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Periodic and almost periodic stability of solutions to degenerate parabolic equations

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Introduction

This paper is concerned with periodic and almost periodic behavior of solutions to the following problem:

$$
(0.1)\qquad \begin{cases} u' - \Delta v = f,\ v \in \beta(u), & \text{in } (0, \infty) \times \Omega, \\ v = g_0 \quad \text{on } (0, \infty) \times \Gamma_0, & \\ \partial_\nu v + p \cdot v = g_1 & \text{on } (0, \infty) \times (\Gamma \backslash \Gamma_0), \\ u(0, \cdot) = u_0 & \text{in } \Omega. \end{cases}
$$

Here $u' = (\partial/\partial t)u$, Ω is a bounded domain in $\mathbf{R}^N$ ($N \geq 1$) with smooth boundary Γ, Γ_0 is a measurable subset of Γ with positive surface measure, p is a non-negative bounded measurable function on Γ, ∂_ν denotes the outward normal derivative on Γ, and β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$. Damlamian and Kenmochi have studied in [8, 9] the global behavior of solutions to (0.1) in the case in which β is Lipschitz continuous. The Lipschitz continuous case is effective for Stefan problems in weak (enthalpy) formulation, but it is in general required to assume that β is multi-valued. In fact, we do have this situation for instance in the weak formulations of free boundary problems arising from Hele-Shaw flows as well as electrochemical machining processes, see [5, 18, 19, 20]. As observed by Damlamian [6, 7], problem (0.1) is formulated as an evolution equation by means of time-dependent subdifferentials in an appropriate Hilbert space. In Kenmochi-Ôtani [14, 15], the periodic and almost periodic stability of solutions to a general class of evolution equations with time-dependent subdifferentials have been studied. However, it does not seem that their result is directly applicable to the problem (0.1) if both β and β^{-1} are multi-valued. In this paper, we extend a part of the result given in [9] to a class of maximal monotone graphs β so that the inverse of the Heaviside function may be contained. This is necessarry to treat the problems for Hele-Shaw flows and electrochemical machining processes, since in these cases β is the inverse of the Heaviside function. The main results of this paper were already announced in [17], and this paper contains their complete proofs.

We shall first establish existence theorems of periodic (resp. almost periodic) solutions of problem (0.1) and then discuss their asymptotic stability, provided that f and g_i $(i = 0, 1)$ are periodic (resp. almost periodic) functions. Our method employed here is similar to that developed in [9], but a new version is presented in Lemma 4, Section 3. We note that our class of functions includes the function $\beta(r) = |r|^{m-1}r$ $(m \geq 1)$ which appears in the porous media equations and functions such that β^{-1} is non-decreasing, bounded and Lipschitz continuous; this case occurs in equations of parabolic-elliptic type. As for the periodic stability of solutions, DiBenedetto-Friedman [10] have investigated evolutionary dam problem, and Kenmochi and Kubo [13] have treated a model problem of partially saturated flows in porous media.

In this paper, given a Banach space Y, we denote by $|\cdot|_Y$ the norm of Y. We use various terminologies related to proper lower-semicontinuous (l.s.c.) convex functions and their subdifferentials; for them we refer to Brézis [3]. Moreover, we use the following notations:

$$H \equiv L^2(\Omega) \text{ with inner product } (\cdot,\cdot), \; X \equiv H^1(\Omega)$$

and

$$V \equiv \{z \in X;\; z = 0 \text{ a.e. on } \Gamma_0\}.$$

By assumption, Γ_0 has positive surface measure and the function p is non-negative and bounded, and so V becomes a Hilbert space with respect to the inner product

$$(0.2) \qquad (z, y)_V \equiv \int_\Omega \nabla z \cdot \nabla y dx + \int_\Gamma p(x)z(x)y(x)d\Gamma(x) \qquad \text{for } z,\; y \in V.$$

We write V^* for the dual space of V and F for the duality mapping of V:

$$(0.3) \qquad \langle Fz,\; y\rangle = (z,\; y)_V \qquad \text{for } z,\; y \in V,$$

where $\langle\cdot,\cdot\rangle$ stands for the duality paring between V^* and V. V^* becomes a Hilbert space with respect to the inner product $(z,\; y)_* \equiv \langle z,\; F^{-1}y\rangle$ and the norm is defined by $|z|_* \equiv (z,\; z)_*^{1/2}$. It should be noted that $V \subset H \subset V^*$ and the injections are dense and compact.

1. Theorems

We study the problem (0. 1) for functions β belonging to the class defined below.

DEFINITION 1. Let β be a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$ and $\hat{\beta}$: $\mathbf{R} \to \mathbf{R} \cup \{\infty\}$ be a proper l.s.c. convex function with $\beta = \partial\hat{\beta}$. Given constants $\alpha > 0$ and $b \geq 0$, we say that β belongs to the class $B(a, b)$ if $\hat{\beta}$ satisffies

$$\hat{\beta}(r) \geq a|r|^2 - b \qquad \text{for all } r \in \mathbf{R}.$$

For example, if β is the inverse of the Heaviside function, then $\hat{\beta}(r) = 0$ for $0 \leq r \leq 1$, $= \infty$ otherwise, and $\beta \in B(1, 1)$

LEMMA 1 (cf. Damlamian [6, 7]). *Let $\beta \in B(a, b)$, g: $\mathbf{R} \to H$, $t \in \mathbf{R}$, and define the function φ^t: $V^* \to \mathbf{R} \cup \{\infty\}$ by*

$$
(1.1) \qquad \varphi^t(z) = \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z(x))dx - (g(t), z) & \text{for } z \in H, \\ \infty & \text{for } z \in V^* \setminus H. \end{cases}
$$

Then φ^t is proper, l.s.c. and convex on V^*, and the effective domain of φ^t is characterized as follows:

$$
(1.2) \qquad D(\varphi^t) = D_\beta \equiv \{z \in H;\ \hat{\beta}(z) \in L^1(\Omega)\}
$$

Moreover for u and u^* in V^*, $u^* \in \partial\varphi^t(u)$ if and only if the following (i) *and* (ii) *hold*:

(i) $u \in D(\varphi^t)$ $(= D_\beta)$.

(ii) *There is a function v in X such that $u - g \in V$, $u^* = F(v - g)$ and $v(x) \in \beta(u(x))$ for a.e. $x \in \Omega$.*

We now reformulate problem (0.1) as an evolution equation of the form

$$
(1.3) \qquad u'(t) + \partial\varphi^t(u(t)) \ni f(t) \quad \text{for a.e. } t \in \mathbf{R}_+ \ (= [0, \infty))
$$

with the initial condition

$$
u(0) = u_0.
$$

In fact, system (0.1) excluding the initial condition is written in the following variational form (see [6, 7, 8, 9] for the details):

$$
(1.4)_a \qquad v(t, x) \in \beta(u(t, x)) \quad \text{for a.e. } (t, x) \in \mathbf{R}_+ \times \Omega,
$$

$$
(1.4)_b \qquad v(t) - g(t) \in V \quad \text{for a.e. } t \in \mathbf{R}_+,
$$

$$
(1.4)_c \qquad \langle u'(t), z\rangle + \int_\Omega \nabla(v(t, x) - g(t, x)) \cdot \nabla z(x)dx
$$
$$
+ \int_\Gamma p(x)(v(t, x) - g(t, x))z(x)d\Gamma(x) = (f(t), z)
$$
$$
\text{for all } z \in V \text{ and a.e. } t \in \mathbf{R}_+.
$$

Here $g(t)$ is determined by $g_0(t)$ and $g_1(t)$ as the solution of the elliptic problem

$$
\begin{cases} -\Delta g(t, \cdot) = 0 & \text{in } \Omega, \\ g(t, \cdot) = g_0(t, \cdot) & \text{on } \Gamma_0, \\ \partial_\nu g(t, \cdot) + p \cdot g(t, \cdot) = g_1(t, \cdot) & \text{on } \Gamma \setminus \Gamma_0 \end{cases}
$$

for each t. With the mapping F(cf. (0.2), (0.3)), variational form (1.4) is expressed as

$$u'(t) + F(v(t) - g(t)) = f(t) \quad \text{and} \quad v \in \beta(u) \text{ for a.e. } t \in \mathbf{R}_+. \tag{1.5}$$

In view of Lemma 1-(ii), we see that (1.3) is equivalent to (1.5) and therefore to (1.4). In the sequel, we denote equation (1.3) (or (1.5)) by $E(\beta, g, f)$ and treat the problem (0.1) in this form

DEFINITION 2. Let $\beta \in B(a, b)$, $g \in W^{1,1}_{loc}(\mathbf{R}; H)$ and $f \in L^2_{loc}(\mathbf{R}; V^*)$. A function u: $J = [t_0, t_1] \to V_*$ is called *a solution to $E(\beta, g, f)$ on J*, if u satisfies the following three conditions (a), (b) and (c):

(a) $u \in C(J; V^*) \cap W^{1,1}_{loc}((t_0, t_1]; V^*)$.

(b) The function $t \mapsto \varphi^t(u(t))$ belongs to $L^1(J)$, where φ^t is given by (1.1).

(c) $u'(t) + \partial\varphi^t(u(t)) \ni f(t)$ for a.e. $t \in J$.

Also for a general interval J in $\mathbf{R}$, u: $J \to V^*$ is said to be *a solution to $E(\beta, g, f)$ on J*, if u is a solution to $E(\beta, g, f)$ on each compact subinterval of J in the above sense.

We note that if u is a solution to $E(\beta, g, f)$ on an interval J (in the above sense), then by (1.2) $u(t)$ belongs to the closure of D_β with respect to the topology of V^* for all $t \in J$, since $u \in C(J; V^*)$ and $u(t) \in D(\varphi^t)$ for a.e. $t \in J$. As to the solvability of the Cauchy problem for $E(\beta, g, f)$, we have the following result which is obtained by applying the abstract theories developed in [6,7,11] and their slight modifications

LEMMA 2. *Let β, g and f be as in Definition 2. Let u_0 belong to the closure of D_β with respect to the topology of V^* and let $t_0 \in \mathbf{R}$. Then there exists a unique solution u to $E(\beta, g, f)$ on $[t_0, \infty)$ with $u(t_0) = u_0$ such that $t \mapsto (t - t_0)^{1/2} u' \in L^2_{loc}([t_0, \infty); V^*)$, $t \mapsto (t - t_0)^{1/2} u \in L^\infty_{loc}([t_0, \infty); H)$ and $t \mapsto \varphi^t(u(t)) \in W^{1,1}_{loc}((t_0, \infty))$. If in particular $u_0 \in D_\beta$, then $u' \in L^2_{loc}([t_0, \infty); V^*)$, $u \in L^\infty_{loc}([t_0, \infty); H)$ and $t \mapsto \varphi^t(u(t)) \in W^{1,1}_{loc}([t_0, \infty))$.*

In what follows, we shall discuss solutions in the sence of Definition 2 and Lemma 2. Our first result is concerned with the periodic stability of solutions.

THEOREM 1. *Let $\beta \in B(a, b)$, $g \in W^{1,1}_{loc}(\mathbf{R}; H)$ and $f \in L^2_{loc}(\mathbf{R}; V^*)$. Assume that there exists a constant $T > 0$ such that*

$$g(t + T) = g(t) \quad \textit{and} \quad f(t + T) = f(t) \qquad \textit{for a.e. } t \in \mathbf{R}.$$

Then we have the assertions (i) *through* (iii) *below.*

(i) *There exists a T-periodic solution to $E(\beta, g, f)$ on $\mathbf{R}$. In other words, the set $P_T \equiv \{u: \mathbf{R} \to V^*; u$ is a solution to $E(\beta, g, f)$ on $\mathbf{R}$ satisfying $u(t + T) = u(t)$ for all $t \in \mathbf{R}\}$ is not empty.*

(ii) *For each solution u to $E(\beta, g, f)$ on $[t_0, \infty)$, $t_0 \in \mathbf{R}$, there exists $\omega \in P_T$ such that*

$$u(t) - \omega(t) \longrightarrow 0 \quad in\ V^* \quad and \quad weakly\ in\ H\ as\ t \longrightarrow \infty.$$

(iii) *Let* $\omega_1, \omega_2 \in P_T$, *and let* $\eta_i \in \beta(\omega_i)$ *be such that* $\omega_i' + F(\eta_i - g) = f$ *a.e. on* $\mathbf{R}$, $i = 1, 2$. *Then* $\eta_1(t) = \eta_2(t)$ *for a.e.* $t \in \mathbf{R}$ *and there is an element* α *in* H *such that* $\omega_1(t) = \omega_2(t) + \alpha$ *for all* $t \in \mathbf{R}$.

We shall prove the above theorem as a corollary to the second theorem which is concerned with the almost periodic stability.

DEFINITION 3 (cf. [1]). Let E be a Banach space. A continuous function $f: \mathbf{R} \to E$ is said to be *E-almost periodic* if for any sequence $\{t_n\}$ in $\mathbf{R}$ there is a subseqence $\{t_{n_k}\}$ of $\{t_n\}$ such that $f(t + t_{n_k})$ converges in E uniformly in $t \in \mathbf{R}$.

Given a function $f \in L^2_{loc}(\mathbf{R}; E)$, we define $f: \to L^2(0,1;E)$ by $[f(t)](s) = f(t+s)$ for $t \in \mathbf{R}$ and a.e. $s \in (0, 1)$. We say that a function $f \in L^2_{loc}(\mathbf{R}; E)$ is *E-almost periodic in the sense of Stepanov* if the mapping $\hat{f}: \mathbf{R} \to L^2(0,1; E)$ *is* $L^2(0,1; E)$-almost periodic.

THEOREM 2. *Let* $\beta \in B(a,b)$. *Assume that* $g \in W^{1,1}_{loc}(\mathbf{R}; H)$ *is H-almost periodic and that* $f \in L^2_{loc}(\mathbf{R}; V^*)$ *is* V^**-almost periodic in the sense of Stepanov. Suppose that the following two conditions* (a) *and* (b) *hold*:

(a) $\sup_{t\in\mathbf{R}} |g'|_{L^1(t,t+1:H)} < \infty$.

(b) *If* $\{t_n\}$ *is a sequence in* $\mathbf{R}$ *such that* $g(t + t_n) \to \hat{g}(t)$ *in* H *uniformly on* $\mathbf{R}$, *then* $\hat{g} \in W^{1,1}_{loc}(\mathbf{R}; H)$ *and* $\sup_{t\in\mathbf{R}} |\hat{g}'|_{L^1(t,t+1:H)} < \infty$.

Then we have the assertions (i) *through* (iii) *below*:

(i) *There exists a* V^**-almost periodic solution to* $E(\beta, g, f)$ *on* $\mathbf{R}$. *In other words, the set* $AP \equiv \{u: \mathbf{R} \to V^*;\ u$ *is a solution to* $E(\beta, g, f)$ *on* $\mathbf{R}$ *and* V^**-almost periodic*$\}$ *is not empty.*

(ii) *For each solution* u *to* $E(\beta, g, f)$ *on* $[t_0, \infty)$, $t_0 \in \mathbf{R}$, there exists $\omega \in AP$ *such that*

$$u(t) - \omega(t) \longrightarrow 0 \quad in\ V^*\ and\ weakly\ in\ H\ as\ t \longrightarrow \infty$$

(iii) *Let* $\omega_1, \omega_2 \in AP$, *and let* $\eta_i \in \beta(\omega_i)$ *be such that* $\omega_i' + F(\eta_i - g) = f$ *a.e. on* $\mathbf{R}$, $i = 1, 2$. *Then* $\eta_1(t) = \eta_2(t)$ *for a.e.* $t \in \mathbf{R}$ *and there is an element* α *in* H *such that* $\omega_1(t) = \omega_2(t) + \alpha$ *for all* $t \in \mathbf{R}$.

REMARK 1. (a) Assumption (b) of Theorem 2 is valid if, for instance, the following holds:

$$\sup_{t\in\mathbf{R}} |g'|_{L^p(t,t+1:H)} < \infty \qquad \text{for some } 1 < p \leq \infty.$$

(b) In general AP as well as P_T is not necessarily a singleton set.

The third theorem of this paper is concerned with the characterization of periodicity and almost periodicity of solutions by means of global boundedness.

THEOREM 3. *Under the same assumptions as those in Theorem* 1 (*resp. Theorem* 2), *a solution* u *to* $E(\beta, g, f)$ *on* $\mathbf{R}$ *is T-periodic* (*resp.* V^**-almost periodic*) *if and only if u is V^*-bounded on* $\mathbf{R}$ *in the sense that* $\sup_{t\in\mathbf{R}}|u(t)|_* < \infty$.

In the next section, we recall some known results. Some key lemmas are prepared in Section 3. And Theorems 1, 2 and 3 are proved in Section 4.

2. Continuous dependence of solutions on boundary, forcing and initial data

In this section, we collect some results concerning key estimates for solutions to $E(\beta, g, f)$ and the continuous dependence of solutions on the data g, f and u_0 given in [8] and [9]. Although the following results are stated in [8] and [9] under the assumption that β is Lipschitz continuous, the proof for the case in which $\beta \in B(a, b)$ is almost the same as in those of [8, 9]. The same results can also be obtained from the abstract results due to Kenmochi [11] and Kenmochi and Kubo [12].

PROPOSITION A. *Let* $\beta \in B(a,b)$ *and let* $g \in W^{1,1}_{loc}(\mathbf{R}; H) \cap L^\infty(\mathbf{R}; H)$ *and* $f \in L^2_{loc}(\mathbf{R}; V^*)$. *Assume that*

$$S^{1,1}(g) \equiv |g|_{L^\infty(\mathbf{R};H)} + \sup_{t\in\mathbf{R}}|g'|_{L^1(t,t+1;H)} < \infty$$

and

$$S^{0,2}_*(f) \equiv \sup_{t\in\mathbf{R}}|f|_{L^2(t,t+1;V^*)} < \infty.$$

Let u be a solution to $E(\beta, g, f)$ *on* $[t_0, \infty)$. *Then the following holds*:

$$\begin{aligned}
&\sup_{t\ge t_0}|u(t)|_* + \sup_{t_0<t\le t_0+1}(t-t_0)|u(t)|_H + \sup_{t\ge t_0+1}|u(t)|_H \\
&\quad + \sup_{t\ge t_0}|u|_{L^2(t,t+1:H)} + |(\cdot - t_0)^{1/2}\ u'|_{L^2(t_0,t_0+1;V^*)} \\
&\quad + \sup_{t\ge t_0+1}|u'|_{L^2(t,t+1;V^*)} + \sup_{t_0<t\le t_0+1}(t-t_0)|\varphi^t(u(t))| \\
&\quad + \sup_{t\ge t_0+1}|\varphi^t(u(t))| + \sup_{t\ge t_0}|\varphi^{(\cdot)}(u(\cdot))|_{L^1(t,t+1)} \\
&\le M_1 = M_1(a,\ b,\ S^{1,1}(g),\ S^{0,2}_*(f),\ |u(t_0)|_*) < \infty,
\end{aligned}$$

where M_1: $(\mathbf{R}_+\backslash\{0\}) \times \mathbf{R}^4_+ \to \mathbf{R}_+$ $(\mathbf{R}_+ = [0, \infty))$ *is an appropriate locally bounded function. If in particular* $u(t_0) \in D_\beta$, *then we have*

$$\begin{aligned}
&\sup_{t\ge t_0}|u(t)|_H + \sup_{t\ge t_0}|u'|_{L^2(t,t+1:V^*)} + \sup_{t\ge t_0}|\varphi^t(u(t))| \\
&\quad \le M_2 = M_2(a,\ b,\ S^{1,1}(g),\ S^{0,2}_*(f),\ |u(t_0)|_*,\ |\varphi^{t_0}(u(t_0))|) < \infty,
\end{aligned}$$

where M_2: $(\mathbf{R}_+\backslash\{0\}) \times \mathbf{R}^5_+ \to \mathbf{R}_+$ *is an appropriate locally bounded function.*

PROPOSITION B. *Let* $\beta \in B(a, b)$, g_n, $g \in W^{1,1}_{loc}(\mathbf{R}: H)$, *and let* $f_n, f \in L^2_{loc}(\mathbf{R}; H)$, $n = 1, 2, \cdots$. *Assume that*

$$\sup_{n\in\mathbf{N}} S^{1,1}(g_n) + S^{1,1}(g) + \sup_{n\in\mathbf{N}} S_*^{0,2}(f_n) + S_*^{0,2}(f) < \infty,$$

$$g_n(t) \longrightarrow g(t) \qquad \textit{in } H \textit{ for all } t\in\mathbf{R}$$

and that

$$f_n \longrightarrow f \qquad \textit{in } L^2(J; V^*) \textit{ for each compact set } J \textit{ in } \mathbf{R}.$$

Let u_n, $n = 1, 2, \cdots$, *and* u *be solutions to* $E(\beta, g_n, f_n)$ *and* $E(\beta, g, f)$ *on* $[t_0, \infty)$, *respectively. Assume that*

$$\{\varphi^{t_0}(u_n(t_0))\}_{n\in\mathbf{N}} \textit{ is bounded}$$

and that

$$u_n(t_0) \longrightarrow u(t_0) \quad \textit{in } V^*.$$

Then we have:

(i) $u_n(t) \to u(t)$ *in* V^* *and weakly in* H *uniformly on each compact interval in* $[t_0, \infty)$.

(ii) $u_n' \to u'$ *weakly in* $L^2(t_0, t_1; V^*)$ *for all* $t_1 > t_0$.

(iii) $v_n \to v$ *weakly in* $L^2(t_0, t_1; X)$ *for all* $t_1 > t_0$,

where v_n, v *are the functions such that*

$$u_n' + F(v_n - g_n) = f_n,\ v_n \in \beta(u_n), \qquad n = 1, 2, \cdots,$$

$$u' + F(v - g) = f,\ v \in \beta(u).$$

These propositions will be used in various limit processes in the proofs of Theorems 1, 2 and 3.

3. Lemmas

In this section, we prepare four lemmas which will be used in the proofs of Theorems 1, 2 and 3.

LEMMA 3. *Let* u_1 *and* u_2 *be solutions to* $E(\beta, g, f)$ *on* $J \equiv [t_0, t_1]$ *and set*

$$u_i' + F(v_i - g) = f, \quad v_i \in \beta(u_i), \quad i = 1, 2. \tag{3.1}$$

Then the following equality holds:

$$2^{-1}|u_1(t) - u_2(t)|_*^2 + \int_s^t (v_1(\tau) - v_2(\tau), u_1(\tau) - u_2(\tau))d\tau \tag{3.2}$$

$$= 2^{-1}|u_1(s) - u_2(s)|_*^2 \qquad \textit{for all } t_0 \le s \le t \le t_1.$$

PROOF. Multiplying the identity derived from (3.1)

$$u_1' - u_2' + F(v_1 - v_2) = 0$$

by $u_1 - u_2$, we obtain

$$2^{-1}(d/dt)|u_1(t) - u_2(t)|_*^2 + (F(v_1(t) - v_2(t)),\ u_1(t) - u_2(t))_* = 0$$

$$\text{for a.e. } t \in J.$$

Since $u_1(t),\ u_2(t) \in D(\varphi^t) \subset H$ for a.e. $t \in J$, $(F(v_1(t) - v_2(t)),\ v_1(t) - u_2(t))_* = (v_1(t) - v_2(t),\ u_1(t) - u_2(t))$. Hence we have (3.2). q.e.d.

REMARK 2. We note that in the above lemma $(v_1 - v_2, v_1 - u_2) \geq 0$ by the monotonicity of β. Hence $t \mapsto |u_1(t) - u_2(t)|_*$ is non-increasing.

LEMMA 4. *Let* $\beta \in B(a, b)$, $g \in W_{loc}^{1,1}(\mathbf{R}; H)$ *and* $f \in L_{loc}^2(\mathbf{R}; V^*)$. *Let* u_1 *and* u_2 *be solutions to* $E(\beta, g, f)$ *on* $J \equiv [t_0, t_1]$. *Assume furthermore that*

(3.3) $$t \mapsto |u_1(t) - u_2(t)|_* \quad \text{is constant on } J.$$

Then the following (i) *and* (ii) *hold*:

(i) *There is an element* α *in* H *such that*

$$u_1(t) = u_2(t) + \alpha \qquad \text{for all } t \in J.$$

(ii) $\xi u_1 + (1 - \xi)u_2$ *is an solution to* $E(\beta, g, f)$ *on* J *for all* $0 \leq \xi \leq 1$.

PROOF OF LEMMA 4-(ii). We first show assertion (ii) by assuming that assertion (i) of Lemma 4 is valid. We give the proof of (i) after this. From (i) it follows that $v_1 = F^{-1}(f - u_1') + g = F^{-1}(f - u_2') + g = v_2$ a.e. on J. Therefore if we put $w = \xi u_1 + (1 - \xi)u_2$ ($\xi \in [0, 1]$), then the relation $v_i \in \beta(u_i)$, $i = 1, 2$, and the maximal monotonicity of β together imply that $v \equiv v_1 = v_2 \in \beta(w)$ a.e. on $J \times \Omega$. On the other hand we infer from (3.1)

$$\begin{aligned} w' = \xi u_1' + (1 - \xi)u_2' &= \xi\{f - F(v_1 - g)\} + (1 - \xi)\{f - F(v_2 - g)\} \\ &= f - F(v - g) \qquad \text{for a.e. } t \in J. \end{aligned}$$

Hence w is a solution to $E(\beta, g, f)$. q.e.d.

Before proving (i) of Lemma 4, we observe that the proof is immediate provided that β or β^{-1} is single-valued. In fact, under the assumptions of Lemma 4, (3.2) and (3.3) together imply

$$\int_{t_0}^{t_1}\int_\Omega \{v_1(t,x) - v_2(t,x)\}\{u_1(t,x) - u_2(t,x)\}dtdx = 0.$$

The integrand on the left hand side is non-negative by the monotonicity of β. Hence

$$\{v_1(t,x) - v_2(t,x)\}\{u_1(t,x) - u_2(t,x)\} = 0 \quad \text{for a.e. } (t,x) \in J \times \Omega.$$

Consequently, (a) if β is single-valued, then $v_1 = v_2$ a.e. on $J \times \Omega$; (b) if β^{-1} is single-valued, then $u_1 = u_2$ a.e. on $J \times \Omega$. In both cases (i) of Lemma 4 follows. However, as mentioned in Introduction we are interested in the case where both β and β^{-1} are multi-valued.

PROOF OF LEMMA 4-(i) (cf. Remark 3). Let φ^t be defined by (1.1). First using Kenmochi-Ôtani [15; Lemma 4.4] and (3.3) we have

(3.4) $\quad f(t) - u_2'(t) \in \partial\varphi^t(u_1(t))$ and $f(t) - u_1'(t) \in \partial\varphi^t(u_2(t))$ for a.e. $t \in J$.

Next we see from (3.1) and $\beta = \partial\hat{\beta}$ that the following inequality holds for $i = 1, 2$

$$\begin{aligned}
&\varphi^{t+h}(u_i(t+h)) - \varphi^t(u_i(t))\\
&= \int_\Omega \hat{\beta}(u_i(t+h,x))dx - \int_\Omega \hat{\beta}(u_i(t,x))dx - (g(t+h), u_i(t+h)) + (g(t), u_i(t))\\
&\geq (v_i(t), u_i(t+h) - u_i(t)) - (g(t+h), u_i(t+h)) + (g(t), u_i(t))\\
&= (v_i(t) - g(t), u_i(t+h) - u_i(t)) - (g(t+h) - g(t), u_i(t+h))\\
&= \langle v_i(t) - g(t), u_i(t+h) - u_i(t)\rangle - (g(t+h) - g(t), u_i(t+h))\\
&= (F(v_i(t) - g(t)), u_i(t+h) - u_i(t))_* - (g(t+h) - g(t), u_i(t+h))\\
&= (f(t) - u_i'(t), u_i(t+h) - u_i(t))_* - (g(t+h) - g(t), u_i(t+h)).
\end{aligned}$$

Dividing both sides of the above inequalities by h, letting $h \to 0$, and noting that $t \mapsto u_i(t)$ is weakly continuous in H, we have

$$(d/dt)\varphi^t(u_i(t)) = (f(t) - u_i'(t),\ u_i'(t))_* - (g'(t),\ u_i(t)) \quad \text{for a.e. } t \in J \text{ and } i = 1, 2. \tag{3.5}$$

Next it follows from (3.4) and Lemma 1 that for a.e. $t \in J$ there exist $\hat{u}_1(t)$ and $\hat{u}_2(t) \in X$ such that

$$f(t) - u_2'(t) = F(\hat{u}_1(t) - g(t)), \quad \hat{u}_1(t) \in \beta(u_1(t))$$

and

$$f(t) - u_1'(t) = F(\hat{u}_2(t) - g(t)), \quad \hat{u}_2(t) \in \beta(u_2(t)).$$

Therefore

$$v_2 = F^{-1}(f - u_2') + g = \hat{u}_1 \in \beta(u_1) \tag{3.6}$$

and

$$v_1 = F^{-1}(f - u_1') + g = \hat{u}_2 \in \beta(u_2). \tag{3.7}$$

Using (3.6), we obtain as before

$$\begin{aligned}
&\varphi^{t+h}(u_1(t+h)) - \varphi^t(u_1(t)) \\
&\quad \geq (v_2(t),\ u_1(t+h) - u_1(t)) - (g(t+h),\ u_1(t+h)) + (g(t),\ u_1(t)) \\
&\quad = (F(v_2(t) - g(t)),\ u_1(t+h) - u_1(t))_* - (g(t+h) - g(t),\ u_1(t+h)) \\
&\quad = (f(t) - u_2'(t),\ u_1(t+h) - u_1(t))_* - (g(t+h) - g(t),\ u_1(t+h)).
\end{aligned}$$

Therefore, in the same way as in the derivation of (3.5), we get

$$(3.8) \qquad (d/dt)\varphi^t(u_1(t)) = (f(t) - u_2'(t),\ u_1'(t))_* - (g'(t),\ u_1(t)).$$

Similarly, by (3.7),

$$(3.9) \qquad (d/dt)\varphi^t(u_2(t)) = (f(t) - u_1'(t),\ u_2'(t))_* - (g'(t),\ u_2(t)).$$

From (3.5) with $i = 1$ and (3.8) it follows that

$$(u_1'(t),\ u_1'(t))_* = (u_2'(t),\ u_1'(t))_*.$$

Similarly, (3.5) with $i = 2$ and (3.9) together imply

$$(u_2'(t),\ u_2'(t))_* = (u_1'(t),\ u_2'(t))_*.$$

Hence

$$\begin{aligned}
|u_1'(t) - u_2'(t)|_*^2 &= (u_1'(t),\ u_1'(t))_* - (u_2'(t),\ u_1'(t))_* \\
&\quad + (u_2'(t),\ u_2'(t))_* - (u_1'(t),\ u_2'(t))_* \\
&= 0 \qquad \text{for a.e.} \quad t \in J.
\end{aligned}$$

Thus there is $\alpha \in V^*$ such that $u_1(t) = u_2(t) + \alpha$ for all $t \in J$. But, note that $u_i \in L^\infty(J; H) \cap W^{1,2}(J; V^*)$ by Lemma 2, and so $\alpha \in H$. This proves the desired assertion (i). q.e.d.

REMARK 3. Our proof of Lemma 4–(i) is a modification of that of Baillon-Haraux [2], in which the periodic behavior of solutions to evolution equations formulated with time-independent subdifferential is studied, and they used the equality $(d/dt)\varphi(u(t)) = (u'(t), u^*(t))$ with $u^*(t) \in \partial\varphi(u(t))$. In the time-dependent case, this type of equality can no longer be expected. But for our $\{\varphi^t\}$, it is possible to compute $(d/dt)\varphi^t(u(t))$ explicitly and apply the same technique as in [2].

Now the application of Lemma 4 implies the following two lemmas, which play an important role in the proofs of the theorems.

LEMMA 5. *Let $\beta \in B(a, b)$, $g, g^* \in W^{1,1}_{loc}(\mathbf{R}; H)$ and $f, f^* \in L^2_{loc}(\mathbf{R}; V^*)$. Let u_i and u_i^*, $i = 1, 2$, be solutions to $E(\beta, g, f)$ on $[t_0, \infty)$, $t_0 \in \mathbf{R}$, and to $E(\beta, g^*, f^*)$ on $\mathbf{R}$, respectively. Suppose that $\{t_n\}$ is a sequence in $\mathbf{R}$ such that $t_n \to \infty$ and*

(3.10) $\quad u_i(t+t_n) \longrightarrow u_i^*(t)$ *in* V^* *as* $n \longrightarrow \infty \qquad$ *for all* $t \in \mathbf{R}$ $i=1,2.$

Then we have the following (i), (ii) *and* (iii):

(i) $$|u_1^*(t)-u_2^*(t)|_* = d \qquad \textit{for all } t \in \mathbf{R},$$

where

(3.11) $$d = \lim_{s\to\infty}|u_1(s)-u_2(s)|_* \ (<\infty).$$

(ii) *There is an element* α *in* H *such that*

$$u_1^*(t) = u_2^*(t) + \alpha \qquad \textit{for all } t \in \mathbf{R}.$$

(iii) $\xi u_1^* + (1-\xi)u_2^*$ *is a solution to* $E(\beta, g^*, f^*)$ *on* $\mathbf{R}$ *for all* $0 \le \xi \le 1$.

PROOF. We first observe that the limit (3.11) exists (cf. Remark 2). Therefore by (3.10)

$$\begin{aligned}|u_1^*(t)-u_2^*(t)|_* &= \lim_{n\to\infty}|u_1(t+t_n)-u_2(t+t_n)|_*\\ &= \lim_{s\to\infty}|u_1(s)-u_2(s)|_*\\ &= d<\infty \qquad \text{for all } t\in\mathbf{R}.\end{aligned}$$

Thus (i) is obtained, so that (ii) and (iii) follow from Lemma 4. q.e.d.

LEMMA 6. *Let* β, g, g^*, f *and* f^* *be as in Lemma 5. Let* u_i *and* u_i^*, $i=1,2$, *be solutions to* $E(\beta, g, f)$ *on* $\mathbf{R}$ *and to* $E(\beta, g^*, f^*)$ *on* $\mathbf{R}$, *respectively. Suppose that* $\{t_n\}$ *is a sequence in* $\mathbf{R}$ *such that* $t_n \to -\infty$ *as* $n\to\infty$ *and* (3.10) *holds. Furthermore, assume that*

(3.12) $$\sup_{t\in\mathbf{R}}|u_i(t)|_* < \infty, \qquad i=1,2.$$

Then we have the same assertions (ii) *and* (iii) *as in the statement of Lemma* 5 *and the following* (i)′:

(i)′ $$|u_1^*(t)-u_2^*(t)|_* = d' \qquad \textit{for all } t\in\mathbf{R},$$

where

(3.13) $$d' = \lim_{s\to-\infty}|u_1(s)-u_2(s)|_* \qquad (<\infty).$$

PROOF. By (3.12) and Remark 2, the limit (3.13) exists and is finite. Therefore the lemma is proved in the same way as Lemma 5. q.e.d.

4. Proofs of Theorems

In this section, we prove Theorems 1, 2 and 3 by applying the argument in [9]. We first prove Theorem 2, then Theorem 1 as a special case of Theorem 2, and then Theorem 3.

PROOF OF THEOREM 2-(i). We employ the min-max principle. Fix $z_0 \in D_\beta$. For $n = 1, 2, \cdots$, let w_n be the solution to $E(\beta, g, f)$ on $[-n, \infty)$ with $w_n(-n) = z_0$. Note that the existence of w_n follows from Lemma 2, since $\sup_{n \in \mathbf{N}} \varphi^{-n}(z_0) < \infty$. In view of Propositions A and B in Section 2 and the fact that the injection from H into V^* is compact, we see that there is a subsequence $\{w_{n_k}\}$ such that

$$w_{n_k}(t) \longrightarrow w(t) \text{ strongly in } V^* \text{ and weakly in } H$$

and the convergence is uniform in t on each compact set in $\mathbf{R}$, and that w is a solution to $E(\beta, f, g)$ on $\mathbf{R}$. Moreover

$$C_1 \equiv \sup_{t \in \mathbf{R}} |w(t)|_H + \sup_{t \in \mathbf{R}} |w'|_{L^2(t, t+1; V^*)} + \sup_{t \in \mathbf{R}} \varphi^t(w(t)) < \infty.$$

Therefore the following set K is not empty.

$K \equiv \{u;\ u$ is a solution to $E(\beta, g, f)$ on $\mathbf{R}$ and condition $(*)$ below holds$\}$,

$$(*) \qquad \sup_{t \in \mathbf{R}} |u(t)|_H + \sup_{t \in \mathbf{R}} |u'|_{L^2(t, t+1; V^*)} + \sup_{t \in \mathbf{R}} \varphi^t(u(t)) \le C_1.$$

We then put

$$I(u) \equiv \sup_{t \in \mathbf{R}} |u(t)|_* \qquad \text{for } u \in L^\infty(\mathbf{R}; V^*)$$

and

$$I_0 \equiv \inf\{I(u);\ u \in K\}$$

We want to show that there is a unique $u^* \in K$ satisfying $I(u^*) = I_0$. To this end, we take a sequence $\{u_n\}$ in K so that $I(u_n) \downarrow I_0$ as $n \to \infty$. As before, there is a subsequence $\{u_{n_k}\}$ and a solution u^* to $E(\beta, g, f)$ on $\mathbf{R}$ such that

$$u_{n_k}(t) \longrightarrow u^*(t) \text{ strongly in } V^* \text{ and weakly in } H$$

and the convergence is uniform in t on each compact set in $\mathbf{R}$. Clearly $u^* \in K$ and $I(u^*) = I_0$. Next let u_1, u_2 be two elements in K such that $I(u_1) = I(u_2) = I_0$. By the almost periodicity of f and g, there is a sequence $t_n \to -\infty$ in $\mathbf{R}$ such that (cf. [1])

$$g(t + t_n) \longrightarrow g(t) \text{ in } H \text{ uniformly in } t \in \mathbf{R}$$

and

$$f(t + t_n + \cdot) \longrightarrow f(t + \cdot) \text{ in } L^2(0, 1; V^*) \text{ uniformly in } t \in \mathbf{R}.$$

Moreover, taking a subsequence of $\{t_n\}$ if necessary and referring to Propositions A and B in Section 2, we may assume that for $i = 1, 2$ there exist solutions w_i to $E(\beta, g, f)$ on $\mathbf{R}$ and

$$u_i(t + t_n) \longrightarrow w_i(t) \text{ strongly in } V^* \text{ and weakly in } H,$$

where the convergence is uniform in t on each compact set in $\mathbf{R}$. Clearly w_1, $w_2 \in K$ and $I_0 = I(w_1) = I(w_2)$. Hence by Lemma 6

$$|w_1(t) - w_2(t)|_* = d \equiv \lim_{s\to-\infty} |u_1(s) - u_2(s)|_* \qquad \text{for all } t \in \mathbf{R}$$

and

$$w \equiv 2^{-1}(w_1 + w_2) \in K.$$

On the other hand

$$\begin{aligned} |w(t)|_*^2 + |2^{-1}\{w_1(t) - w_2(t)\}|_*^2 &= 2^{-1}\{|w_1(t)|_*^2 + |w_2(t)|_*^2\} \\ &\le 2^{-1}(I_0^2 + I_0^2) = I_0^2. \end{aligned}$$

Therefore $I_0^2 \le I(w)^2 \le I_0^2 - 2^{-2}d^2$. Hence $d = 0$ and

$$|u_1(t) - u_2(t)|_* \le \lim_{s\to-\infty} |u_1(s) - u_2(s)|_* = d = 0 \qquad \text{for all } t \in \mathbf{R},$$

which shows that $u_1 = u_2$. Thus it is concluded that there is one and only one element u^* in K such that $I(u^*) = I_0$. Once this is proved, the V^*-almost periodicity of u^* is derived in the same way as in [9] or [1]. We refer to [9] or [1] for the rest of the proof. q.e.d.

Proof of Theorem 2-(ii). Fix $\bar{\omega} \in AP$. Then by the almost periodicity of g, f and $\bar{\omega}$, there is a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\begin{aligned} &g(t + t_n) \longrightarrow g(t) && \text{in } H \text{ uniformly in } t \in \mathbf{R}, \\ &f(t + t_n + \cdot) \longrightarrow f(t + \cdot) && \text{in } L^2(0, 1; V^*) \text{ uniformly in } t \in \mathbf{R}, \\ &\bar{\omega}(t + t_n) \longrightarrow \bar{\omega}(t) && \text{in } V^* \text{ uniformly in } t \in \mathbf{R}. \end{aligned}$$

We now put $u_n(t) \equiv u(t + t_n)$ for $t \ge t_0 - t_n$. Then using Proposition A in Section 2 and taking a subsequence of $\{t_n\}$ if necessary, we find a function $\omega\colon \mathbf{R} \to V^*$ such that

$$u_n(t) \longrightarrow \omega(t) \text{ in } V^* \text{ uniformly in } t \text{ on each compact subset in } \mathbf{R}.$$

Moreover by Proposition B in Section 2, we observe that ω is a solution to $E(\beta, g, f)$ on $\mathbf{R}$. Therefore by Lemma 5, there is an element α in H such that

$$\omega(t) = \bar{\omega}(t) + \alpha \qquad \text{for all } t \in \mathbf{R}.$$

Hence ω belongs to AP and

$$\omega(t + t_n) = \bar{\omega}(t + t_n) + \alpha \longrightarrow \bar{\omega}(t) + \alpha = \omega(t) \text{ in } V^* \text{ uniformly in } t \in \mathbf{R}.$$

Moreover

$$\begin{aligned} \lim_{t\to\infty} |u(t) - \omega(t)|_* &= \lim_{n\to\infty} |u(t_n) - \omega(t_n)|_* = \lim_{n\to\infty} |u_n(0) - \omega(t_n)|_* \\ &= |\omega(0) - \omega(0)|_* = 0. \end{aligned}$$

From this and the fact that $\sup_{t\geq t_0+1}|u(t)|_H < \infty$, we infer (ii). q.e.d.

PROOF OF THEOREM 2–(iii). Let $\omega_1, \omega_2 \in AP$. Then $\lim_{t\to\infty}|\omega_1(t)-\omega_2(t)|_* \equiv d$ exists. Since $t \mapsto |\omega_1(t)-\omega_2(t)|_*$ is $\mathbf{R}$-almost periodic, we conclude that $|\omega_1(t)-\omega_2(t)|_* \equiv d$ on $\mathbf{R}$. Therefore, by Lemma 4, there is an element $\alpha \in H$ such that

$$\omega_1(t) = \omega_2(t) + \alpha \qquad \text{for all } t \in \mathbf{R}.$$

From this we obtain the relation

$$\eta_1(t) = g(t) + F^{-1}(f(t) - \omega_1'(t)) = g(t) + F^{-1}(f(t) - \omega_2'(t)) = \eta_2(t)$$

$$\text{for a.e. } t \in \mathbf{R}.$$ q.e.d.

PROOF OF THEOREM 1. By Proposition A in Section 2, any solution u to $E(\beta, g, f)$ on $[0, \infty)$ is V^*-bounded, that is $\sup_{t\geq 0}|u(t)|_* < \infty$. Therefore, the assertion (i) follows from Remark 2 in Section 2 and the well-known theorem by Browder-Petryshyn [4]. Next note that all the assumptions of Theorem 2 are satisfied. It is clear that $P_T \subset AP$. This relation and the assertion (iii) of Theorem 2 together imply that $P_T = AP$. Hence assertions (ii) and (iii) of Theorem 1 follows from those of Theorem 2. q.e.d.

PROOF OF THEOREM 3. The "only if" part is evident, so it suffices to prove the "if" part. First we treat the almost periodic case. Fix any $\omega \in AP$. Then, since u is V^*-bounded, we have (cf. Remark 2 in Section 2)

$$d \equiv \lim_{t\to\infty}|u(t) - \omega(t)|_* < \infty.$$

By the almost periodicity of g, f and ω, there is a sequence $t_n \to -\infty$ such that

$$\begin{aligned} &g(t+t_n) \longrightarrow g(t) && \text{in } H \text{ uniformly in } t \in \mathbf{R},\\ &f(t+t_n+\cdot) \longrightarrow f(t+\cdot) && \text{in } L^2(0,1;V^*) \text{ uniformly in } t \in \mathbf{R},\\ &\omega(t+t_n) \longrightarrow \omega(t) && \text{in } V^* \text{ uniformly in } t \in \mathbf{R}. \end{aligned}$$

Moreover, putting $u_n(t) = u(t+t_n)$, we may assume that there exists a solution u^* to $E(\beta, g, f)$ on $\mathbf{R}$ and

$$u_n(t) \longrightarrow u^*(t) \text{ in } V^* \text{ uniformly on each compact set in } \mathbf{R}.$$

Therefore, by Lemma 6, one finds $\alpha \in H$ such that

$$u^*(t) = \omega(t) + \alpha \qquad \text{for all } t \in \mathbf{R}.$$

Consequently u^* belongs to AP and

$$u^*(t+t_n) = \omega(t+t_n) + \alpha \longrightarrow \omega(t) + \alpha = u^*(t) \text{ in } V^* \text{ uniformly in } t \in \mathbf{R}.$$

Hence

$$\begin{aligned}|u(t) - u^*(t)|_* &\leq \lim_{s\to -\infty}|u(s) - u^*(s)|_* \\ &= \lim_{n\to\infty}|u(t_n) - u^*(t_n)|_* \\ &= \lim_{n\to\infty}|u_n(0) - u^*(t_n)|_* \\ &= |u^*(0) - u^*(0)|_* \\ &= 0 \qquad \text{for all } t\in\mathbf{R}.\end{aligned}$$

Thus we have $u = u^* \in AP$.

As for the periodic case, repeat the above argument with ω in P_T instead of AP and with $t_n = -nT$, $n = 1, 2, \cdots$. Then, exactly in the same way as above, we have $u = u^* = \omega + \alpha \in P_T$. q.e.d.

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