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Asymptotic and integral equivalence of multivalued differential systems

Dedicated to Professor Marko Švec on the occasion of his seventieth birthday

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The aim of this paper is to study the asymptotic and (ψ, p)-integral equivalence of differential systems of the form

$$x'(t) \in A(t)x(t) + F(t, x(t), Sx(t)), \tag{a}$$

$$y'(t) = A(t)y(t), \tag{b}$$

where $A(t)$ is an $n \times n$ matrix-function defined on $J = [0, \infty)$ whose elements are integrable on compact subsets of J; x and y are n-dimensional vectors, S is a continuous operator mapping the set $B_\psi(J)$ of continuous and ψ-bounded functions defined on J to $B_\psi(J)$ in the sense that if $x_n \xrightarrow{q} x$ then $Sx_n \xrightarrow{q} Sx$ (precise definitions are given below) e.g.

$$Sx(t) := \int_0^t K(t, s)x(s)\, ds,$$

under certain conditions on the function $K(t, s)$, and $F(t, u, v)$ is a nonempty, compact and convex subset of R^n for each $(t, u, v) \in J \times R^n \times R^n$.

By a solution of (a), we mean an absolutely continuous function $x(t)$ on some nondegenerate subinterval of J which satisfies (a) almost everywhere (a.e.).

DEFINITION 1 (A. Haščák and M. Švec [10]). Let $\psi(t)$ be a positive continuous function on an interval $[t_0, \infty)$ and let $p > 0$. We shall say that two systems (a) and (b) are (ψ, p)-integral equivalent on $[t_0, \infty)$ iff for each solution $x(t)$ of (a) there exists a solution $y(t)$ of (b) such that

$$\psi^{-1}(t)|x(t) - y(t)| \in L_p([t_0, \infty)) \tag{c}$$

and conversely, for each solution $y(t)$ of (b) there exists a solution $x(t)$ of (a) such that (c) holds.

By a restricted (ψ, p)-integral equivalence between (a) and (b) we shall mean that the relation (c) is satisfied for some subsets of solutions of (a) and (b), e.g. for the ψ-bounded solutions.

We shall say that a function $z(t)$ is ψ-bounded on the interval $[t_0, \infty)$ iff

$$\sup_{t \geqq t_0} |\psi^{-1}(t) z(t)| < \infty .$$

REMARK 1. In [17], two examples are given which demonstrate that, in general, integral equivalence does not imply asymptotic equivalence, and conversely, asymptotic equivalence does not in general imply integral equivalence.

Now we shall define some notions and give preliminary results which will be needed in the sequel.

We shall write $|\cdot|$ for any convenient matrix (vector) norm. Let A be a subset of R^n. Then $|A| := \sup \{|a| : a \in A\}$. $L_p^n(J)$ will denote n-th Cartesian product of $L_p(J)$. $B(I)$ will denote the space of all continuous functions from $I := [t_0, \infty)$ to R^n. Let $\psi(t)$ be a positive continuous function on $[t_0, \infty)$. For $z \in B(I)$, we denote

$$|z|_\psi := \sup_{t \geqq t_0} |\psi^{-1}(t) z(t)| .$$

Let $B_\psi(J) := \{z \in B(J) : |z|_\psi < \infty\}$. Then $B_\psi(J)$ with the norm $|\cdot|_\psi$ is a Banach space. For $\rho > 0$, we denote

$$B_{\psi,\rho}(J) := \{z \in B(J) : |z|_\psi \leqq \rho\} .$$

Further, let $\varphi(t)$ be a positive continuous function defined on J. By $L_{p,\varphi}(J)$ $(1 \leqq p < \infty)$ we shall denote the set of all real-valued measurable functions $y(t)$ defined on J such that

$$|y|_{p,\varphi} := \left(\int_0^\infty |\varphi^{-1}(s) y(s)|^p \, ds \right)^{1/p} < \infty .$$

$L_{p,\varphi}(J)$ with the norm $|\cdot|_{p,\varphi}$ is also a Banach space.

It is easy to prove the following lemma.

LEMMA 1. *Let $g: J \times J \to J$ be a function such that*

i) *$g(t, x)$ is monotone nondecreasing in x for each fixed $t \in J$;*

ii) *$g(t, c) \in L_{p'}(J)$ for each $c \geqq 0$.*

Then the set

$$L_{p',\varphi,g}(J) := \{y \in L_{p',\varphi}(J): \text{there are nonnegative constants } c, K \text{ such that } |y(t)| \leqq K\varphi(t) g(t, c) \text{ a.e. on } J\}$$

is a linear subspace of $L_{p',\varphi}(J)$.

DEFINITION 2. Let a function g fulfil the hypothesis i) of Lemma 1. A set

A of functions which are defined on J is g-bounded iff there are two nonnegative constants c and K such that

$$|y(t)| \leqq K\varphi(t)g(t, c) \qquad \text{a.e. on } J$$

for each $y \in A$.

COROLLARY 1. *Let a function g fulfil the hypotheses of Lemma 1. Then each g-bounded set A is also bounded in the space $L_{p',\varphi}(J)$.*

The converse of Corollary 1 is not true: boundedness in $L_{p',\varphi}(J)$ does not imply g-boundedness, as the following example shows.

EXAMPLE 1. Let $\varphi(t) \equiv 1$ and $g(t, x) = e^{-t}$, $t \in J$. It is easy to see that φ and g fulfil the assumptions of Lemma 1. Then the set $\{y_n\}$ of functions

$$y_n(t) := \begin{cases} 1, & t \in A_n := \bigcup_{k=1}^{n} [k - 2^{-k}, k] \\ 0, & t \in R - A_n \end{cases}$$

is bounded in $L_{p'}(J)$, $p' \in [1, \infty)$, but it is not g-bounded.

Let X and Y be topological spaces. Let us denote by 2^Y the family of all nonempty subsets of the space Y and let $\mathrm{cf}(Y)$ be the set of all nonempty closed and convex subsets of Y.

DEFINITION 3 (C. Berge [2]). A mapping $F: X \to 2^Y$ is upper semicontinuous at a point $x \in X$ iff for an arbitrary neighbourhood $O_{F(x)}$ of the set-image $F(x)$ there exists such a neighbourhood O_x of the point x that $F(O_x) \subset O_{F(x)}$, where

$$F(O_x) := \bigcup_{z \in O_x} F(z).$$

This mapping is said to be upper semicontinuous iff it is upper semicontinuous at each point $x \in X$.

DEFINITION 4 (W. Sobieszek [14]). A mapping $F: X \to 2^Y$ is upper semicompact (sequentially upper semicontinuous) at the point $x \in X$ iff from the assumptions $x_n \to x$, $x_n \in X$, $y_n \in F(x_n)$ it follows that there exists a subsequence of the sequence $\{y_n\}$ which converges to some $y \in F(x)$.

DEFINITION 4′ (W. Sobieszek and P. Kowalski [15]). A mapping F is said to be upper semicompact at a point $x \in X$ iff it is upper semicontinuous at the point x and the set $F(x)$ is compact.

It turns out (see Theorem 1 and Corollary 2 below) that Definitions 4 and 4′ are equivalent under some additional assumptions.

THEOREM 1. *Let X fulfil the first axiom of countability and let Y be such that compactness and sequential compactness are equivalent in Y.*

Then F is upper semicompact at a point $x \in X$ in the sense of Definition 4 if and only if it is upper semicompact at x in the sense of Definition 4′.

Theorem 1 is a generalization of Theorem 4 of [15], but its proof is formally the same. The hypotheses of Theorem 4 of [15] will not be fulfilled in our case and thus it is not applicable in this case.

COROLLARY 2. *Let X and Y be metric spaces. Then F is upper semicompact at a point $x \in X$ in the sense of Definition 4 if and only if it is upper semicompact at x in the sense of Definition 4′.*

Let $I \subset R$ be an arbitrary interval (finite or infinite). By $B_0(I)$ we shall denote the Banach space of all bounded continuous functions on I with the norm

$$|f|_{B_0(I)} := \sup_{x \in I} |f(x)| .$$

DEFINITION 5 (M. Švec [16]). A sequence $f_k \in B_0(I)$ quasi-converges (q-converges) to $f \in B_0(I)$ iff

$$\lim_{k \to \infty} f_k(x) = f(x) \qquad \text{for every } x \in I .$$

This convergence will be denoted by $f_k \xrightarrow{q} f$.

DEFINITION 6. A set $M \subset B_0(I)$ is said to be q-closed iff, for $f_k \in M$, $f_k \xrightarrow{q} f$ implies $f \in M$.

DEFINITION 7 (A. Haščák [6]). Let Y be a normed linear space. An operator $T: B_0(I) \to 2^Y$ is upper q-continuous iff from the assumptions $f_k \xrightarrow{q} f$, f_k, $f \in B_0(I)$, $y_k \in T(f_k)$ it follows that there exists a subsequence of $\{y_k\}$ converging to some $y \in T(f)$ (in the norm of Y).

COROLLARY 3. *If T is upper q-continuous, then T is upper semicompact (and hence it is upper semicontinuous).*

DEFINITION 8 (A. Haščák [6]). An operator $T: B_0(I) \to 2^Y$ is weakly upper q-continuous iff from the assumptions $f_k \xrightarrow{q} f$, f_k, $f \in B_0(I)$, $y_k \in T(f_k)$ it follows that there exists a subsequence of $\{y_k\}$ converging weakly to some $y \in T(f)$.

THEOREM 2 (A. Haščák [6]). *Suppose that $D \subset B_0(I)$ is a nonempty, convex and q-closed set and $T: D \to \mathrm{cf}(D)$ is an upper q-continuous operator such that TD is a uniformly bounded set of functions which are equicontinuous on every compact subinterval of I. Then there is a point $x \in D$ such that $x \in Tx$.*

THEOREM 3 (A. Haščák [9]). *Let $w\text{-}\lim_{n\to\infty} x_n = x_0$ (i.e. $x_n \to x_0$ weakly) in $L_1([a, \infty))$ and let there exist a function $g \in L_1([a, \infty))$ such that*

$$|x_n(t)| \leqq g(t) \qquad \text{a.e. on } [a, \infty),\ n = 1, 2, \dots .$$

Then there exists a subsequence $\{x_{1n}\}$ of the sequence $\{x_n\}$ such that

$$\frac{1}{k}(x_{11} + x_{12} + \cdots + x_{1k})$$

converges to x_0 in the norm of $L_1([a, \infty))$.

LEMMA 2 (A. Haščák [7]). *Let $p \geqq 1$ and $f(t)$ be a nonnegative function for $t \geqq 0$. Then*

$$\left(\int_0^\infty \left(\int_t^\infty f(s)\,ds\right)^p dt\right)^{1/p} \leqq \int_0^\infty s^{1/p} f(s)\,ds .$$

LEMMA 2′ (A. Haščák and M. Švec [10]). *Let $g(t) \geqq 0$ be continuous on $0 \leqq t < \infty$ and such that*

$$\int_0^\infty s g(s)\,ds < \infty .$$

Then

$$\int_t^\infty g(s)\,ds \in L_p([0, \infty)) , \qquad p \geqq 1 .$$

LEMMA 3 (A. Haščák [8]). *Let $K \subset L_1([t_0, \infty))$ and suppose that there exists $g: [t_0, \infty) \to [0, \infty)$, $g \in L_1([t_0, \infty))$ such that for each $f \in K$*

$$|f(t)| \leqq g(t) \qquad \text{a.e. on } [t_0, \infty) .$$

Then K is weakly relative compact in $L_1([t_0, \infty))$.

LEMMA 4 (A. Haščák and M. Švec [10]). *Let $\psi(t)$ and $\varphi(t)$ be positive functions for $t \geqq 0$, $Y(t)$ a nonsingular matrix and P a projection. Further suppose that*

$$\left(\int_0^t |\psi^{-1}(t) Y(t) P Y^{-1}(s) \varphi(s)|^p\,ds\right)^{1/p} \leqq K$$

for $t \geqq 0$, $K > 0$, $p > 0$ *and*

$$\int_0^\infty \exp\left(-K^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\,ds\right)dt < \infty .$$

Then

$$\lim_{t\to\infty} \psi^{-1}(t)|Y(t)P| = 0$$

and

$$|\psi^{-1}(t)Y(t)P| \in L_p([0, \infty)) .$$

Now we shall prove the following theorem.

THEOREM 4. *Let* $\psi(t)$ *and* $\varphi(t)$ *be positive continuous functions on* $J := [0, \infty)$ *and let the mapping* $F\colon J \times R^n \times R^n \to \mathrm{cf}(R^n)$ *satisfy the following conditions*:

(c_0) $F(t, u, v)$ *is a nonempty, compact and convex subset of* R^n *for each* $(t, u, v) \in J \times R^n \times R^n$;

(c_1) *for every fixed* $t \in J$, *the function* $F(t, u, v)$ *is upper semicontinuous*;

(c_2) *for each* $x \in B_\psi(J)$ *there exists a measurable function* $f_x\colon J \to R^n$ *such that* $f_x(t) \in F(t, x(t), Sx(t))$ *a.e. on* J;

(c_3) *there is a constant* $k \in (0, \infty)$ *such that* $|Sz|_\psi \leqq k|z|_\psi$, $z \in B_\psi(J)$.

Further suppose that there exists $g\colon J \times J \times J \to J$ *such that*

i) $g(t, u, v)$ *is monotone nondecreasing in* u *for each fixed* $t \in J$, $v \in J$, *and monotone nondecreasing in* v *for each fixed* $t \in J$, $u \in J$;

ii) $g(t, c, c) \in L_{p'}(J)$ *for any constant* $c \geqq 0$ *and some* $p' \in [1, \infty)$;

iii) *for each* $u, v \in R^n$ $|F(t, u, v)| \leqq \varphi(t)g(t, \psi^{-1}(t)|u|, \psi^{-1}(t)|v|)$ *a.e. on* J.

Given a function $x \in B_\psi(J)$ *denote by* $M(x)$ *the set of all measurable functions* $y\colon J \to R^n$ *such that*

$$y(t) \in F(t, x(t), Sx(t)) \qquad \text{a.e. on } J .$$

Then the correspondence $x \to M(x)$ *defines a bounded mapping of* $B_{\psi,p}(J)$ *into* $\mathrm{cf}\,(L^n_{p',\varphi,g}(J))$.

PROOF. We have to show that for every $x \in B_{\psi,p}(J)$ (a) $M(x)$ is nonempty; (b) $M(x)$ is convex; (c) $M(x)$ is closed; (d) $M(x) \subset L^n_{p',\varphi,g}(J)$; (e) for every $\delta > 0$ there is a constant $K > 0$ such that $|x|_\psi \leqq \delta$ implies $|y|_{p',\varphi} \leqq K$ for every $y \in M(x)$.

The statements (a) and (b) are trivial. (e) follows from assumptions (ii) and (iii) and obviously implies (d). Thus we have to prove (c) only. Let $\{y_n\}$, $y_n \in M(x)$ be a sequence such that $|y_n - y|_{p',\varphi} \to 0$ as $n \to \infty$, $p' \in [1, \infty)$. By the Riesz Theorem there is a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ such that $\{y_{1n}(t)\}$ converges a.e. on J to $y(t)$ as $n \to \infty$. On the other hand

$$y_{1n}(t) \in F(t, x(t), Sx(t)) \qquad \text{a.e. on } J .$$

Because of (c_0)

$$y(t) \in F(t, x(t), Sx(t)) \qquad \text{a.e. on } J .$$

Thus $y \in M(x)$ and the proof of Theorem 4 is complete.

THEOREM 5. *Let the hypotheses of Theorem 4 be satisfied. Then the mapping* $M\colon B_{\psi,\rho}(J) \to \mathrm{cf}\,(L^n_{p',\varphi,g}(J))$ *is weakly upper q-continuous.*

PROOF. Let $x_n \xrightarrow{q} x$, x_n, $x \in B_{\psi,\rho}(J)$ and $y_n \in M(x_n)$. The existence of a subsequence $\{y_{1n}\}$ of the sequence $\{y_n\}$ which converges weakly to some $y \in L^n_{p',\varphi}(J)$ is implied by Lemma 3 in the case $p' = 1$. For $p' > 1$ it follows from

$$|y_n|_{p',\varphi} \leqq C := \left(\int_0^\infty g^{p'}(s, c, c)\, ds\right)^{1/p'} < \infty ,$$

where $c = \max(\rho, k\rho)$. Thus we have only to prove that $y \in M(x)$. By Theorem 3 (in the case $p' = 1$) or by the Banach-Saks Theorem (in the case $p' > 1$), there is a subsequence $\{y_{2n}\}$ of $\{y_{1n}\}$ such that

$$\left|\frac{1}{n}\sum_{k=1}^{n} y_{2k} - y\right|_{p',\varphi} \to 0 \qquad \text{as} \qquad n \to \infty .$$

Now, by the Riesz Theorem, there is a sequence $\{\sigma_n\}$, $\sigma_n \in N$, $\sigma_n \geqq n$ such that

$$\frac{1}{\sigma_n}\sum_{k=1}^{\sigma_n} y_{2k}(t) \to y(t) \qquad \text{a.e. on } J \qquad \text{for } n \to \infty .$$

On the other hand, by the assumption (c_1), for almost every fixed $t \in J$ and any $\varepsilon > 0$ there is an integer $N(\varepsilon, t)$ such that

$$\begin{aligned} &F(t, x_i(t), Sx_i(t)) \subset F(t, x(t), Sx(t)) + K_\varepsilon \\ &\quad := \{u + v\colon u \in F(t, x(t), Sx(t)), |v| \leqq \varepsilon\} \qquad \text{for } i \geqq N(\varepsilon, t) . \end{aligned}$$

Thus

$$y_{2k}(t) \in F(t, x(t), Sx(t)) + K_\varepsilon,\ 2k \geqq N(\varepsilon, t)$$

and by the convexity of $F(t, x(t), Sx(t))$ we get

$$\frac{1}{\sigma_n}\sum_{k=1}^{\sigma_n} y_{2k}(t) \in F(t, x(t), Sx(t)) + K_\varepsilon,\ 2\sigma_n \geqq N(\varepsilon, t) ,$$

so that

$$y(t) \in F(t, x(t), Sx(t)) \quad \text{a.e. on } J\,.$$

The proof is complete.

DEFINITION 8. An operator T mapping a subset A of $L^n_{p,\varphi}(J)$ into a Banach space Y is said to be g-compact iff it is continuous and maps each g-bounded subset of A into a relatively compact set in Y.

COROLLARY 4. *Let a function g fulfil the hypotheses of Lemma 1. Then each compact operator is g-compact. The converse is not true.*

COROLLARY 5. *The identity operator $I\colon L_2(J) \to L_2(J)$ is not e^{-t}-compact as the following example shows.*

EXAMPLE 2. Since

$$|y_k(t)| = |e^{-t} \sin kt| \leqq e^{-t} =: g(t) \qquad \text{for } t \in J$$

and $k = 1, 2, \dots$, the set $\{e^{-t} \sin kt\colon k = 1, 2, \dots\}$ is e^{-t}-bounded in $L_2(J)$. However, no subsequence of the sequence $\{y_k\}$ converges to any $y \in L_2(J)$ in the norm of $L_2(J)$ since

$$\begin{aligned} |y_m - y_n|_2^2 &= \int_0^\infty e^{-2t}(\sin mt - \sin nt)^2\, dt \\ &> e^{-4\pi} \int_0^{2\pi} (\sin mt - \sin nt)^2\, dt = 2\pi e^{-4\pi} \qquad \text{for } m \neq n\,. \end{aligned}$$

THEOREM 6. *Let the hypotheses of Theorem 4 be satisfied and D be a Banach space. Suppose that $T\colon L^n_{p',\varphi}(J) \to D$ is a g-compact linear operator (g satisfies i) of Theorem 4). Then the operator TM defined by*

$$TMx := \{z \in D : z = Ty \text{ and } y \in M(x)\}$$

maps $B_{\psi,\rho}(J)$ into cf (D) *and is upper q-continuous.*

PROOF. First we shall prove that the operator TM is upper q-continuous. Let $x_n \xrightarrow{q} x$, x_n, $x \in B_{\psi,\rho}(J)$ and $z_n \in TMx_n$. We have to show that there is a subsequence of the sequence $\{z_n\}$ that converges (in the norm of D) to some $z \in TMx$. Let $z_i = Ty_i$, $y_i \in M(x_i)$. Since M is weakly upper q-continuous (by Theorem 5), there is a subsequence $\{y_{1i}\}$ of $\{y_i\}$ which converges weakly to some $y \in M(x)$. Since $\{y_{1i}\colon i = 1, 2, \dots\} \subset \bigcup_{x \in B_{\psi,\rho}(J)} M(x)$ is a g-bounded set and T is a g-compact operator, there is a subsequence $\{y_{2i}\}$ of $\{y_{1i}\}$ such that

$Ty_{2i} \to z \in D$ as $i \to \infty$. We shall show that $z = Ty \in TMx$. Because $\{y_{1i}\}$ converges weakly to y, we infer that also $\{y_{2i}\}$ converges weakly to y. By Theorem 3 (in the case $p' = 1$) or by the Banach-Saks Theorem (in the case $p' > 1$) there is a subsequence $\{y_{3i}\}$ of the sequence $\{y_{2i}\}$ such that

$$\frac{y_{31} + y_{32} + \cdots + y_{3i}}{i} \to y \qquad \text{as} \qquad i \to \infty$$

in the norm of $L^n_{p',\varphi}(J)$. Since T is g-compact, T is continuous and we have

$$(*) \qquad T\left(\frac{y_{31} + y_{32} + \cdots + y_{3i}}{i}\right) \to Ty \qquad \text{as} \qquad i \to \infty$$

On the other hand, since $Ty_{3i} \to z \in D$ and T is linear, we have

$$(**) \qquad z = \lim_{i\to\infty} Ty_{3i} = \lim_{i\to\infty} \frac{Ty_{31} + Ty_{32} + \cdots + Ty_{3i}}{i}$$
$$= \lim_{i\to\infty} T\left(\frac{y_{31} + y_{32} + \cdots + y_{3i}}{i}\right).$$

By $(*)$ and $(**)$ we infer that $z = Ty \in TMx$. Thus the operator TM is upper q-continuous. From this we conclude that TMx is closed. Further, $M(x)$ is a convex set and T is a linear operator. Thus TMx is also a convex set.

COROLLARY 5. *Let the hypotheses of Theorem 4 be satisfied and D be a Banach space. Suppose that $T: L^n_{p',\varphi}(J) \to D$ is a g-compact linear operator. Then the operator TM maps $B_{\psi,\rho}(J)$ into* cf (D) *and is upper semicompact (upper semicontinuous).*

REMARK 2. There are a few papers on asymptotic and integral equivalence of differential systems which are based on a theorem similar to Theorem 6 with only linearity hypothesis about T (but not compactness). However, such a theorem is not valid, as shown by the following example.

EXAMPLE 3. Let the mapping $F: [0, \infty) \times (-\infty, \infty) \to \mathrm{cf}((-\infty, \infty))$ be defined by the formula

$$F(t, x) := [-e^{-t}, e^{-t}].$$

It is easy to see that F fulfils all the assumptions of Theorem 4 (with $n = 1$, for arbitrary number $p' \geqq 1$, $\psi(t) = \varphi(t) \equiv 1$ and $g(t, u) = e^{-t}$) and thus the mapping $M: B_{1,\rho}(J) \to \mathrm{cf}(L_{p',1}(J))$ is bounded and upper q-continuous.

Now suppose that $p' = 2$ and that $T: L_2(J) \to L_2(J)$ is the identity operator (thus T is a linear operator but T is neither compact nor e^{-t}-compact (see

Corollary 5)). However, the operator $TM: B_{1,\rho}(J) \to \mathrm{cf}\,(L_2(J))$ is neither upper q-continuous nor upper semicompact. In fact, if $f_k \xrightarrow{q} f$, f_k, $f \in B_{1,\rho}(J)$ (or $f_k \to f$, f_k, $f \in B_{1,\rho}(J)$ respectively), then $y_k := e^{-t} \sin kt \in TMf_k$ and no subsequence of the sequence $\{y_k\}$ converges to any $y \in TMf$ in the norm of $L_2(J)$ (see Example 2).

LEMMA 5. *Let the function g fulfil the hypotheses of Lemma* 1. *Let $Y(t)$ be a continuous matrix for $t \geqq 0$ with* $\det Y(t) \neq 0$ *for each $t \geqq 0$ and P be an $n \times n$ constant matrix with $P^2 = P$.*

Suppose that there exist constants $K_1 > 0$ and $1 < p < \infty$ such that

$$\text{(v)} \qquad \left(\int_0^t |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)|^p\, ds\right)^{1/p} \leqq K_1 \qquad \textit{for all } t \geqq 0\,;$$

$$\text{(c)} \qquad \int_0^\infty \exp\left\{-K_1^p \int_0^t \varphi^p(s)\psi^{-p}(s)\right\} dt < \infty\,;$$

$$\text{(d)} \qquad \int_0^\infty |PY^{-1}(s)\varphi(s)|g(s, c)\, ds < \infty \qquad \textit{for any constant } c \geqq 0\,.$$

Then the linear operator $T_1: L^n_{p',\varphi}(J) \to B_\psi(J)$, $(1/p) + (1/p') = 1$ *defined by the formula*

$$(T_1 y)(t) := \int_0^t Y(t)PY^{-1}(s)y(s)\, ds$$

is g-compact.

PROOF. For each $y \in L^n_{p',\varphi}(J)$ we have

$$\begin{aligned} |T_1 y|_\psi &:= \sup_{t \geqq 0} |\psi^{-1}(t)(T_1 y)(t)| \\ &\leqq \sup_{t \geqq 0} \int_0^t |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)|\,|\varphi^{-1}(s)y(s)|\, ds \\ &\leqq K_1\left(\int_0^\infty |\varphi^{-1}(s)y(s)|^{p'}\, ds\right)^{1/p'} = K_1|y|_{p',\varphi}\,, \end{aligned}$$

which implies that T_1 is bounded and hence continuous on $L^n_{p',\varphi}(J)$. Further, take any g-bounded sequence $\{y_k\}$ from $L^n_{p',\varphi}(J)$. We have to show that the sequence $\{T_1 y_k\}$ contains a subsequence which is convergent in $B_\psi(J)$. Let

$$z_i(t) := \int_0^t Y(t)PY^{-1}(s)y_i(s)\, ds\,, \qquad i = 1, 2, \ldots.$$

Since $\{y_i: i = 1, 2, \dots\}$ is g-bounded subset of $L^n_{p',\varphi}(J)$, $p' \in (1, \infty)$ there is a subsequence $\{y_{1i}\}$ of $\{y_i\}$ which converges weakly to an element $y \in L^n_{p',\varphi}(J)$, i.e.

$$z_{1i}(t) := (T_1 y_{1i})(t) \to (T_1 y)(t) := \int_0^t Y(t) P Y^{-1}(s) y(s)\, ds =: z(t)\,.$$

Further, there are nonnegative constants c and K such that

$$(1) \qquad |y_{1i}(t)| \leqq K\varphi(t) g(t, c) \qquad \text{a.e. on } J, \qquad i = 1, 2, \dots .$$

Using this fact, the Hölder inequality and (v), we have

$$|\psi^{-1}(t) z_{1i}(t)| \leqq K_1 K \left(\int_0^\infty g^{p'}(s, c)\, ds \right)^{1/p'}, \qquad i = 1, 2, \dots .$$

The the functions z_{1i}, $i = 1, 2, \dots$ are uniformly ψ-bounded, and from the inequalities holding for $0 \leqq t_1 \leqq t_2$

$$\begin{aligned}
&|\psi^{-1}(t_2) z_{1i}(t_2) - \psi^{-1}(t_1) z_{1i}(t_1)| \\
&\leqq \int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2) P Y^{-1}(s) \varphi(s)|\, |\varphi^{-1}(s) y_{1i}(s)|\, ds \\
&\qquad + |\psi^{-1}(t_2) Y(t_2) - \psi^{-1}(t_1) Y(t_1)|\, |P|\, |\psi(t_1) Y^{-1}(t_1)| \\
&\cdot \int_0^{t_1} |\psi^{-1}(t_1) Y(t_1) P Y^{-1}(s) \varphi(s)|\, |\varphi^{-1}(s) y_{1i}(s)|\, ds \\
&\leqq \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2) P Y^{-1}(s) \varphi(s)|^p\, ds \right)^{1/p} \left(\int_{t_1}^{t_2} |\varphi^{-1}(s) y_{1i}(s)|^{p'}\, ds \right)^{1/p'} \\
&\qquad + |\psi^{-1}(t_2) Y(t_2) - \psi^{-1}(t_1) Y(t_1)|\, |P|\, |\psi(t_1) Y^{-1}(t_1)| \\
&\cdot \left(\int_0^{t_1} |\psi^{-1}(t_1) Y(t_1) P Y^{-1}(s) \varphi(s)|^p\, ds \right)^{1/p} \left(\int_0^{t_1} |\varphi^{-1}(s) y_{1i}(s)|^{p'}\, ds \right)^{1/p'} \\
&\leqq K \left(\int_0^\infty g^{p'}(s, c)\, ds \right)^{1/p'} \left(\int_{t_1}^{t_2} |\psi^{-1}(t_2) Y(t_2) P Y^{-1}(s) \varphi(s)|^p\, ds \right)^{1/p} \\
&\qquad + K K_1 \left(\int_0^\infty g^{p'}(s, c)\, ds \right)^{1/p'} |\psi(t_1) Y^{-1}(t_1)|\, |\psi^{-1}(t_2) Y(t_2) - \psi^{-1}(t_1) Y(t_1)|
\end{aligned}$$

it follows that the functions $\psi^{-1}(t) z_{1i}(t)$, $i = 1, 2, \dots$ are equicontinuous on every compact subinterval of J. By the Ascoli theorem as well as by Cantor's diagonalization process, the sequence $\{z_{1i}\}$ contains a subsequence $\{z_{2i}\}$ such that $\{\psi^{-1}(t) z_{2i}(t)\}$ is uniformly convergent on every compact subinterval of J.

This fact together with the inequality

$$|\psi^{-1}(t)z_{2i}(t)| \leqq |\psi^{-1}(t)Y(t)P|K\int_0^\infty |PY^{-1}(s)\varphi(s)|g(s,c)\,ds$$

(note that, by Lemma 4, $|\psi^{-1}(t)Y(t)P| \to 0$ as $t \to \infty$ and use d)) guarantees the convergence of $\{z_{2i}\}$ on J in the norm of $B_\psi(J)$.

COROLLARY 6. *Let $p = \infty$ (and $p' = 1$). Let the conditions* (v) *and* (c) *of Lemma* 5 *be replaced by*

$$\sup_{0 \leqq s \leqq t} |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)| \leqq K_2$$

and

$$\lim_{t\to\infty} |\psi^{-1}(t)Y(t)P| = 0$$

and let the other assumptions of Lemma 5 *hold. Then the linear operator $T_2\colon L^n_{1,\varphi}(J) \to B_\psi(J)$ defined by the formula*

$$(T_2y)(t) := \int_0^t Y(t)PY^{-1}(s)y(s)\,ds$$

is g-compact.

LEMMA 6. *Let the function g fulfil the hypotheses of Lemma* 1. *Let $Y(t)$ be a continuous matrix for $t \geqq 0$ with* det $Y(t) \neq 0$ *for each $t \geqq 0$ and P be an $n \times n$ constant matrix with $P^2 = P$.*

If there exist constants $K_3 > 0$ and $1 < p < \infty$ such that

$$\text{(vi)} \qquad \left(\int_t^\infty |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)|^p\,ds\right)^{1/p} \leqq K_3 \quad \text{for all } t \geqq 0\,,$$

then the linear operator $T_3\colon L^n_{p',\varphi}(J) \to B_\psi(J)$, $(1/p) + (1/p') = 1$, defined by the formula

$$(T_3y)(t) := \int_t^\infty Y(t)PY^{-1}(s)y(s)\,ds$$

is g-compact.

PROOF. The proof of Lemma 6 proceeds analogously to that of Lemma 5.

COROLLARY 7. *Let $p = \infty$ (and $p' = 1$). Let the condition* (vi) *of Lemma* 6 *be replaced by*

$$\sup_{t \leqq s < \infty} |\psi^{-1}(t)Y(t)PY^{-1}(s)\varphi(s)| \leqq K_4$$

and let the other assumptions of Lemma 6 *hold. Then the linear operator* $T_4\colon L^n_{1,\varphi}(J) \to B_\psi(J)$ *defined by the formula*

$$(T_4 y)(t) := \int_t^\infty Y(t)PY^{-1}(s)y(s)\,ds$$

is g-compact.

COROLLARY 8. *Let the mappings* $F_i\colon X \to 2^Y$, $i = 1, 2$ *be upper q-continuous. Then the mappings* $-F_i$ $(i = 1, 2)$ *and* $F_1 + F_2$ *are upper q-continuous.*

THEOREM 7. *Let* $\varphi(t)$ *and* $\psi(t)$ *be positive continuous functions for* $t \geqq 0$, $Y(t)$ *a fundamental matrix of* (b) *and let the hypotheses* (c_0), (c_1), (c_2) *and* (c_3) *of Theorem* 4 *be satisfied.*

Suppose that

a) *there exist supplementary projections* P_1, P_2 *and constants* $K > 0$ *and* $2 \leqq p < \infty$ *such that*

$$\int_0^t |\psi^{-1}(t)Y(t)P_1 Y^{-1}(s)\varphi(s)|^p\,ds + \int_t^\infty |\psi^{-1}(t)Y(t)P_2 Y^{-1}(s)\varphi(s)|^p\,ds \leqq K^p$$

for all $t \geqq 0$.

b) *there exists* $g\colon J \times J \times J \to J$ *such that*

(i) $g(t, u, v)$ *is monotone nondecreasing in* u *for each fixed* $t \in J$, $v \in J$; *monotone nondecreasing in* v *for each fixed* $t \in J$, $u \in J$ *and integrable on compact subsets of* J *for fixed* $u \in J$, $v \in J$;

(ii) $\int_0^\infty s^{p'/p} g^{p'}(s, c, c)\,ds < \infty$ *for any constant* $c \geqq 0$, *where* $(1/p) + (1/p') = 1$;

(iii) *for each* $u, v \in R^n |F(t, u, v)| \leqq \varphi(t)g(t, \psi^{-1}(t)|u|, \psi^{-1}(t)|v|)$ *a.e. on* J.

c) $\int_0^\infty \exp\{-K^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\,ds\}\,dt < \infty$.

d) $\int_0^\infty |P_1 Y^{-1}(s)\varphi(s)| g(s, c, c)\,ds < \infty$.

Then the set of ψ*-bounded solutions of* (a) *and of* (b) *are* (ψ, p)*-integral equivalent.*

PROOF. Let $y(t)$ be a ψ-bounded solution of (b) on $[t_0, \infty)$, $t_0 \geqq 0$. Then there is $\rho > 0$ such that $y \in B_{\psi,\rho}([t_0, \infty))$. Define for $x \in B_{\psi,2\rho}([t_0, \infty))$ the operator

$$TMx := \left\{z\colon z(t) := y(t) + \int_{t_0}^t Y(t)P_1 Y^{-1}(s)f_x(s)\,ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f_x(s)\,ds,\ f_x \in M(x)\right\}.$$

By Lemmas 5 and 6, Theorem 6 and Corollary 8 the operator TM maps

$B_{\psi,2\rho}([t_0,\infty))$ into $\mathrm{cf}(B_\psi([t_0,\infty))$ and is upper q-continuous on $B_{\psi,2\rho}([t_0,\infty))$. Further, for each $z \in TMx$, $x \in B_{\psi,2\rho}([t_0,\infty))$, we have

$$\begin{aligned}
|\psi^{-1}(t)z(t)| &\leqq |\psi^{-1}(t)y(t)| + \int_{t_0}^{t} |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|g(s,2\rho,2k\rho)\,ds \\
&\quad + \int_{t}^{\infty} |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|g(s,2\rho,2k\rho)\,ds \\
&\leqq \rho + \left(\int_{t_0}^{t} |\psi^{-1}(t)Y(t)P_1Y^{-1}(s)\varphi(s)|^p\,ds\right)^{1/p}\left(\int_{t_0}^{t} g^{p'}(s,2\rho,2k\rho)\,ds\right)^{1/p'} \\
&\quad + \left(\int_{t}^{\infty} |\psi^{-1}(t)Y(t)P_2Y^{-1}(s)\varphi(s)|^p\,ds\right)^{1/p}\left(\int_{t}^{\infty} g^{p'}(s,2\rho,2k\rho)\,ds\right)^{1/p'} \\
&\leqq \rho + K\left(\int_{t_0}^{\infty} g^{p'}(s,2\rho,2k\rho)\,ds\right)^{1/p'} \\
&\leqq \rho + K\left(\int_{t_0}^{\infty} g^{p'}(s,c,c)\,ds\right)^{1/p'}, \qquad \text{where} \qquad c = \max(2\rho, 2k\rho).
\end{aligned}$$

If we choose t_0 so that

$$K\left(\int_{t_0}^{\infty} g^{p'}(s,c,c)\,ds\right)^{1/p'} \leqq \rho,$$

then we see that TM maps $B_{\psi,2\rho}([t_0,\infty))$ into $\mathrm{cf}(B_{\psi,2\rho}([t_0,\infty))$. The functions in $TB_{\psi,2\rho}([t_0,\infty))$ are evidently uniformly bounded for each $t \geqq t_0$ because $TMB_{\psi,2\rho} \subset \mathrm{cf}(B_{\psi,2\rho})$.

Let $x \in B_{\psi,2\rho}([t_0,\infty))$ and $z \in TMx$. Then there is $f_x \in M(x)$ such that

$$z'(t) = A(t)z(t) + f_x(t) \qquad \text{a.e. on } [t_0,\infty).$$

Therefore by (iii) of b) we have for $t_0 \leqq t_1 \leqq t_2$

$$\begin{aligned}
|z(t_2) - z(t_1)| &\leqq \int_{t_1}^{t_2} |A(s)||z(s)|\,ds + \int_{t_1}^{t_2} |f_x(s)|\,ds \\
&\leqq 2\rho\int_{t_1}^{t_2} \psi(s)|A(s)|\,ds + \int_{t_1}^{t_2} \varphi(s)g(s,c,c)\,ds.
\end{aligned}$$

Thus the functions in $TMB_{\psi,2\rho}([t_0,\infty))$ are equicontinuous on every compact subinterval of $[t_0,\infty)$. Then Theorem 2 ensures the existence of $x \in B_{\psi,2\rho}([t_0,\infty))$ such that $x \in TMx$. Clearly this fixed point $x(t)$ is a ψ-bounded solution of (a).

Conversely, let $x(t)$ be a ψ-bounded solution of (a). Define

$$y(t) := x(t) - \int_{t_0}^{t} Y(t)P_1 Y^{-1}(s)f_x(s)\,ds + \int_{t}^{\infty} Y(t)P_2 Y^{-1}(s)f_x(s)\,ds\,,$$

where

$$f_x(t) := x'(t) - A(t)x(t) \in F(t, x(t), Sx(t)) \qquad \text{a.e. on } [t_0, \infty)\,.$$

It is easy to prove that $y(t)$ is a ψ-bounded solution of (b). It remains to prove that

$$\psi^{-1}(t)|x(t) - y(t)| \in L_p([t_0, \infty))\,.$$

Since

$$\psi^{-1}(t)(x(t) - y(t)) = \int_{t_0}^{t} \psi^{-1}(t)Y(t)P_1 Y^{-1}(s)f_x(s)\,ds - \int_{t}^{\infty} \psi^{-1}(t)Y(t)P_2 Y^{-1}(s)f_x(s)\,ds\,,$$

it is sufficient to show that the terms on the right-hand side belong to $L_p([t_0, \infty))$. By the assumptions of the theorem and the Hölder inequality we get

$$\left|\int_{t_0}^{t} \psi^{-1}(t)Y(t)P_1 Y^{-1}(s)f_x(s)\,ds\right| \leqq \int_{t_0}^{t} |\psi^{-1}(t)Y(t)P_1 Y^{-1}(s)|\varphi(s)g(s, c, c)\,ds$$
$$\leqq |\psi^{-1}(t)Y(t)P_1| \int_{t_0}^{t} |P_1 Y^{-1}(s)\varphi(s)g(s, c, c)|\,ds\,.$$

Since (from Lemma 4)

$$|\psi^{-1}(t)Y(t)P_1| \in L_p([t_0, \infty))$$

and d) holds, it is evident that this first term belongs to $L_p([t_0, \infty))$. For the second term we have

$$\int_{t}^{\infty} |\psi^{-1}(t)Y(t)P_2 Y^{-1}(s)|\,|f_x(s)|\,ds$$
$$\leqq \int_{t}^{\infty} |\psi^{-1}(t)Y(t)P_2 Y^{-1}(s)|\varphi(s)g(s, c, c)\,ds$$
$$\leqq \left(\int_{t}^{\infty} |\psi^{-1}(t)Y(t)P_2 Y^{-1}(s)\varphi(s)|^p\,ds\right)^{1/p} \left(\int_{t}^{\infty} g^{p'}(s, c, c)\,ds\right)^{1/p'}$$
$$\leqq K\left(\int_{t}^{\infty} g^{p'}(s, c, c)\,ds\right)^{1/p'}.$$

Thus from (ii) of b) and Lemma 2 we see that this term also belongs to $L_p([t_0, \infty))$. The proof of the theorem is thus complete.

REMARK 3. If we substitute in Theorem 7 the condition (ii) of b) by the condition

$$\left(\int_t^\infty g^{p'}(s, c, c)\, ds\right)^{1/p'} \in L_p([0, \infty))$$

with p such that $1 < p < \infty$, then the conclusion of Theorem 7 still holds.

REMARK 4. Let $p = \infty$ (and $p' = 1$). Let the conditions a), c) and (ii) of b) of Theorem 7 be replaced by

$$\sup_{0 \leqq s \leqq t} |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s) \varphi(s)| + \sup_{t \leqq s < \infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s) \varphi(s)| \leqq K,$$

$$\lim_{t \to \infty} |\psi^{-1}(t) Y(t) P_1| = 0,$$

$$|\psi^{-1}(t) Y(t) P_1| \in L_v([0, \infty)), \qquad v > 1,$$

$$\int_0^\infty s g^{p'}(s, c, c)\, ds < \infty \qquad \text{for any constant } c \geqq 0,$$

and let all the other assumptions of Theorem 7 hold. Then the sets of ψ-bounded solutions of (a) and (b) are (ψ, v)-integral equivalent.

THEOREM 8. *Let $Y(t)$ be a fundamental matrix of* (b), *$\varphi(t)$ and $\psi(t)$ be positive continuous functions for $t \geqq 0$, and let the hypotheses* (c_0), (c_1), (c_2) *and* (c_3) *of Theorem* 4 *be satisfied. Suppose that*

a) *there exist supplementary projections P_1, P_2 and constants $K > 0$ and $1 < p < \infty$ such that*

$$\int_0^t |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s) \varphi(s)|^p\, ds + \int_t^\infty |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s) \varphi(s)|^p\, ds \leqq K^p$$

for all $t \geqq 0$;

b) *there exists $g\colon J \times J \times J \to J$ such that*

(i) *$g(t, u, v)$ is monotone nondecreasing in u for each fixed $t \in J$, $v \in J$; monotone nondecreasing in v for each fixed $t \in J$, $u \in J$ and integrable on compact subsets of J for fixed $u \in J$, $v \in J$;*

(ii) *$\int_0^\infty g^{p'}(s, c, c)\, ds < \infty$ for any constant $c \geqq 0$, where $(1/p) + (1/p') = 1$;*

(iii) *for each $u, v \in R^n$*

$$|F(t, u, v)| \leqq \varphi(t) g(t, \psi^{-1}(t)|u|, \psi^{-1}(t)|v|) \qquad \textit{a.e. on } J;$$

c) $\int_0^\infty \exp\{-K^{-p}\int_0^t \varphi^p(s)\psi^{-p}(s)\,ds\}\,dt < \infty$;

d) $\int_0^\infty |P_1 Y^{-1}(s)\varphi(s)| g(s, c, c)\,ds < \infty$.

Then, to each ψ-bounded solution $y(t)$ of (b) *there exists a solution $x(t)$ of* (a) *such that*

(e) $$\psi^{-1}(t)|x(t) - y(t)| \to 0 \qquad as \qquad t \to \infty\,,$$

and conversely, to each ψ-bounded solution $x(t)$ of (a) *there exists a solution $y(t)$ of* (b) *such that* (e) *holds.*

Proof. The proof of Theorem 8 is essentially the same as that of Theorem 7.

Remark 5. Let $p = \infty$ (and $p' = 1$). Let the conditions a) and c) of Theorem 8 be replaced by

$$\sup_{0 \leqq s \leqq t} |\psi^{-1}(t) Y(t) P_1 Y^{-1}(s)\varphi(s)| + \sup_{t \leqq s < \infty} |\psi^{-1}(t) Y(t) P_2 Y^{-1}(s)\varphi(s)| \leqq K\,,$$

and

$$\lim_{t \to \infty} |\psi^{-1}(t) Y(t) P_1| = 0\,,$$

and let the other assumptions of Theorem 8 hold. Then, to each ψ-bounded solution $y(t)$ of (b) there exists a solution $x(t)$ of (a) such that (e) holds.

Remark 6. For many years the problem of existence of oscillatory solutions of differential equations has been extensively studied. It turns out that the asymptotic equivalence is a good tool for solving this problem for differential inclusions.

Definition 9. A vector-function $y(t) = \operatorname{col}(y_1(t), \ldots, y_n(t)) \in B_0(J)$ is *s*-oscillatory iff there is $\varepsilon > 0$ such that for each its component $y_i(t)$, $i = 1, \ldots, n$ there is an increasing sequence $\{t_{ik}\}$ such that $\lim_{k\to\infty} t_{ik} = \infty$, $y_i(t_{ik})y_i(t_{i,k+1}) < 0$ for $k = 1, 2, \ldots$, and $|y_i(t_{ik})| > \varepsilon$ for $k = 1, 2, \ldots$.

It is easy to prove the following theorem

Theorem 9. *Let the systems* (a) *and* (b) *be* 1*-asymptotically equivalent and let $y = y(t)$, $t \in J$ be an s-oscillatory solution of* (b). *Then there is an s-oscillatory solution $x = x(t)$ of* (a), *and conversely.*

As a consequence of this theorem we have

Theorem 10. *Let the systems* (a) *and* (b) *be* 1*-asymptotically equivalent. Then the system* (a) *is s-oscillatory if and only if the system* (b) *is s-oscillatory.*

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