

Oscillations of mixed neutral equations

I. P. STAVROULAKIS

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1. Introduction and preliminaries

A first order functional differential equation in which the present value of $\dot{x}(t)$ is expressed in terms of both past and future values of x is said to be of “mixed” type. A first order equation in which the expression for $\dot{x}(t)$ involves $\dot{x}(\tau(t))$ for some $\tau(t) \neq t$ is said to be of “neutral” type. So, when both of these characteristics are present, the equation is of mixed neutral type or a neutral equation with mixed arguments or simply a mixed neutral equation. See Driver [6].

Consider the neutral differential equation

$$(*) \quad \frac{d}{dt} [x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) = 0$$

where $c, r, p_i, \tau_i, i = 1, \dots, k$ are real numbers.

Observe that when $c = 0$ or $r = 0$ the above equation reduces to a non-neutral equation whose oscillatory character has been studied by several authors. See for example the papers by Ladas and Stavroulakis [22, 23], Arino, Györi and Jawhari [1], Hunt and Yorke [16] and Fukagai and Kusano [7]. Also in the case where $p_i = 0, i = 1, \dots, k$ equation (*) reduces to

$$\frac{d}{dt} [x(t) + cx(t-r)] = 0$$

and there exists a (nonoscillatory) solution of the form $x(t) = c, c$ a constant. Thus we will assume that

$$c \neq 0, \quad r \neq 0, \quad \text{and} \quad p_i \neq 0 \quad \text{for all } i = 1, \dots, k.$$

The following (duality) lemma is easily established (cf. [11, Lemma 5]).

LEMMA 1.1. *Suppose that $c \neq 0$ and $p_i \neq 0, i = 1, \dots, k$. Then $x(t)$ satisfies the neutral equation (*) if and only if $x(t)$ satisfies the neutral equation*

$$\frac{d}{dt} \left[x(t) + \frac{1}{c} x(t - (-r)) \right] + \frac{1}{c} \sum_{i=1}^k p_i x(t - (\tau_i - r)) = 0.$$

If $c < 0$ and $rp_i < 0$ or if $c > 0$, $rp_i > 0$ and $p_i\tau_i < 0$, $i = 1, \dots, k$ then it is easily seen that the characteristic equation of (*)

$$\lambda + c\lambda e^{-\lambda r} + \sum_{i=1}^k p_i e^{-\lambda\tau_i} = 0$$

has a real root and therefore Eq. (*) has a nonoscillatory solution.

On the basis of the above observations in order to obtain sufficient conditions which guarantee the oscillation of all solution of Eq. (*) we need only to consider Eq. (*) for all quadruplets (c, r, p_i, τ_i) such that $rp_i > 0$ or $c > 0$ and $p_i\tau_i > 0$, $c \in (-\infty, 0) \cup (0, \infty)$, $r \in (-\infty, 0) \cup (0, \infty)$, $p_i \in (-\infty, 0) \cup (0, \infty)$ and $\tau_i \in \mathbf{R}$, $i = 1, \dots, k$. Using the (duality) Lemma 1.1 and arguments similar to those in [13] we see that it suffices to consider the following cases only

$$(C^+) \quad c \in \mathbf{R} - \{0\} \quad \text{and} \quad r > 0, \quad p_i > 0, \quad \tau_i > 0, \quad i = 1, \dots, k,$$

and

$$(C^-) \quad c \in \mathbf{R} - \{0\} \quad \text{and} \quad r < 0, \quad p_i < 0, \quad \tau_i < 0, \quad i = 1, \dots, k.$$

These two cases correspond to the neutral differential equations of the retarded

$$(1) \quad \frac{d}{dt} [x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) = 0$$

and the advanced type

$$(2) \quad \frac{d}{dt} [x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t + \tau_i) = 0$$

where condition (C^+) is satisfied.

Thus in this paper we develop oscillatory results for the above neutral equations (1) and (2). We also combine them to obtain oscillation results for the solutions of the neutral equations of mixed type

$$(3) \quad \frac{d}{dt} [x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) + \sum_{j=1}^{\ell} q_j x(t + \sigma_j) = 0$$

and

$$(4) \quad \frac{d}{dt} [x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t + \tau_i) - \sum_{j=1}^{\ell} q_j x(t - \sigma_j) = 0,$$

where

$$(C) \quad \begin{aligned} &c \in \mathbf{R}, \quad r \in (0, \infty), \quad p_i, q_j \in (0, \infty) \quad \text{and} \\ &\tau_i, \sigma_j \in [0, \infty) \quad \text{for } i = 1, \dots, k; \quad j = 1, \dots, \ell. \end{aligned}$$

To the best of the author's knowledge this is the only paper at the present time dealing with oscillation of mixed neutral equations.

In the special case where $k = 1$, Ladas and Sficas [21] studied the asymptotic and oscillatory behavior of the solutions of Eq. (1) when $c \in [-1, 0]$, and then Grammatikopoulos, Grove and Ladas [10] investigated the other possible cases $c < -1$ and $c > 0$. In this special case Sficas and Stavroulakis [26] obtained a necessary and sufficient condition for the oscillation of all solutions of Eq. (1) in terms of its characteristic equation, and then Grove, Ladas and Meimaridou [13] extended this result for Eq. (*). Kulenovic, Ladas and Meimaridou [19] proved that all solutions of Eq. (1) oscillate if and only if the characteristic equation

$$(5) \quad \lambda + c\lambda e^{-\lambda r} + \sum_{i=1}^k p_i e^{-\lambda \tau_i} = 0$$

has no real roots. For some oscillation results concerning Eq. (1) or some special cases of it see also Ruan [17], Gopalsamy [8] and Gopalsamy and Zhang [9].

Our aim in this paper is to obtain sufficient conditions, involving the coefficients and the argument only, under which all solutions of (1), (2), (3) and (4) oscillate. The advantage of working with these conditions rather than the characteristic equations associated with the equations under consideration is that they are explicit and therefore easily verifiable, while determining whether or not a real root to the characteristic equation exists may be quite a problem in itself. Furthermore our technique is given in such a way that it can be extended in a straightforward manner to the case of equations with variable coefficients.

The problem of asymptotic and oscillatory behavior of solutions of neutral differential equations is of both theoretical and practical interest. Note that equations of this type appear in networks containing lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits (see [3, 27]).

It is to be noted that, in general, the theory of neutral differential equations presents complications, and results which are true for delay differential equations may not be true for neutral equations. For example, see [3, 4, 14, 15, 27, 28].

Let $T = \max \{r, \tau_1, \dots, \tau_k\}$. We say that $x(t)$ is a solution of (1) provided there exists $t_0 \in \mathbf{R}$ such that $x \in C([t_0 - T, \infty), \mathbf{R})$, $x(t) + cx(t - r)$ is continuously differentiable for $t \geq t_0$ and (1) holds for $t \geq t_0$. Solutions for (2), (3) and (4) are defined analogously. See Driver [5, 6], Bellman and Cooke [2] and Hale [15] for questions of existence and uniqueness.

As is customary, a solution is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In the sequel all functional inequalities that we write are assumed to hold eventually, that is for all sufficiently large t .

Also, for convenience, we will assume without loss of generality that

$$\tau_1 < \cdots < \tau_k.$$

A solution x of (1) [(2), (3) or (4)] may not be differentiable. However, as the following lemma indicates, from every solution of (1) [(2), (3) or (4)] we can construct auxiliary solutions with some "nice" asymptotic properties. Given a solution x of (1) [(2), (3) or (4)], set

$$(6) \quad z(t) = x(t) + cx(t - r)$$

and

$$(7) \quad w(t) = z(t) + cz(t - r)$$

in the case of equations (1) and (3), while set

$$(8) \quad z(t) = x(t) + cx(t + r)$$

and

$$(9) \quad w(t) = z(t) + cz(t + r)$$

in the case of equations (2) and (4). Then the following lemma is easily established by using arguments similar to those in [10] and [21]. See also Lemma 1 in [19] and Lemma 2 in [12].

LEMMA 1.2. *Let $x(t)$ be an eventually positive solution of (1) [(2), (3) or (4)]. Then*

(a) *$z(t)$ is a differentiable solution and $w(t)$ is a twice differentiable solution of (1) [(2), (3) or (4)];*

(b) *$c \neq -1$;*

(c) *if $-1 < c$ in Eqs (1) and (3) or $c < -1$ in Eqs (2) and (4), then*

$$(10) \quad w(t) > 0, \quad \dot{w}(t) < 0, \quad \ddot{w}(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} w(t) = 0 = \lim_{t \rightarrow \infty} \dot{w}(t);$$

(d) *if $c < -1$ in Eqs (1) and (3) or $-1 < c$ in Eqs (2) and (4), then*

$$(11) \quad w(t) > 0, \quad \dot{w}(t) > 0, \quad \ddot{w}(t) > 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} w(t) = \infty = \lim_{t \rightarrow \infty} \dot{w}(t).$$

Combining results due to Ladas and Stavroulakis [22, 23] and to Hunt and Yorke [16], the following lemma is derived.

LEMMA 1.3. *Consider the inequality with retarded arguments*

$$(12) \quad \dot{x}(t) + \sum_{i=1}^n p_i x(t - \tau_i) \leq 0$$

and the inequality with advanced arguments

$$(13) \quad \dot{x}(t) - \sum_{i=1}^n p_i x(t + \tau_i) \geq 0$$

where p_i and τ_i , $i = 1, 2, \dots, n$ are positive constants. Then any of the following conditions

$$(14) \quad \sum_{i=1}^n p_i \tau_i > \frac{1}{e},$$

or

$$(15) \quad \left[\prod_{i=1}^n p_i \right]^{1/n} \left(\sum_{i=1}^k \tau_i \right) > \frac{1}{e},$$

implies that (12) and (13) have no eventually positive solutions.

It is easily seen, see also [16], that the above conditions (14) and (15) are independent. They are also sharp in that the lower bound $1/e$ cannot be improved.

2. Retarded arguments

In this section we study the neutral equation (1) and we obtain sufficient conditions under which all solutions of (1) oscillate.

THEOREM 2.1. Consider equation (1) under condition (C). Assume that

$$(i) \quad -1 < c, r < \tau_1,$$

or

$$(ii) \quad c < -1, r > \tau_k.$$

Then any of the following conditions

$$(16) \quad \frac{1}{1+c} \sum_{i=1}^k p_i (\tau_i - r) > \frac{1}{e}$$

or

$$(17) \quad \frac{1}{1+c} \left[\prod_{i=1}^k p_i \right]^{1/k} \left(\sum_{i=1}^k (\tau_i - r) \right) > \frac{1}{e}$$

implies that all solutions of (1) oscillate. Moreover if

$$(iii) \quad -1 < c < 0$$

then any of the following conditions

$$(18) \quad \sum_{i=1}^k p_i \tau_i > \frac{1}{e}$$

or

$$(19) \quad [\prod_{i=1}^k p_i]^{1/k} (\sum_{i=1}^k \tau_i) > \frac{1}{e}$$

implies that all solutions of (1) oscillate.

PROOF. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $x(t)$. Then, by Lemma 1.2,

$$(20) \quad \dot{w}(t) + c\dot{w}(t-r) + \sum_{i=1}^k p_i w(t-\tau_i) = 0.$$

Consider now the following cases:

(i) $-1 < c$ and $r < \tau_1$. In this case (10) holds and therefore

$$\dot{w}(t-r) + c\dot{w}(t-r) + \sum_{i=1}^k p_i w(t-\tau_i) < 0$$

or

$$\dot{w}(t) + \frac{1}{1+c} \sum_{i=1}^k p_i w(t - (\tau_i - r)) < 0.$$

But, by Lemma 1.3, any of the conditions (16) or (17) implies that the last inequality can not have an eventually positive solution. This is a contradiction.

(ii) $c < -1$ and $r > \tau_k$. In this case (11) holds and so (20) yields

$$\dot{w}(t-r) + c\dot{w}(t-r) + \sum_{i=1}^k p_i w(t-\tau_i) < 0.$$

Hence

$$\dot{w}(t-r) + \frac{1}{1+c} \sum_{i=1}^k p_i w(t-\tau_i) > 0$$

or

$$\dot{w}(t) - \left(-\frac{1}{1+c} \right) \sum_{i=1}^k p_i w(t + (r - \tau_i)) > 0$$

which, as before, leads to a contradiction.

(iii) $-1 < c < 0$. Then (10) holds and therefore

$$c\dot{w}(t-r) > 0.$$

From (20), we have

$$\dot{w}(t) + \sum_{i=1}^k p_i w(t-\tau_i) < 0.$$

This is a contradiction since by Lemma 1.3 any of the conditions (18) or (19) implies that the last inequality can not have an eventually positive solution.

The proof of the theorem is complete.

3. Advanced arguments

In this section we obtain sufficient conditions under which all solutions of the neutral equation (2) oscillate.

THEOREM 3.1. *Consider equation (2) and assume that condition (C) is satisfied. Under the hypotheses of Theorem 2.1 all solutions of (2) oscillate.*

PROOF. Assume, for the sake of contradiction, that Eq. (2) has an eventually positive solution $x(t)$. Then, by Lemma 1.2,

$$(20) \quad \dot{w}(t) + c\dot{w}(t+r) - \sum_{i=1}^k p_i w(t + \tau_i) = 0.$$

Consider now the following cases:

(i) $-1 < c$ and $r < \tau_1$. In this case (11) holds and therefore

$$\dot{w}(t+r) + c\dot{w}(t+r) - \sum_{i=1}^k p_i w(t + \tau_i) > 0$$

or

$$\dot{w}(t) - \frac{1}{1+c} \sum_{i=1}^k p_i w(t + (\tau_i - r)) > 0$$

which, in view of Lemma 1.3, leads to a contradiction.

(ii) $c < -1$ and $r > \tau_k$. In this case (10) holds and therefore

$$\dot{w}(t+r) + c\dot{w}(t+r) - \sum_{i=1}^k p_i w(t + \tau_i) > 0$$

or

$$\dot{w}(t) + \frac{-1}{1+c} \sum_{i=1}^k p_i w(t - (r - \tau_i)) < 0$$

and again we are led to a contradiction.

(iii) $-1 < c < 0$. Then (11) holds and therefore

$$c\dot{w}(t+r) < 0.$$

From (20)', we have

$$\dot{w}(t) - \sum_{i=1}^k p_i w(t + \tau_i) > 0$$

which again leads to a contradiction.

The proof of the theorem is complete.

4. Mixed arguments

In this section we employ the oscillatory character of retarded and advanced differential equations to obtain sufficient conditions under which all solutions of certain mixed neutral equations are oscillatory. To the best of the author's knowledge this is the only paper at the present time dealing with oscillation of mixed neutral equations.

THEOREM 4.1. *Consider equations (3) and (4) and assume that condition (C) is satisfied. Under the hypotheses of Theorem 2.1 all solutions of (3) and (4) oscillate.*

PROOF. First we consider Eq. (3) and assume, for the sake of contradiction, that it has an eventually positive solution $x(t)$. Then, by Lemma 1.2,

$$\dot{w}(t) + c\dot{w}(t-r) + \sum_{i=1}^k p_i w(t-\tau_i) + \sum_{j=1}^{\ell} q_j w(t+\sigma_j) = 0$$

or

$$\dot{w}(t) + c\dot{w}(t-r) + \sum_{i=1}^k p_i w(t-\tau_i) < 0$$

since $w(t)$ is eventually positive. From the last inequality we are led to a contradiction as from equation (20) in the proof of Theorem 2.1.

Consider now Eq. (4) and assume that it has an eventually positive solution $x(t)$. Then, by Lemma 1.2,

$$\dot{w}(t) + c\dot{w}(t+r) - \sum_{i=1}^k p_i w(t+\tau_i) - \sum_{j=1}^{\ell} q_j w(t-\sigma_j) = 0$$

or

$$\dot{w}(t) + c\dot{w}(t+r) - \sum_{i=1}^k p_i w(t+\tau_i) > 0$$

since $w(t)$ is positive. From the last inequality we are led to a contradiction as from equation (20)' in the proof of Theorem 3.1.

The proof of the theorem is complete.

5. Summary and examples

Combining the above results into a single statement we have the following.

THEOREM 5.1. *Consider the neutral equations*

$$(1) \quad \frac{d}{dt} [x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t-\tau_i) = 0,$$

$$(2) \quad \frac{d}{dt} [x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t+\tau_i) = 0,$$

$$(3) \quad \frac{d}{dt}[x(t) + cx(t-r)] + \sum_{i=1}^k p_i x(t - \tau_i) + \sum_{j=1}^{\ell} q_j x(t + \sigma_j) = 0,$$

and

$$(4) \quad \frac{d}{dt}[x(t) + cx(t+r)] - \sum_{i=1}^k p_i x(t + \tau_i) - \sum_{j=1}^{\ell} q_j x(t - \sigma_j) = 0,$$

where

$$(C) \quad \begin{aligned} &c \in \mathbf{R}, \quad r \in (0, \infty), \quad p_i, q_j \in (0, \infty) \quad \text{and} \\ &\tau_i, \sigma_j \in [0, \infty) \quad \text{for } i = 1, \dots, k; j = 1, \dots, \ell. \end{aligned}$$

Then in any of the following cases all solutions of (1), (2), (3) and (4) oscillate:

- (i) $c = -1$;
- (ii) $-1 < c, r < \tau_1$ or $c < -1, r > \tau_k$ and furthermore

$$(16) \quad \frac{1}{1+c} \sum_{i=1}^k p_i (\tau_i - r) > \frac{1}{e}$$

or

$$(17) \quad \frac{1}{1+c} (\prod_{i=1}^k p_i)^{1/k} (\sum_{i=1}^k (\tau_i - r)) > \frac{1}{e}$$

is satisfied;

- (iii) $-1 < c < 0$ and

$$(18) \quad \sum_{i=1}^k p_i \tau_i > \frac{1}{e}$$

or

$$(19) \quad (\prod_{i=1}^k p_i)^{1/k} (\sum_{i=1}^k \tau_i) > \frac{1}{e}$$

is satisfied.

Observe that

$$(21) \quad \lambda + \lambda c e^{-\lambda r} + \sum_{i=1}^k p_i e^{-\lambda \tau_i} = 0$$

is the characteristic equation of (1) and (2), and

$$(22) \quad \lambda + \lambda c e^{-\lambda r} + \sum_{i=1}^k p_i e^{-\lambda \tau_i} + \sum_{j=1}^{\ell} q_j e^{\lambda \sigma_j} = 0$$

is the characteristic equation of (3) and (4). As a corollary of Theorem 5.1 we can obtain sufficient conditions under which the characteristic equations (21)

and (22) have no real roots something that cannot be so easily determined by investigating directly the exponential equations (21) and (22).

COROLLARY 5.1. *Assume that condition (C) is satisfied. Then in any of the following cases:*

- (i) $c = -1$;
 - (ii) $-1 < c$, $r < \tau_1$ or $c < -1$, $r > \tau_k$ and furthermore (16) or (17) is satisfied;
 - (iii) $-1 < c < 0$ and (18) or (19) is satisfied,
- the characteristic equations (21) and (22) have no real roots.

REMARK 5.1. In view of the above corollary and the Theorem in [19] the conclusion of Theorem 3.1 is obvious. Indeed, under the hypotheses of Theorem 2.1 the characteristic equation (21) has no real roots. But this is also a sufficient condition for all solutions of Eq. (2) (associated with the characteristic equation (21)) to oscillate.

EXAMPLE 5.1. In the following mixed neutral equation

$$\frac{d}{dt} \left[x(t) - x \left(t - \frac{\pi}{2} \right) \right] + x \left(t - \frac{3\pi}{2} \right) + x(t + \pi) = 0,$$

$c = -1$; and by Theorem 5.1 all solutions of this equation oscillate. For example, $\sin t$ and $\cos t$ are oscillatory solutions.

The following examples illustrate that the above conditions (16) and (17) for oscillations are independent. They are chosen in such a way that only one of the two conditions (16) and (17) is satisfied. As we have already mentioned, conditions (18) and (19) are also independent. Furthermore in all these conditions (16), (17), (18) and (19) the lower bound $1/e$ cannot be improved.

EXAMPLE 5.2. (Only condition (16) is satisfied.) For the equation

$$\frac{d}{dt} \left[x(t) + \frac{1}{10^3} x \left(t - \frac{\pi}{2} \right) \right] + \frac{1}{10^3} x(t - \pi) + x \left(t - \frac{5\pi}{2} \right) = 0,$$

$$\frac{1}{1+c} [p_1(\tau_1 - r) + p_2(\tau_2 - r)] = \frac{1000}{1001} \left[\frac{1}{10^3} \frac{\pi}{2} + 2\pi \right] > \frac{1}{e},$$

that is, condition (16) is satisfied. Therefore, by Theorem 5.1, all solutions of this equation oscillate. For example, $\sin t$ and $\cos t$ are oscillatory solutions of the above equation. Note, however, that condition (17) is not satisfied.

EXAMPLE 5.3. (Only condition (17) is satisfied.) Consider the equation

$$\frac{d}{dt} [x(t) - 2x(t + 3)] - \frac{1}{4}x\left(t + \frac{5}{2}\right) - \frac{1}{9}x(t + 1) = 0.$$

Observe that

$$\frac{1}{1+c} \sqrt{p_1 p_2} [(\tau_1 - r) + (\tau_2 - r)] = \frac{5}{12} > \frac{1}{e},$$

that is, condition (17) is satisfied. Therefore all solutions of this equation oscillate. Note, however, that condition (16) is not satisfied.

EXAMPLE 5.4. (None of the conditions is satisfied.) The equation

$$\frac{d}{dt} [x(t) + x(t - 1)] + \frac{1}{e^3}x(t - 4) + \frac{1}{e^2}x(t - 2) = 0$$

has the nonoscillatory solution $x(t) = e^{-t}$, while the equation

$$\frac{d}{dt} [x(t) - 3e^3x(t - 3)] + ex(t - 1) + e^2x(t - 2) = 0$$

has the nonoscillatory solution $x(t) = e^t$. As expected none of the conditions of Theorem 5.1 is satisfied for these equations.

6. Generalizations

In this section we generalize our results to differential equations with variable coefficients of the form

$$(1) \quad \frac{d}{dt} [x(t) + cx(t - r)] + \sum_{i=1}^k P_i(t)x(t - \tau_i) = 0,$$

$$(2) \quad \frac{d}{dt} [x(t) + cx(t + r)] - \sum_{i=1}^k P_i(t)x(t + \tau_i) = 0,$$

$$(3) \quad \frac{d}{dt} [x(t) + cx(t - r)] + \sum_{i=1}^k P_i(t)x(t - \tau_i) + \sum_{j=1}^{\ell} Q_j(t)x(t + \sigma_j) = 0,$$

$$(4) \quad \frac{d}{dt} [x(t) + cx(t + r)] - \sum_{i=1}^k P_i(t)x(t + \tau_i) - \sum_{j=1}^{\ell} Q_j(t)x(t - \sigma_j) = 0$$

where

$$(C) \quad c \in \mathbf{R}, r \in (0, \infty), \tau_i, \sigma_j \in [0, \infty) \text{ and} \\ P_i, Q_j \in C([t_0, \infty), \mathbf{R}^+) \text{ for } i = 1, \dots, k; j = 1, \dots, \ell.$$

The following lemma is the variable coefficient analogue of Lemma 1.3 and it can be derived by combining results in [16, 18, 22, 23].

LEMMA 6.1. *Consider the inequalities*

$$(12') \quad \dot{x}(t) + \sum_{i=1}^n P_i(t)x(t - \tau_i) \leq 0$$

and

$$(13') \quad \dot{x}(t) - \sum_{i=1}^n P_i(t)x(t + \tau_i) \geq 0$$

where τ_i are positive constants and P_i are positive and continuous functions for $i = 1, \dots, n$. Then any of the following conditions

$$(14') \quad \liminf_{t \rightarrow \infty} \sum_{i=1}^n \tau_i P_i(t) > \frac{1}{e}$$

or

$$(15') \quad \left[\prod_{i=1}^n \left(\sum_{j=1}^n \liminf_{t \rightarrow \infty} \int_{t-\tau_j}^t P_i(s) ds \right) \right]^{1/n} > \frac{1}{e}$$

implies that (12)' and (13)' have no eventually positive solutions.

The proofs of Theorems 2.1, 3.1 and 4.1 are given in such a way that they can be extended in a straightforward manner to the case of differential equations (1)'–(4)'.

The following theorems provide sufficient conditions for all solutions of (1)', (2)', (3)' and (4)' to oscillate.

THEOREM 6.1. *Assume that $-1 < c < 0$. Then any of the following conditions*

$$(23) \quad \liminf_{t \rightarrow \infty} \sum_{i=1}^k \tau_i P_i(t) > \frac{1}{e}$$

or

$$(24) \quad \left[\prod_{i=1}^k \left(\sum_{j=1}^k \liminf_{t \rightarrow \infty} \int_{t-\tau_j}^t P_i(s) ds \right) \right]^{1/k} > \frac{1}{e}$$

implies that all solutions of (1)', (2)', (3)' and (4)' oscillate.

PROOF. We present the proof for Eq. (1)'. The proof for equations (2)', (3)' and (4)' can be treated in a similar fashion. To this end suppose that there exists an eventually positive solution $x(t)$ of (1)'. Set

$$z(t) = x(t) + cx(t - r).$$

Then, by (1)',

$$\dot{z}(t) = -\sum_{i=1}^k P_i(t)x(t - \tau_i) < 0$$

which implies that $z(t)$ is decreasing and, as in Lemma 1.2, we can easily show that $z(t)$ is eventually positive. Furthermore $z(t) < x(t)$ and therefore

$$\dot{z}(t) + \sum_{i=1}^k P_i(t)z(t - \tau_i) < 0.$$

In view of (23) or (24) and Lemma 6.1, the last inequality cannot have an eventually positive solution. This is a contradiction and the proof is complete.

In the sequel we will assume that there exist $p_i > 0, i = 1, \dots, k$ so that $P_i(t) \geq p_i$ eventually.

THEOREM 6.2. *Assume that $c < -1$ and $r > \tau_k$. Then any of the following conditions*

$$(25) \quad -\frac{1}{c} \liminf_{t \rightarrow \infty} \left(\sum_{i=1}^k (r - \tau_i) P_i(t) \right) > \frac{1}{e}$$

or

$$(26) \quad -\frac{1}{c} \left[\prod_{i=1}^k \left(\sum_{i=1}^k \liminf_{t \rightarrow \infty} \int_t^{t+(r-\tau_i)} P_i(s) ds \right) \right]^{1/k} > \frac{1}{e}$$

implies that all solutions of (1)', (2)', (3)' and (4)' oscillate.

PROOF. We present the proof for Eq. (2)'. The other cases can be treated in a similar way. To this end suppose that there exists an eventually positive solution $x(t)$ of (2)'. Set

$$z(t) = -[x(t) + cx(t + r)].$$

Then, by (2)',

$$\dot{z}(t) = -\sum_{i=1}^k P_i(t)x(t + \tau_i) < 0$$

which implies that $z(t)$ is decreasing and as in Lemma 1.2, we can show that $z(t)$ is eventually positive. Moreover

$$z(t) = -x(t) - cx(t + r) < (-c)x(t + r).$$

From this inequality and Eq. (2)', we find that eventually

$$\dot{z}(t) + \left(\frac{1}{-c} \right) \sum_{i=1}^k P_i(t)z(t - (r - \tau_i)) < 0$$

and, in view of (25) or (26), we are led to a contradiction.

THEOREM 6.3. Assume that $-1 < c$, $r < \tau_1$ and P_i, Q_j for $i = 1, \dots, k$; $j = 1, \dots, \ell$ are r -periodic. Then any of the following conditions

$$(27) \quad \frac{1}{1+c} \liminf_{t \rightarrow \infty} \left(\sum_{i=1}^k (\tau_i - r) P_i(t) \right) > \frac{1}{e}$$

or

$$(28) \quad \frac{1}{1+c} \left[\prod_{i=1}^k \left(\sum_{j=1}^k \liminf_{t \rightarrow \infty} \int_{t-(\tau_j-r)}^t P_i(s) ds \right) \right]^{1/k} > \frac{1}{e}$$

implies that all solutions of (1)', (2)', (3)' and (4)' oscillate.

PROOF. We present the proof for Eq. (3)'. To this end suppose that there exists an eventually positive solution $x(t)$ of (3)'. Set $z(t)$ and $w(t)$ as in (6) and (7) and observe that since P_i, Q_j are r -periodic it follows that z and w are continuously differentiable solutions of (3)' and $w(t) > 0, \dot{w}(t) > 0$. Thus

$$\dot{w}(t) + c\dot{w}(t-r) + \sum_{i=1}^k P_i(t)w(t-\tau_i) + \sum_{j=1}^{\ell} Q_j(t)w(t+\sigma_j) = 0$$

or

$$\dot{w}(t) + c\dot{w}(t-r) + \sum_{i=1}^k P_i(t)w(t-\tau_i) < 0.$$

Since $\dot{w}(t)$ is increasing the last inequality yields

$$\dot{w}(t-r) + c\dot{w}(t-r) + \sum_{i=1}^k P_i(t)w(t-\tau_i) < 0$$

or

$$\dot{w}(t) + \frac{1}{1+c} \sum_{i=1}^k P_i(t)w(t-(\tau_i-r)) < 0.$$

In view of (27) or (28) and Lemma 6.1, this is impossible and the proof is complete.

THEOREM 6.4. Assume that $c < -1$, $r > \tau_k$ and P_i, Q_j for $i = 1, \dots, k$; $j = 1, \dots, \ell$ are r -periodic. Then any of the following conditions

$$(29) \quad -\frac{1}{1+c} \liminf_{t \rightarrow \infty} \left(\sum_{i=1}^k (r - \tau_i) P_i(t) \right) > \frac{1}{e}$$

or

$$(30) \quad -\frac{1}{1+c} \left[\prod_{i=1}^k \left(\sum_{j=1}^k \liminf_{t \rightarrow \infty} \int_t^{t+(r-\tau_j)} P_i(s) ds \right) \right]^{1/k} > \frac{1}{e}$$

implies that all solutions of (1)', (2)', (3)' and (4)' oscillate.

PROOF. In this theorem we present the proof for Eq. (4)'. To this end suppose that there exists an eventually positive solution $x(t)$ of (4)'. Set $z(t)$ and $w(t)$ as in (8) and (9) and observe that z and w are solutions of (4)' and moreover $w(t) > 0$, $\dot{w}(t) > 0$. Thus

$$\dot{w}(t) + c\dot{w}(t+r) - \sum_{i=1}^k P_i(t)w(t+\tau_i) - \sum_{j=1}^{\ell} Q_j w(t-\sigma_j) = 0$$

or

$$\dot{w}(t) + c\dot{w}(t+r) - \sum_{i=1}^k P_i(t)w(t+\tau_i) > 0$$

and since $\dot{w}(t)$ is increasing

$$\dot{w}(t) + \left(-\frac{1}{1+c}\right) \sum_{i=1}^k P_i(t)w(t-(r-\tau_i)) < 0$$

which in view of (29) or (30) leads to a contradiction. The proof is complete.

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*Department of Mathematics,
University of Ioannina
(Ioannina, Greece)*