.13758	−0.01867
42	SA(7;6,1,0,0,0,1,1,1)		0.96289	0.13756	−0.01869
43	SA(7;1,1,0,0,0,1,2,0)		0.77656	0.11094	−0.01406
44	SA(7;1,1,0,0,1,0,0,1)		0.23090	0.03299	0.00868
45	SA(7;2,1,0,0,1,0,0,1)		0.21723	0.03103	0.00673
46	SA(7;2,1,0,0,1,0,0,2)		0.21115	0.03016	0.00586
47	SA(7;3,1,0,0,1,0,0,2)		0.20660	0.02951	0.00521
48	SA(7;4,1,0,0,1,0,0,2)		0.20432	0.02919	0.00488
49	SA(7;4,1,0,0,1,0,0,3)		0.20229	0.02890	0.00459
50	SA(7;1,1,0,0,1,0,1,0)		0.18302	0.02615	0.00184
*51	SA(7;2,1,0,0,1,0,1,0)		0.17960	0.02566	0.00135
52	SA(7;2,2,0,0,1,0,0,1)		0.17470	0.02496	0.00412
53	SA(7;2,2,0,0,1,0,0,2)		0.16862	0.02409	0.00326
54	SA(7;3,2,0,0,1,0,0,2)		0.16406	0.02344	0.00260
55	SA(7;4,2,0,0,1,0,0,2)		0.16178	0.02311	0.00228
56	SA(7;4,2,0,0,1,0,0,3)		0.15976	0.02282	0.00199
*57	SA(7;1,2,0,0,1,0,1,0)		0.15094	0.02156	0.00073
58	SA(7;2,2,0,0,1,0,1,0)		0.14752	0.02107	0.00024
59	SA(7;3,2,0,0,1,0,1,0)		0.14638	0.02091	0.00008
*60	SA(7;4,2,0,0,1,0,1,0)		0.14581	0.02083	0.00000
*61	SA(7;5,2,0,0,1,0,1,0)		0.14547	0.02078	−0.00005
*62	SA(7;6,2,0,0,1,0,1,0)		0.14524	0.02075	−0.00008
*63	SA(7;7,2,0,0,1,0,1,0)		0.14508	0.02073	−0.00011

TABLE 4.6. PA-optimal 2^8-BFF designs having resolution $R^*(\{1\}|P)$ $(45 \leqq N \leqq 92)$

N	SA$(8;\ \lambda_0, \lambda_1, \ldots, \lambda_8)$	Resolution	S_1	$c_0^{(1,1)}$	$c_1^{(1,1)}$
45	SA(8; 1, 1, 0, 0, 0, 0, 1, 1, 0)	$R(\{0,1\}|P)$	1.78667	0.22333	−0.02667
46	SA(8; 2, 1, 0, 0, 0, 0, 1, 1, 0)		1.77972	0.22247	−0.02753
47	SA(8; 3, 1, 0, 0, 0, 0, 1, 1, 0)		1.77741	0.22218	−0.02782
48	SA(8; 4, 1, 0, 0, 0, 0, 1, 1, 0)		1.77625	0.22203	−0.02797
49	SA(8; 5, 1, 0, 0, 0, 0, 1, 1, 0)		1.77556	0.22194	−0.02806
*50	SA(8; 6, 1, 0, 0, 0, 0, 1, 1, 0)		1.77509	0.22189	−0.02811
*51	SA(8; 7, 1, 0, 0, 0, 0, 1, 1, 0)		1.77476	0.22185	−0.02815
*52	SA(8; 8, 1, 0, 0, 0, 0, 1, 1, 0)		1.77451	0.22181	−0.02819
53	SA(8; 1, 1, 0, 0, 0, 0, 1, 2, 0)		1.29361	0.16170	−0.01799
54	SA(8; 2, 1, 0, 0, 0, 0, 1, 2, 0)		1.28667	0.16083	−0.01885
55	SA(8; 3, 1, 0, 0, 0, 0, 1, 2, 0)		1.28435	0.16054	−0.01914
56	SA(8; 4, 1, 0, 0, 0, 0, 1, 2, 0)		1.28319	0.16040	−0.01929
57	SA(8; 5, 1, 0, 0, 0, 0, 1, 2, 0)		1.28250	0.16031	−0.01938
58	SA(8; 6, 1, 0, 0, 0, 0, 1, 2, 0)		1.28204	0.16025	−0.01943
*59	SA(8; 6, 1, 0, 0, 0, 0, 1, 2, 1)		1.28166	0.16021	−0.01948
*60	SA(8; 7, 1, 0, 0, 0, 0, 1, 2, 1)		1.28136	0.16017	−0.01952
61	SA(8; 1, 1, 0, 0, 0, 0, 1, 3, 0)		1.12926	0.14116	−0.01509
62	SA(8; 2, 1, 0, 0, 0, 0, 1, 3, 0)		1.12231	0.14029	−0.01596
63	SA(8; 3, 1, 0, 0, 0, 0, 1, 3, 0)		1.12000	0.14000	−0.01625
64	SA(8; 4, 1, 0, 0, 0, 0, 1, 3, 0)		1.11884	0.13986	−0.01639
65	SA(8; 1, 0, 1, 0, 0, 0, 1, 1, 0)	$R(\{0,1,2\}|P)$	0.85449	0.10681	−0.01428
*66	SA(8; 2, 0, 1, 0, 0, 0, 1, 1, 0)		0.85417	0.10677	−0.01432
*67	SA(8; 3, 0, 1, 0, 0, 0, 1, 1, 0)		0.85406	0.10676	−0.01434
*68	SA(8; 4, 0, 1, 0, 0, 0, 1, 1, 0)		0.85401	0.10675	−0.01434
*69	SA(8; 5, 0, 1, 0, 0, 0, 1, 1, 0)		0.85398	0.10675	−0.01435
*70	SA(8; 6, 0, 1, 0, 0, 0, 1, 1, 0)		0.85396	0.10674	−0.01435
*71	SA(8; 7, 0, 1, 0, 0, 0, 1, 1, 0)		0.85394	0.10674	−0.01435
72	SA(8; 0, 1, 1, 0, 0, 0, 1, 1, 0)		0.55250	0.06906	−0.00906
73	SA(8; 1, 1, 0, 0, 0, 1, 0, 1, 0)	$R(\{0,1\}|P)$	0.24000	0.03000	0.00266
74	SA(8; 2, 1, 0, 0, 0, 1, 0, 1, 0)		0.23000	0.02875	0.00141
75	SA(8; 3, 1, 0, 0, 0, 1, 0, 1, 0)		0.22667	0.02833	0.00099
*76	SA(8; 4, 1, 0, 0, 0, 1, 0, 1, 0)		0.22500	0.02813	0.00078
*77	SA(8; 5, 1, 0, 0, 0, 1, 0, 1, 0)		0.22400	0.02800	0.00066
*78	SA(8; 6, 1, 0, 0, 0, 1, 0, 1, 0)		0.22333	0.02792	0.00057
*79	SA(8; 7, 1, 0, 0, 0, 1, 0, 1, 0)		0.22286	0.02786	0.00051
*80	SA(8; 7, 1, 0, 0, 0, 1, 0, 1, 1)		0.22245	0.02781	0.00046
81	SA(8; 1, 2, 0, 0, 0, 1, 0, 1, 0)		0.19703	0.02463	0.00168
82	SA(8; 2, 2, 0, 0, 0, 1, 0, 1, 0)		0.18703	0.02338	0.00043

TABLE 4.6. (continued)

N	SA(8; $\lambda_0, \lambda_1, \ldots, \lambda_8$)	Resolution	S_1	$c_0^{(1,1)}$	$c_1^{(1,1)}$
83	SA(8; 3, 2, 0, 0, 0, 1, 0, 1, 0)		0.18370	0.02296	0.00001
84	SA(8; 4, 2, 0, 0, 0, 1, 0, 1, 0)		0.18203	0.02275	−0.00020
85	SA(8; 5, 2, 0, 0, 0, 1, 0, 1, 0)		0.18103	0.02263	−0.00032
86	SA(8; 6, 2, 0, 0, 0, 1, 0, 1, 0)		0.18036	0.02255	−0.00040
87	SA(8; 1, 1, 0, 0, 1, 0, 0, 1, 0)	$R(\{0, 1, 2\} \mid P)$	0.15401	0.01925	0.00710
88	SA(8; 1, 1, 0, 0, 1, 0, 0, 1, 1)		0.13194	0.01649	0.00434
89	SA(8; 1, 1, 0, 0, 1, 0, 0, 1, 2)		0.12720	0.01590	0.00375
90	SA(8; 2, 1, 0, 0, 1, 0, 0, 1, 2)		0.12413	0.01552	0.00336
91	SA(8; 2, 1, 0, 0, 1, 0, 0, 1, 3)		0.12270	0.01534	0.00318
92	SA(8; 3, 1, 0, 0, 1, 0, 0, 1, 3)		0.12153	0.01519	0.00304

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