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Maximal tori and the center in an analytic group

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§ 1. Introduction

Let G be an analytic group (= connected Lie group), and Z the center of G. Let G' denote the factor group G/Z, which can be identified with the adjoint group of G. In [1] and [2], the author introduced notions of "*generalized maximal tori*" and "*standard Cartan subgroups*" of G, in terms of the adjoint group of G. They played important roles in these papers. Each of these subgroups is connected with a maximal torus of the adjoint group, and contains the center and a maximal torus of G. The purpose of this paper is to give a direct relation between maximal tori and the center in G and maximal tori in G', as follows.

THEOREM. *Let G be an analytic group and Z the center of G. Let α denote the natural homomorphism $G \to G' = G/Z$. Let H be an analytic subgroup of G containing Z. Then H contains a maximal torus of G if and only if α (H) contains a maximal torus of G'.*

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In § 2 and § 3, we recall results on the automorphism group of G and on maximal tori of G, respectively, for the later use. We divide the proof of Theorem into two parts essentially, Proposition 1 in § 4, and Proposition 2 in § 5, such that Theorem follows from them directly. In § 6, an alternate definition of standard Cartan subgroups will be given as an application.

§ 2. Aut (G)

For an analytic group and for its Lie algebra, we shall use the same capital Roman and capital script letter, respectively. Let G be analytic group, and let Aut (G) denote the group of all bicontinuous automorphisms of G. For the Lie algebra $\mathscr{G}$ of G, let Aut ($\mathscr{G}$) denote the group of all Lie algebra automorphisms of $\mathscr{G}$. Then Aut ($\mathscr{G}$) is an algebraic subgroup in the general linear group $GL(\mathscr{G})$, and for any ρ in Aut (G), there corresponds a unique automorphism $d\rho$ of $\mathscr{G}$ such that

$$\rho(\exp X) = \exp(d\rho\, X) \qquad \text{for } X \in \mathscr{G}\,.$$

After this in this paper, we shall identify Aut (G) with a subgroup of Aut $(\mathscr{G})$ by the map $\rho \mapsto d\rho$, and denote $d\rho$ also by ρ. If, in particular, G is simply connected, then Aut (G) = Aut $(\mathscr{G})$ by virtue of the monodromy principle. In general, Aut (G) is a closed subgroup of Aut $(\mathscr{G})$, and is a Lie group.

The adjoint group of G (or $\mathscr{G}$) is denoted by Ad (G) = Ad $(\mathscr{G})$, and is identified with the group of all inner automorphisms of G. Let Z be the center of G. Then we have an immersion (-injective continuous homomorphism) from $G' = G/Z$ into Aut (G), such that the image coincides with Ad (G). Hence Ad (G) is an analytic subgroup of Aut (G).

§3. Maximal tori

Let G be an analytic group. We shall list up results on maximal tori in G. (See [1] pp. 737–8 and [2] p.257.)

(1) *All maximal tori in G are conjugated to each other with respect to inner automorphisms.*

(2) *A maximal torus T is a maximal compact abelian subgroup of G.*

(3) *For a maximal torus T, the factor space G/T is simply connected.*

(4) *Let Z be the center of G, and let T' be a maximal torus in the factor group $G' = G/Z$. Let $\alpha: G \to G'$ be the natural homomorphism. Then $\alpha^{-1}(T')$ is a closed connected abelian subgroup of G.*

Remark. The adjoint group Ad (G) is an analytic subgroup of the Lie group Aut (G), and is not a Lie group in general with respect to the relative topology in Aut (G). But by the following lemma, there is no ambiguity in the definition of maximal tori in Ad (G).

Lemma. *Let L be a locally compact σ-compact group, and M a topological group. Let $\psi: L \to M$ be an immersion. Let K be a subgroup of L. If $\psi(K)$ is (locally) compact, then ψ_K is a homeomorphism, and in particular K is (locally) compact.*

Proof. If $\psi(K)$ is locally compact, then $\psi(K)$ is closed in M, and so is K in L. Hence K is locally compact and σ-compact. Thus by a category argument ψ_K is a homeomorphism. ||

§ 4. Proof of Proposition 1

Retaining the notations in the previous sections, we shall prove the following

PROPOSITION 1. *For any maximal torus T' in G', we can find a maximal torus T in G, such that $\alpha^{-1}(T')$ is a minimal analytic subgroup of G containing both T and Z.*

PROOF. Let T_1 be a maximal torus in G. Then $\alpha(T_1)$ is a torus in G. Hence we can find $a \in G$ such that $\alpha(a)\alpha(T_1)\alpha(a)^{-1} \subset T'$, *i.e.* $a\ T_1 a^{-1} \subset \alpha^{-1}(T')$. Denoting $\alpha^{-1}(T') = A$ and $a\ T_1 a^{-1} = T$, we see that T is a maximal torus in G, and is the largest compact subgroup in the abelian analytic group A. Hence we can find a vector group V, a closed subgroup isomorphic with $\boldsymbol{R}^m$ for a suitable m such that

$$A = V \times T \qquad \text{(direct product)}.$$

Let B be an analytic subgroup of A containing T. Then

$$B = V_2 \times T, \qquad V_2 = B \cap V \cong \boldsymbol{R}^n \qquad (n \leqq m).$$

Then we can find $V_1 \subset V$, $V_1 \cong \boldsymbol{R}^{m-n}$ such that $V = V_1 \times V_2$. Thus we have

$$A = V_1 \times B$$

Suppose that $B \supset Z$. Then $A/Z \cong V_1 \times (B/Z)$, which cannot be compact unless $m - n = 0$. Hence $A = B$. ||

§ 5. Proof of Proposition 2

PROPOSITION 2. *Let T be a maximal torus in G, and let C be a minimal analytic subgroup of G containing Z and T. Then there exists a maximal torus T' of G' such that $C = \alpha^{-1}(T')$.*

PROOF. Let Z_C denote the center of C, and let $\beta: C \to C/Z_C = C'$ be the natural homomorphism. Let T'_C be a maximal torus of C' containing $\beta(T)$. Then $\beta^{-1}(T'_C)$ is a closed connected abelian subgroup of C, and is an analytic subgroup of G. Since C contains T and $Z(\subset Z_C)$, we have $C = \beta^{-1}(T'_C)$ *and* C is abelian.

Let Z^0 denote the identity component of Z. Then

$$Z^0 \cong \boldsymbol{R}^a \times \boldsymbol{T}^b,$$

where $\boldsymbol{T} = \boldsymbol{R}/\boldsymbol{Z}$ and a, $b = 0, 1, 2, \ldots$, and Z^0 is a divisible group. Also the factor group Z/Z^0 is known to be finitely generated. Hence $Z/Z^0 \cong \boldsymbol{Z}^c \times F$, $c = 0, 1, 2, \ldots$, and $Z \cong \boldsymbol{R}^a \times \boldsymbol{T}^b \times \boldsymbol{Z}^c \times F$, where F is a finite group. Since $\boldsymbol{T}^b \times F$ is compact, we have $\boldsymbol{T}^b \times F \subset T$, and $ZT = \boldsymbol{R}^a \times \boldsymbol{Z}^c \times T$. Let $\{x_1, \ldots, x_c\}$ denote a (minimal) system of generators of $\boldsymbol{Z}^c$. Because the exponential map of an abelian analytic group is surjective, there exists X_i in the Lie algebra of C such that $x_i = \exp X_i$, $i = 1, 2, \ldots, c$. Then the subgroup $D = \boldsymbol{R}^a \exp(\Sigma\, \boldsymbol{R}X_i)\cdot T$ is a continuous homomorphic image of $\boldsymbol{R}^{a+c} \times T$ and is an analytic subgroup of C. Since D contains T and Z, we have that $C = D$. Because the subgroup $\boldsymbol{R}^a \exp(\Sigma\, \boldsymbol{Z}X_i)$ is contained in Z, we have that C/ZT is compact. Then by

$$(C/Z)/(ZT/Z) \cong C/ZT,$$

C/Z is compact, and is a torus. Hence there exists a maxinal torus T' in G' with $C \subset \alpha^{-1}(T')$. Then by Proposition 1, we have $C = \alpha^{-1}(T')$. ||

§6. Standard Cartan subgroups

About this section the reader may refer [2].

Let G be an analytic group, and Ad(G) the adjoint group. Let $\alpha: G \to$ Ad(G) be the natural homomorphism. Let H be a Cartan subgroup of G. Then $\alpha(H)$ is a Cartan subgroup of the analytic group Ad(G). The author named H *standard* if $\alpha(H)$ contains a maximal torus in Ad(G). By Theorem of this paper we have in particular

COROLLARY. *A Cartan subgroup H of G is standard if and only if H contains the center and a maximal torus of G.*

Thus we have an alternate definition of standard Cartan subgroups.

Bibliography

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