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The maximal codegree of the quaternionic projective spaces

Dedicated to Professor Akio Hattori on his sixtieth birthday

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§1. Introduction

For a k-dimensional oriented vector bundle α over a connected finite CW-complex X, the codegree $cd(X^\alpha)$ of the Thom space X^α is defined by

$$cd(X^\alpha) = |\mathrm{Coker}\,[h : \pi_s^k(X^\alpha) \to H^k(X^\alpha; Z)]|\,,$$

the order of the cokernel of the stable Hurewicz homomorphism h of the stable cohomotopy group to the integral cohomology group. We study this codegree by restricting our attention to

$$cd_2(X^\alpha) = \nu_2(cd(X^\alpha))\,,$$

the exponent of 2 in the prime power decomposition of $cd(X^\alpha)$. The cohomology groups are always assumed to be reduced.

Let kO (resp. $kSpin$) be the -1 (resp. 3) connective cover of the KO-spectrum KO. Then the spectrum j is defined to represent the fiber of

$$\psi : kO^*(\ )_{(2)} \to kSpin^*(\ )_{(2)} \quad (G_{(2)} \text{ is the localization of } G \text{ at } 2)$$

which is a unique lift of the stable Adams operation $\psi^3 - 1 : KO^*(\ )_{(2)} \to KO^*(\ )_{(2)}$ (cf. [16], [6], [17]); and we have the Hurewicz homomorphism

$$h_j : j^k(X^\alpha) \to H^k(X^\alpha; Z_{(2)})$$

which factors $h : \pi_s^k(X^\alpha) \to H^k(X^\alpha; Z_{(2)})$. Thus we have the j-codegree

$$cd_2^j(X^\alpha) = \nu_2(|\mathrm{Coker}\,(h_j)|) \qquad \text{with} \qquad cd_2^j(X^\alpha) \leqq cd_2(X^\alpha)\,,$$

which has another description being available for calculations (see Corollary 2.7).

Now, M. C. Crabb and K. Knapp introduced the notion of the maximal codegree given as follows:

THEOREM A (Crabb-Knapp[7]). *For any integer n, put*

(1.1) $m_2(n) = [n/2]$ *if* $n \equiv 0, 1, 2, 6, 7 \bmod 8$, $= [n/2] + 1$ *otherwise*.

(i) *Then,* $cd_2(X^\alpha) \leqq m_2(n)$ *if* $\dim X \leqq n$.

(ii) *For a complex vector bundle* α *over the complex projective space* CP^r,

$$cd_2((CP^r)^\alpha) = cd_2^j((CP^r)^\alpha) \quad \text{if} \quad cd_2^j((CP^r)^\alpha) \geqq m_2(2r) - 4\,.$$

The object of this paper is to study $cd_2(X^\alpha)$ when X is the quaternionic projective space HP^r. Let ξ_r be the canonical quaternionic line bundle over HP^r. Then, up to a homeomorphism, we have

$$X^\alpha = HP_n^{n+r} = HP^{n+r}/HP^{n-1} \quad \text{when} \quad X = HP^r \quad \text{and} \quad \alpha = n\xi_r$$

(cf. [3]). Here, the stunted space HP_n^{n+r} is a CW-complex with one $4i$-cell for each $n \leqq i \leqq n + r$. Thus, we consider a finite CW-spectrum W of the form

$$(1.2) \qquad W = S^0 \cup e^{4a_1} \cup \cdots \cup e^{4a_t} \quad \text{with} \quad 1 \leqq a_1 \leqq \cdots \leqq a_t = r$$

in general, and study the codegrees

$$cd_2(W) = \nu_2(|\mathrm{Coker}\,[h: \pi^0(W) \to H^0(W; Z) = Z]|)\,,$$

$$cd_2^j(W) = \nu_2(|\mathrm{Coker}\,[h_j: j^0(W) \to H^0(W; Z_{(2)}) = Z_{(2)}]|)\,,$$

where h and h_j are the Hurewicz homomorphisms. Then, in the above case,

$$cd_2(X^\alpha) = cd_2(W)\,, \qquad cd_2^j(X^\alpha) = cd_2^j(W) \qquad \text{for} \qquad W = \Sigma^{-4n}Y$$

where Y is the suspension spectrum of the CW-complex $Y = HP_n^{n+r}$.

Now, as an analogy of Theorem A (ii), we can prove the following main result:

THEOREM 1. *Let* W *be a* CW*-spectrum given in* (1.2), *and put*

$$\varepsilon(r) = 3 \text{ if } r \text{ is even}, = 5 \text{ if } r \text{ is odd}\,.$$

(i) *If* $cd_2^j(W) \geqq 2r - \varepsilon(r)$, *then* $cd_2(W) = cd_2^j(W)$.

(ii) *For* $\varepsilon < \varepsilon(r)$, $cd_2(W) = 2r - \varepsilon$ *if and only if* $cd_2^j(W) = 2r - \varepsilon$.

We also consider the vector bundle ζ_r over HP^r defined by the mixing construction of the adjoint representation of S^3 with the canonical principal S^3-bundle $S^{4r+3} \to HP^r$, and the quaternionic quasi-projective space Q_{r+1} which is the Thom space of ζ_r. Then

$$X^\beta = Q_n^{n+r} = Q_{n+r}/Q_{n-1} \quad \text{when} \quad X = HP^r \quad \text{and} \quad \beta = \zeta_r \oplus (n-1)\xi_r$$

(cf. [3]), and we can apply Theorem 1 also for $W = \Sigma^{-4n+1}Q_n^{n+r}$.

By Theorem 1 and by tractable calculations on j-codegrees, we can determine the codegree $cd_2(X^\alpha)$ for $X = HP^r$ and $\alpha = n\xi_r$ or $\zeta_r \oplus (n-1)\xi_r$ when it is near maximal. To describe the concrete results, we put

(1.3) $\binom{n}{r} \equiv a \bmod 8$, $\binom{n}{r-1} \equiv b \bmod 4$ and $n\binom{n+1}{r-1} \equiv c \bmod 2$

for given integers n and $r \geqq 1$, where $0 \leqq a < 8$, $0 \leqq b < 4$ and $0 \leqq c < 2$. Then we have the following theorems.

THEOREM 2. *Put* $cd_2(HP_n^{n+r}) = m_2(4r) - \varepsilon$ *for* $m_2(4r) = 3r - 2[r/2]$. *Then* $\varepsilon \geqq 0$, *and* $\varepsilon = 0$, 1 *or* 2 *if and only if the following* (0), (1) *or* (2) *holds for* a, b *and* c *in* (1.3), *respectively*:

(0) *a is odd.*

(1) $a = 2$ *or* 6 *for any* r; *or* a *is even and* b *is odd when* r *is even.*

(2) $a = 4 - 4c$ *when* r *is odd*;
$a = 4$, b *is even and* $c = 0$, *or* $a = 0$ *or* 4 *and* $b = 2$, *when* r *is even.*

THEOREM 3. *Put* $cd_2(Q_{n+1}^{n+1+r}) = m_2(4r) - \varepsilon'$. *Then* $\varepsilon' \geqq 0$, *and* $\varepsilon' = 0$, 1 *or* 2 *if and only if the following* (0), (1') *or* (2') *holds for* a, b *and* c *in* (1.3), *respectively*:

(0) *a is odd.*

(1') *a is even and* $(a/2) + b$ *is odd for any* r; *or* a *is even and* b *is odd when* r *is even.*

(2') $a = n \equiv 2 \bmod 8$ *when* $r = 1$;
a is even, $a/2$ *and* b *are odd and* $(a/2) + b + 2\binom{n+1}{r-1} \equiv 2 \bmod 4$, *or*
$(a, b, c) = (0, 0, 1), (0, 2, 0)$ *or* $(4, 0, 0)$, *when* r *is odd* $\geqq 3$;
$a = 0$ *or* 4, $b = 0$ *or* 2 *and* $\binom{n}{r-2} - 1 \equiv b/2 \bmod 2$, *or*
$(a, b, c) = (0, 2, 0)$ *or* $(4, 0, 0)$, *when* r *is even.*

We prove these theorems for any n by defining HP_n^{n+r} and Q_{n+1}^{n+1+r} to be the Thom spaces of $n\xi_r$ and $\zeta_r \oplus n\xi_r$, respectively, and by putting $\binom{n}{r} = (-1)^r\binom{r-n-1}{r}$ for $n \leqq 0$ as usual. We note that each condition of these theorems happens really for some n and r. Especially HP_{-1}^{r-1}, HP_{-2}^{2r-2}, Q_0^r and Q_{-1}^{2r-1} take the maximal codegrees; and the results for HP_{-1}^{r-1} and Q_0^r are already proved by Crabb-Knapp [8]. The articles related to the codegree of HP_n^{n+r} or Q_n^{n+r} are also found in [20] and [10–14].

In §2, we prepare some properties for the j-cohomology and the j-codegree. We prove Theorem 1 in §3, Theorems 2 and 3 in §4; and we give some examples in §5.

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§2. Preliminaries

Let j be the spectrum introduced in §1. That is, j is defined to be the fiber spectrum of $\psi: kO^*(\ )_{(2)} \to kSpin^*(\ )_{(2)}$ which is a unique lifting of the stable Adams operation $\psi^{3-1}: KO^*(\ )_{(2)} \to KO^*(\ )_{(2)}$. Thus we have an exact sequence

$$(2.1) \qquad \cdots \to kSpin^{-1}(Y)_{(2)} \xrightarrow{\delta} j^0(Y) \xrightarrow{f} kO^0(Y)_{(2)} \xrightarrow{\psi} kSpin^0(Y)_{(2)} \to \cdots$$

for a finite CW-spectrum Y.

Now, we assume that W is the $4r$-dimensional CW-spectrum given in (1.2), and that W^i is the i-dimensional skeleton of W.

LEMMA 2.2. *Let $0 \leqq i \leqq 4r$. Then we have the following:*
(i) *$kO^0(W^i)$ and $kSpin^0(W^i)$ are the free abelian groups.*
(ii) *$kSpin^{-1}(W^i)$ is a 2-torsion group.*
(iii) *$i^*: kSpin^{-1}(W) \to kSpin^{-1}(W^i)$ is an epimorphism for the inclusion $i: W^i \to W$.*

PROOF. Let $i \geqq j \geqq 0$. Then the Atiyah-Hirzebruch spectral sequence for $KO^*(W^i/W^j)$ collapses, because W^i/W^j has only cells of dimensions divisible by 4. Hence $KO^0(W^i/W^j)$ is a free abelian group and $KO^{-1}(W^i/W^j)$ is a 2-torsion group. It is well known that, for the c-connective cover F of a spectrum E, we have an isomorphism $F^q(Y) \cong \mathrm{Im}\,[E^q(Y/Y^{c+q}) \to E^q(Y/Y^{c+q-1})]$ for a CW-spectrum Y and its i-skeleton Y^i. Since kO (resp. $kSpin$) is the -1 (resp. 3) connective cover of KO, we have isomorphisms $kO^0(W^i) \cong KO^0(W^i)$, $kSpin^0(W^i) \cong KO^0(W^i/S^0)$ and $kSpin^{-1}(W^i) \cong KO^{-1}(W^i/S^0)$. Thus we have (i) and (ii). Since $i^*: kSpin^{-1}(W) \to kSpin^{-1}(W^i)$ is identified with an epimorphism $i^*: KO^{-1}(W/S^0) \to KO^{-1}(W^i/S^0)$, we have (iii). Q.E.D.

Let $ph: KO^0(Y) \to H^{4*}(Y; Q) = \Pi_{i \geqq 0} H^{4i}(Y; Q)$ be the Pontrjagin character. Then the following lemma is well known (cf. [1], [4]):

LEMMA 2.3. *$ph \otimes Q: KO^0(Y) \otimes Q \to H^{4*}(Y; Q)$ is an isomorphism, and the composition $(ph \otimes Q) \circ (\psi^3 - 1) \circ (ph \otimes Q)^{-1}$ maps an element $y \in H^{4k}(Y; Q)$ to $(9^k - 1)y$, where $\psi^3 - 1: KO^0(Y) \otimes Q \to KO^0(Y) \otimes Q$ is the Adams operation.*

Let $\mathrm{Tor}\,(G)$ be the torsion part of a finitely generated abelian group G.

Then the cohomology groups $j^0(W^i)$ satisfy the following:

PROPOSITION 2.4. (i) *For* $i \geqq 0$, $\mathrm{Tor}\,(j^0(W^i)) = \delta(kSpin^{-1}(W^i)_{(2)})$ *and* $j^0(W^i)/\mathrm{Tor}\,(j^0(W^i)) \cong Z_{(2)}$, *where* δ *is the homomorphism in* (2.1) *for* $Y = W^i$.

(ii) $i^*: \mathrm{Tor}\,(j^0(W)) \to \mathrm{Tor}\,(j^0(W^i))$ *is an epimorphism, where* $i: W^i \to W$ *is the inclusion.*

PROOF. Consider the exact sequence (2.1) for $Y = W^i$. Since $kO^0(W^i)$ is a free abelian group by Lemma 2.2 (i), so is $\mathrm{Ker}\,(\psi)$. Thus we have an isomorphism $j^0(W^i) \cong \mathrm{Ker}\,(\psi) \oplus \mathrm{Im}\,(\delta)$, and $\mathrm{Im}\,(\delta)$ is a torsion group by Lemma 2.2 (ii). By Lemma 2.3, we have an isomorphism $\mathrm{Ker}\,(\psi) \otimes Q \cong H^0(W^i; Q) \cong Q$, since ψ is a lifting of $\psi^3 - 1$. Thus we have (i), and $\delta: kSpin^{-1}(W^i) \to \mathrm{Tor}\,(j^0(W^i))$ is an epimorphism. $i^*: kSpin^{-1}(W) \to kSpin^{-1}(W^i)$ is an epimorphism by Lemma 2.2 (iii), and $i^*: \mathrm{Tor}\,(j^0(W)) \to \mathrm{Tor}\,(j^0(W^i))$ factors the epimorphism $\delta \circ i^*$. Thus we have (ii). Q.E.D.

Recall that $cd_2^j(W) = \nu_2(|\mathrm{Coker}\,(h_j)|)$ for the Hurewicz homomorphism $h_j: j^0(W) \to H^0(W; Z_{(2)})$. Let $i_0: S^0 \to W$ be the inclusion to the bottom sphere of W. Then we have $cd_2^j(W) = \nu_2(|\mathrm{Coker}\,(i_0^*)|)$ for the homomorphism $i_0^*: j^0(W) \to j^0(S^0)$. On the other hand, we have the KO-codegree $cd_2^{KO}(W)$ defined below. Let $u \in H^0(W; Z) \cong Z$ be the generator. Then by Lemma 2.3 there is a unique element $V \in KO^0(W) \otimes Q$ such that $(ph \otimes Q)(V) = u$ in $H^0(W; Q)$. We define $cd_2^{KO}(W)$ to be the minimal non negative integer e satisfying $2^e V \in KO^0(W)_{(2)}$. We will show that these two types of codegree agree.

We regard V also as an element of $kO^0(W) \otimes Q$ through the isomorphism $kO^0(W) \cong KO^0(W)$. Then, by Lemmas 2.2 and 2.3 and the minimality of $cd_2^{KO}(W)$, we have the following lemma, where f and ψ are the homomorphisms in (2.1) for $Y = W$:

LEMMA 2.5. *Let* $c = cd_2^{KO}(W)$. *Then* $V \in \mathrm{Im}\,(f \otimes Q)$, *and* $2^c V$ *is a generator of* $\mathrm{Ker}\,(\psi) \cong Z_{(2)}$ *in* $kO^0(W)_{(2)}$.

PROPOSITION 2.6. $cd_2^j(W) = cd_2^{KO}(W)$.

PROOF. Let $i_Q^E: E^0(Y)_{(2)} \to E^0(Y) \otimes Q$ be the canonical map defined by the inclusion $Z_{(2)} \subset Q$ for spectra E and Y. Then $cd_2^{KO}(W) = \mathrm{Min}\,\{e \geqq 0 \mid 2^e V \in \mathrm{Im}(i_Q^{KO})\}$ by definition. By Lemma 2.5, we have an element $V_j \in j^0(W) \otimes Q$ such that $(f \otimes Q)(V_j) = V$. Then we have $(i_0^* \otimes Q)(V_j) = i_Q^j(\iota)$ in $j^0(S^0) \otimes Q$ for some unit $\iota \in j^0(S^0) \cong Z_{(2)}$. Thus we have $cd_2^j(W) = \mathrm{Min}\,\{e \geqq 0 \mid 2^e V_j \in \mathrm{Im}\,(i_Q^j)\}$, since $j^0(W)/\mathrm{Tor}\,(j^0(W)) \cong Z_{(2)}$ by Proposition 2.4 (i). Consider the following commutative diagram:

$$\begin{array}{ccccc} j^0(W) & \xrightarrow{f} & kO^0(W)_{(2)} & \xrightarrow{\psi} & kSpin^0(W)_{(2)} \\ \downarrow{\scriptstyle i_Q^j} & & \downarrow{\scriptstyle i_Q^{KO}} & & \downarrow{\scriptstyle i_Q^{kSpin}} \\ j^0(W)\otimes Q & \xrightarrow{f\otimes Q} & kO^0(W)\otimes Q & \xrightarrow{\psi\otimes Q} & kSpin^0(W)\otimes Q\,. \end{array}$$

Since $kO^0(W) \cong KO^0(W)$ and i_Q^{kO} is identified with i_Q^{KO} through the isomorphism, we have $cd_2^{KO}(W) \leqq cd_2^j(W)$. For $c = cd_2^{KO}(W)$, we have an element $y \in kO^0(W)_{(2)}$ such that $i_Q^{kO}(y) = 2^c V$, and $(i_Q^{kSpin} \circ \psi)(y) = 2^c(\psi \otimes Q)(V) = 0$. But i_Q^{kSpin} is injective by Lemma 2.2 (i). Thus we have an element $z \in j^0(W)$ satisfying $y = f(z)$, and so $((f \otimes Q) \circ i_Q^j)(z) = (f \otimes Q)(2^c V_j)$. Since $f \otimes Q$ is injective by Proposition 2.4 (i), we have $i_Q^j(z) = 2^c V_j$, and this implies $cd_2^j(W) \leqq cd_2^{KO}(W)$. Thus we have the required equality. Q.E.D.

Let X be a connected finite CW-complex which has no cells of dimensions not divisible by 4, α a KO-orientable virtual vector bundle over X of dimension 0, and X^α the Thom spectrum of α. Then we have the KO-Thom class $U_{KO} \in KO^0(X^\alpha)$ and the ordinary Thom class $U_H \in H^0(X^\alpha; Z)$. The multiplicative characteristic class $sh(\alpha) \in H^{4*}(X_+; Q)$ is defined by the equation $ph(U_{KO}) = U_H sh(\alpha)$ (cf. [2]), where X_+ is the disjoint union of X and the base point. Then, in the case $W = X^\alpha$, we can take U_H and $U_{KO} ph^{-1}(sh(-\alpha))$ as the elements u and V respectively, where ph denotes the ring isomorphism $ph \otimes Q$ as in Lemma 2.3.

COROLLARY 2.7. $cd_2^j(X^\alpha) = cd_2^{KO}(X^\alpha) = \mathrm{Min}\,\{e \geqq 0 | 2^e ph^{-1}(sh(-\alpha)) \in KO^0(X_+)_{(2)}\}$.

REMARK 2.8. The properties concerning the j-codegrees and KO-codegrees of the Thom spectra are investigated by M. C. Crabb and K. Knapp in a series of their papers and by H. Ōshima [17]. Proposition 2.6 is a simple analogy of [5; Prop. 3.2] or [9; §4], and Corollary 2.7 is a special case of [17; Th. 3.3].

§3. Proof of Theorem 1

We will apply the method of [7] to the proof of Theorem 1. Then we need some preliminaries about the mod 2 Adams spectral sequence, which has $\mathrm{Ext}_A(Z/2, Z/2)$ as the E_2-term and converges to $\pi_*(S^0)$, where A is the mod 2 Steenrod algebra. In this section, the spectra and the groups are assumed to be localized at 2. Let

$$S^0 = Y_0 \xleftarrow{g_0} Y_1 \xleftarrow{g_1} Y_2 \longleftarrow \cdots \longleftarrow Y_s \xleftarrow{g_s} Y_{s+1} \longleftarrow \cdots$$

be the mod 2 minimal Adams resolution, and $g(s)$ the composition $g_0 g_1 \cdots g_{s-1}$:

$Y_s \to S^0$. Let $[Z, Y_s]$ be the group of the homotopy classes of maps from a spectrum Z to Y_s, and $F^s[Z, S^0]$ denote the image of $g(s)_*: [Z, Y_s] \to [Z, S^0]$. Then the mod 2 Adams filtration of an element $z \in [Z, S^0]$ is the maximal value of s such that $z \in F^s[Z, S^0]$, and we denote it by $F_2(z) = s$. It is known that $\pi_0(Y_s) \cong Z$ with a generator k_0 and the composition $g(s) \circ k_0: S^0 \to S^0$ is of degree 2^s (cf. [7; §2]).

Let $\varepsilon(r) = 3$ if r is even, $= 5$ if r is odd, as in Theorem 1, and $h(j)$: $\pi_{4i-1}(\ ) \to j_{4i-1}(\ )$ the Hurewicz homomorphism. Then by the same way as in the proof of [7; Prop. 4.1] we have the following:

PROPOSITION 3.1. *Assume that* $s \geqq 2r - \varepsilon(r)$. *Then the composition* $h(j) \circ g(s+1)_*: \pi_{4i-1}(Y_{s+1}) \to j_{4i-1}(S^0)$ *is a monomorphism for* $1 \leqq i \leqq r$.

PROOF. Let $E_u^{s,t}(Y)$ be the E_u-term of the mod 2 Adams spectral sequence for $\pi_*(Y)$, which has $\mathrm{Ext}_A(H^*(Y; Z/2), Z/2)$ as the E_2-term. Then we have the connecting homomorphism $\delta_s: E_u^{i,j}(Y_{s+1}) \to E_u^{i+1,j+1}(Y_s)$ which is compatible with the differentials d_u and associated to g_s in E_∞-terms (cf. [18; Chap. 2]). Let $\delta(s): E_2^{i,j}(Y_{s+1}) \to E_2^{i+s+1,j+s+1}(S^0)$ be the composition $\delta_0\delta_1\ldots\delta_s$. Then $\delta(s)$ is an isomorphism for any $i \geqq 0$ and $j \geqq 0$ (cf. [7; (2.4–5)]). In particular $\delta(s)$: $E_2^{0,4i-1}(Y_{s+1}) \to E_2^{s+1,s+4i}(S^0)$ is an isomorphism for any $i \geqq 1$ and $s \geqq 0$. Now we assume that s and i satisfy $s + 1 \geqq 2i - 2$ for even i and $s + 1 \geqq 2i - 5$ for odd i. Let $\mathrm{Im}\,(J)$ be the image of the stable J-homomorphism $J: \pi_{4i-1}(SO) \to \pi_{4i-1}(S^0)$. Then, by [7; Proof of Prop. 4.1], $E_2^{s+1,s+4i}(S^0)$ is isomorphic to 0 or $Z/2$, and generated by a permanent cycle presented by an element of $\mathrm{Im}\,(J)$. Thus $\delta(s)$ induces a monomorphism between E_∞-terms. Note that $\delta(s)$: $E_\infty^{k,k+4i-1}(Y_{s+1}) \to E_\infty^{k+s+1,k+s+4i}(S^0)$ is associated with $g(s+1)_*: F^k\pi_{4i-1}(Y_{s+1}) \to F^{k+s+1}\pi_{4i-1}(S^0)$. Hence we have a monomorphism $g(s+1)_*: \pi_{4i-1}(Y_{s+1}) \to \pi_{4i-1}(S^0)$, and its image is contained in $\mathrm{Im}\,(J)$. Since $h(j)$ is injective on $\mathrm{Im}\,(J)$ (cf. [16], [5]), we have the desired result. Q.E.D.

Let W be the spectrum given in (1.2). Then we have the following:

PROPOSITION 3.2. *If* W *satisfies* $cd_2^j(W) \geqq 2r - \varepsilon(r)$, *then there is an element* $x \in \pi^0(W)$ *satisfying* $\nu_2(h(x)) = F_2(x) = cd_2^j(W)$, *where we regard the Hurewicz image* $h(x) \in H^0(W; Z) = Z$ *as an integer.*

PROOF. We put $s = cd_2^j(W)$. Then $s \geqq 2r - \varepsilon(r)$ by the assumption, and we have an element $z \in j^0(W)$ satisfying $i_0^*(z) = 2^s \in j^0(S^0) = Z_{(2)}$, where i_0 is the inclusion to the bottom sphere of W. Let $k_0: S^0 \to Y_s$ be the generator of $\pi_0(Y_s) \cong Z$. We will construct an extension $k_l: W^{4l} \to Y_s$ of k_0 inductively on l for $0 \leqq l \leqq r$. Then the element $x = g(s) \circ k_r \in \pi^0(W)$ satisfies $\nu_2(h(x)) = F_2(x) = s$, because $s \leqq F_2(x) \leqq \nu_2(h(x)) = s$, and x is the desired element.

So we assume that we have already constructed an extension k_l for some l with $0 \leqq l \leqq r-1$. Let $\phi: \bigvee S^{4l+3} \to W^{4l}$ be the attaching map of the $4(l+1)$-dimensional cells of W. We will show that $k_l \circ \phi = 0$. Then we have an extension k_{l+1} of k_l, and complete the proof. Consider the element $w = h(j) \circ g(s) \circ k_l \in j^0(W^{4l})$. Let $i_{a,b}: W^{4a} \to W^{4b}$ be the inclusion map for $a \leqq b$. Then we have $(i_{0,l})^*(w) = (i_{0,n})^*(z) = 2^s$ in $j^0(S^0)$, and so we have $w - (i_{l,n})^*(z) \in \mathrm{Tor}\,(j^0(W^{4l}))$ by Proposition 2.4 (i). Then we have an element $v \in \mathrm{Tor}\,(j^0(W))$ satisfying $(i_{l,n})_*(z+v) = w$ by Proposition 2.4 (ii). Hence $\phi^*(w) = 0$ in $j^0(\bigvee S^{4l+3})$. Here $\phi^*(w) = (h(j) \circ g(s)_*)(k_l \circ \phi)$, and $k_l \circ \phi \in F^1[\bigvee S^{4l+3}, Y_s]$. Since we have the assumption that $s \geqq 2r - \varepsilon(r)$ and $l \leqq r-1$, $h(j) \circ g(s)_*$ is a monomorphism on $F^1[\bigvee S^{4l+3}, Y_s]$ by Proposition 3.1. Thus we have $k_l \circ \phi = 0$, and the desired result. Q.E.D.

PROOF OF THEOREM 1. (i) By Proposition 3.2 we have $cd_2(W) \leqq cd_2^j(W)$. But it always holds that $cd_2^j(W) \leqq cd_2(W)$, and thus we have the desired result.

(ii) By (i) it is sufficient to show that $cd_2(W) = cd_2^j(W)$ if $cd_2(W) = 2r - \varepsilon$ and $\varepsilon < \varepsilon(r)$. Suppose that $2r - \varepsilon = cd_2(W) \neq cd_2^j(W) = 2r - u$ and $\varepsilon \leqq \varepsilon(r) - 1$. Then we have $u \geqq \varepsilon + 1$, since $cd_2^j(W) < cd_2(W)$. Let $z_1 \in j^0(W)$ be an element satisfying $i_0^*(z_1) = 2^{2r-u} \in j^0(S^0)$, and put $y = 2^{u-\varepsilon-1} z_1 \in j^0(W)$. Then we have $i_0^*(y) = 2^{2r-(\varepsilon+1)} \in j^0(S^0)$, and $\varepsilon + 1 \leqq \varepsilon(r)$. Then we can construct an extension $k': W \to Y_{2r-\varepsilon-1}$ of k_0 by the same way as the proof of Proposition 3.2, using y instead of z. But this is a contradiction, because the Hurewicz image of $g(2r-\varepsilon-1) \circ k' \in \pi^0(W)$ is not divisible by $2^{2r-\varepsilon}$ and it implies that $cd_2(W) \leqq 2r - \varepsilon - 1$. Thus $cd_2^j(W) = cd_2(W)$, and we have completed the proof.

§4. Proof of Theorems 2 and 3

We assume that n and r are integers with $r \geqq 1$. As mentioned in §1, HP_n^{n+r} (resp. Q_{n+1}^{n+1+r}) is considered as the Thom space of $n\xi_r$ (resp. $\zeta_r \oplus n\xi_r$). Let $x \in H^4(HP^r; Z)$ and $X = [\xi_r - 1_H] \in KO^4(HP^r)$ be the Euler classes of ξ_r in the respective cohomology groups, and g_i the generator of $KO^{-4i}(S^0) \cong Z$. We put $Y = (g_1/2)X \in KO^0(HP^r) \otimes Q$. Then it is known that there are ring isomorphisms $H^*(HP_+^r; Z) \cong Z[x]/(x^{r+1})$ and $KO^0(HP_+^r) \otimes Q \cong Q[Y]/((Y)^{r+1})$, and we have $Y^{2i} = g_{2i} X^{2i}$ and $Y^{2i+1} = (g_{2i+1}/2) X^{2i+1}$ for $i \geqq 0$. Consider the power series

$$\sinh(y) = \sum_{i \geqq 0} y^{2i+1}/((2i+1)!) \in Q[\![y]\!] .$$

Then the following is known (cf. [13], [14]):

LEMMA 4.1. *In* $H^*(HP_+^r; Q)$, $sh(\xi_r) = (\sinh(\sqrt{x}/2)/(\sqrt{x}/2))^2$ *and* $sh(\zeta_r) = (d/dy)((2\sinh(\sqrt{y}/2))^2)|_{y=x}$.

Let $\mathrm{Sinh}^{-1}(y) \in Q[\![y]\!]$ be the inverse power series of $\sinh(y)$. That is, it satisfies $\mathrm{Sinh}^{-1}(\sinh(y)) = y$. Consider the power series

$$G(y) = (2\mathrm{Sinh}^{-1}(\sqrt{y}/2))^2 \in Q[\![y]\!] .$$

Then we have an element $G(Y)$ of $KO^0(HP^r_+) \otimes Q$. Let ph denote $ph \otimes Q$ as in Lemma 2.3. Since $ph(Y) = ph(X) = (2\sinh(\sqrt{x}/2))^2$ and ph is a ring isomorphism, we have $ph^{-1}(x) = G(Y)$. On the other hand, by Lemma 4.1 we have $sh(-n\xi_r) = (\sinh(\sqrt{x}/2)/(\sqrt{x}/2))^{-2n}$ and $sh(-(\zeta_r \oplus n\xi_r)) = (n+1)x^n/((d/dy)(2\sinh(\sqrt{y}/2))^{2n+2}|_{y=x})$, since $sh(-\alpha) = sh(\alpha)^{-1}$ and $sh(\alpha+\beta) = sh(\alpha)sh(\beta)$ for KO-orientable vector bundles α and β. Thus we have the following equation:

PROPOSITION 4.2. (i) $ph^{-1}(sh(-n\xi_r)) = (G(Y)/Y)^n$.
(ii) $ph^{-1}(sh(-(\zeta_r \oplus n\xi_r))) = (1/((n+1)Y^n)) \cdot ((d/dy)(G(y)^{n+1})|_{y=Y})$.

By Corollary 2.7 and Proposition 4.2 we have the following:

COROLLARY 4.3. (i) $cd_2^j(HP_n^{n+r}) = \mathrm{Min}\,\{e \geqq 0 \mid 2^e(G(Y)/Y)^n \in KO^0(HP^r_+)_{(2)}\}$.
(ii) $cd_2^j(Q_{n+1}^{n+1+r}) = \mathrm{Min}\,\{e \geqq 0 \mid 2^e(1/((n+1)Y^n)) \cdot ((d/dy)(G(y)^{n+1})|_{y=Y}) \in KO^0(HP^r_+)_{(2)}\}$.

Now, we put

$$G_n(y) = (G(y)/y)^n = \sum_{i=0}^{\infty} a_i(n)y^i \qquad \text{for} \quad a_i(n) \in Q . \tag{4.4}$$

Then we have $a_0(n) = 1$ and the following:

LEMMA 4.5 (F. Sigrist-U. Suter [19; (3.6)]).

$$4^m(1-4^m)a_m(n) = \sum_{i=0}^{m-1} \binom{n+i}{m-i} 4^{2i}a_i(n) \qquad \textit{for} \quad m \geqq 1 .$$

COROLLARY 4.6. $4^m a_m(n) \in Z_{(2)}$ *for* $m \geqq 0$.

For a non zero rational number $a = b/c$, we define $v_2(a) = v_2(b) - v_2(c)$, where b and c are integers, and we put

$$v_2(a_r(n)) = -2r + \varepsilon_1 \qquad \text{and} \qquad v_2(a_{r-1}(n)) = -2(r-1) + \varepsilon_2 \tag{4.7}$$

for given integers n and $r \geqq 1$. Then $\varepsilon_i \geqq 0$ for $i = 1, 2$ by Corollary 4.6, and we have the following:

LEMMA 4.8. *For* $r \geqq 1$, $\varepsilon_1 = 0$, 1, *or* 2 *if and only if the following holds respectively*:

$$\binom{n}{r} \equiv 1 \bmod 2\,, \qquad \binom{n}{r} \equiv 2 \bmod 4 \qquad \textit{or} \qquad \binom{n}{r} - (4n/3)\binom{n+1}{r-1} \equiv 4 \bmod 8\,.$$

PROOF. We have $a_1(n) = -(n/12)$ by Lemma 4.5. Then for $r \geqq 2$ and some $l \in Z_{(2)}$ we have

$$2^{2r}(1-4^r)a_r(n) = \binom{n}{r} - (4n/3)\binom{n+1}{r-1} + 16l$$

by Lemma 4.5 and Corollary 4.6. Thus the desired equivalences are obvious from this equation. Q.E.D.

Using Corollary 4.6 and the minimality of the codegree in Corollary 4.3 (i), we have the following lemma:

LEMMA 4.9. *Put* $cd_2^j(HP_n^{n+r}) = m_2(4r) - \varepsilon$ *for* $m_2(4r) = 3r - 2[r/2]$. *Then* $\varepsilon \geqq 0$, *and* $\varepsilon = 0$, 1 *or* 2 *if and only if the following* (0), (1) *or* (2) *holds for* ε_1 *and* ε_2 *in* (4.7), *respectively*:

(0) $\varepsilon_1 = 0$.

(1) $\varepsilon_1 = 1$ *for any* r; *or* $\varepsilon_1 > 1$ *and* $\varepsilon_2 = 0$ *when* r *is even.*

(2) $\varepsilon_1 = 2$ *when* r *is odd*;

$\varepsilon_1 \geqq 2$, $\varepsilon_2 \geqq 1$ *and* $(\varepsilon_1 - 2)(\varepsilon_2 - 1) = 0$ *when* r *is even.*

PROOF. By Corollary 4.3 (i) and (4.4), $cd_2^j(HP_n^{n+r})$ is equal to the minimal integer e satisfying $2^e a_i(n) \in Z_{(2)}$ (resp. $2^{e-1}a_i(n) \in Z_{(2)}$) for any even (resp. odd) integer i with $0 \leqq i \leqq r$. Then we have $\varepsilon \geqq 0$ by Corollary 4.6, which is also clear by Theorem A (i). We put $M = m_2(4r)$. Since $2^{M-1}a_i(n) \in 2Z_{(2)}$ for $0 \leqq i \leqq r-1$ by Corollary 4.6, $\varepsilon = 0$ if and only if $\varepsilon_1 = 0$. We have $2^{M-2}a_i(n) \in 2Z_{(2)}$ for $0 \leqq i \leqq r-2$, $2^{M-2}a_{r-1}(n) \in 2Z_{(2)}$ for odd $r \geqq 1$, and $2^{M-2}a_{r-1}(n) \in Z_{(2)}$ for even $r \geqq 2$, by Corollary 4.6. Thus $\varepsilon = 1$ if and only if (1) holds. Similarly we have the equivalence for $\varepsilon = 2$ and (2), because we have $2^{M-3}a_i(n) \in 2Z_{(2)}$ for $0 \leqq i \leqq r-2$ and $2^{M-2}a_{r-1} \in 2Z_{(2)}$ for odd $r \geqq 1$. Thus we have completed the proof. Q.E.D.

Consider the power series

$$H_n(y) = (1/((n+1)y^n))\frac{d}{dy}((2\,\mathrm{Sinh}^{-1}(\sqrt{y}/2))^{2n+2}) = \sum_{i=0}^{\infty} b_i(n)y^i\,,$$

where $b_0(n) = 1$ and $b_i(n) \in Q$. Then we have $H_n(y) = G_n(y)(d/dy)(yG_1(y))$, where $G_n(y)$ is the power series of (4.4). Thus we have the following relation between the coefficients $b_i(n)$ and $a_j(n)$:

LEMMA 4.10. $b_m(n) = \Sigma_{i=0}^{m}(i+1)a_{m-i}(n)a_i(1)$ *for* $m \geqq 0$.

COROLLARY 4.11. *$4^m b_m(n) \in Z_{(2)}$ for $m \geqq 0$.*

We also have the following corollary of Lemma 4.10:

COROLLARY 4.12. *Put $\nu_2(b_r(n)) = -2r + \varepsilon_3$. Then $\varepsilon_3 \geqq 0$, and $\varepsilon_3 = 0$, 1 or 2 if and only if the following* (0), (1) *or* (2) *holds for ε_1 and ε_2 in* (4.7), *respectively*:

(0) $\varepsilon_1 = 0$.

(1) $n \equiv 0 \bmod 4$ *when* $r = 1$;
$\varepsilon_1 \geqq 1$, $\varepsilon_2 \geqq 0$, $(\varepsilon_1 - 1)\varepsilon_2 = 0$ *and* $(\varepsilon_1, \varepsilon_2) \neq (1, 0)$, *when* $r \geqq 2$.

(2) $n \equiv 2 \bmod 8$ *when* $r = 1$;
$\varepsilon_1 = 1$, $\varepsilon_2 = 0$ *and* $2^{2r-1}a_r(n) - (1/3)2^{2r-2}a_{r-1}(n) \equiv 2 \bmod 4$, *or*
$\varepsilon_1 \geqq 2$, $\varepsilon_2 \geqq 1$, $(\varepsilon_1 - 2)(\varepsilon_2 - 1) = 0$ *and* $(\varepsilon_1, \varepsilon_2) \neq (2, 1)$, *when* $r \geqq 2$.

PROOF. We notice that $b_1(n) = -(n+2)/12$ by Lemma 4.10. Thus the assertions for $r = 1$ are clear, and so we assume $r \geqq 2$. By Lemma 4.5 we can easily see the values of $a_i(1)$ for $1 \leqq i \leqq 3$, and by Lemma 4.8 we have $\nu_2(a_i(1)) \geqq 4 - 2i$ for $i \geqq 4$. Then the equation in Lemma 4.10 is written as follows:

$$b_r(n) = a_r(n) - (1/6)a_{r-1}(n) + (1/30)a_{r-2}(n) + \sum_{i=3}^{r} (d_i/4^{i-2})a_{r-i}(n),$$

where d_i are some elements of $Z_{(2)}$. Thus the desired results are obvious from this equation and Corollary 4.6. Q.E.D.

PROOF OF THEOREM 2. By Theorem A (i) we have $\varepsilon \geqq 0$. For $0 \leqq \varepsilon \leqq 2$, $cd_2(HP_n^{n+r}) = m_2(4r) - \varepsilon$ if and only if $cd_2^j(HP_n^{n+r}) = m_2(4r) - \varepsilon$ by Theorem 1 (ii). Then we have the desired results by Corollary 4.3 (i) and Lemmas 4.8 and 4.9.

PROOF OF THEOREM 3. By Theorem A (i) we have $\varepsilon' \geqq 0$. For $0 \leqq \varepsilon' \leqq 2$, $cd_2(Q_{n+1}^{n+1+r}) = m_2(4r) - \varepsilon'$ if and only if $cd_2^j(Q_{n+1}^{n+1+r}) = m_2(4r) - \varepsilon'$ by Theorem 1. The assertion for $r = 1$ is clear by Corollary 4.12. Thus, we assume that $r \geqq 2$. In Lemma 4.9, if we replace $cd_2^j(HP_n^{n+r})$ to $cd_2^j(Q_{n+1}^{n+1+r})$, ε_1 to ε_3 given in Corollary 4.12 and ε_2 to $\nu_2(b_{r-1}(n)) + 2(r-1)$, then the analogous results are obtained by using Corollaries 4.3 (ii) and 4.11 instead of Corollaries 4.3 (i) and 4.6 respectively. Then $cd_2^j(Q_{n+1}^{n+1+r}) = m_2(4r)$ if and only if $\varepsilon_3 = 0$, and thus (0) is necessary and sufficient for $\varepsilon' = 0$ by Corollary 4.12 and Lemma 4.8. By the same way, we have the equivalence between the conditions $\varepsilon' = 1$ and (1′). Similarly, as a necessary and sufficient condition for $\varepsilon' = 2$, we obtain a condition formed by those in (2′) and the two other conditions, the latter of which are given using a, b and c in (1.3) as follows:

$$(a, b, c) = (0, 0, 1) \quad \text{when } r \text{ is even, or} \quad (a, b, c) = (4, 2, 1)\,.$$

But each of these conditions has a contradiction in itself, and must be excluded. Thus we have the desired results.

§5. Examples

In this section, we give some examples of (n, r) which satisfy the conditions (0)–(2) of Theorem 2 or (0)–(2′) of Theorem 3.

First we assume that $n \geqq r \geqq 1$. Let $\alpha(i)$ be the number of 1 in the diadic expansion of an integer i. Then it is well known that

$$v_2\left(\binom{k}{l}\right) = \alpha(l) + \alpha(k-l) - \alpha(k) \qquad \text{for} \qquad k \geqq l \geqq 0\,.$$

Using this relation, we can examine (n, r) whether it satisfies the condition. As for (0) of Theorems 2 and 3, a is odd if and only if $\alpha(r) + \alpha(n-r) = \alpha(n)$.

We put

$$(n, r) = [t, s] \qquad \text{if} \qquad n = t + m_1 + m_2 \qquad \text{and} \qquad r = s + m_1$$

for integers $t \geqq s \geqq 1$ and $m_i \geqq 0$ $(i = 1, 2)$ satisfying $\alpha(m_1 + m_2) = \alpha(m_1) + \alpha(m_2)$ and $2^{v_2(m_i)} > t$ if $m_i > 0$.

Lemma 5.1. *Assume that* $k \geqq 1$. *Then* (1), (2) *of Theorem* 2, *or* (1′), (2′) *of Theorem* 3 *holds if* (n, r) *takes the value in the following* (1), (2), (1′) *or* (2′) *respectively*:

(1) $[2^{k+1}+1, 2^k+1]$ (*when* r *is odd*);
$[2^{k+1}, 2^k]$ *or* $[2^l + 2^k - 1, 2^k]$ *for* $l > k$ (*when* r *is even*).

(2) $[2^{k+1}+2, 2^k+1]$ *or* $[17, 3]$ (*odd* r);
$[2^{k+2}+4, 2^{k+1}+2]$ *or* $[9, 6]$ (*even* r).

(1′) $[2^{k+1}+2, 2^k+1]$ *or* $[2^{k+2}+2^{k+1}+1, 2^{k+1}+1]$ (*odd* r);
$[l, 2^k]$ *for* $l = 2^{k+1}+1$ *or* 2^k (*even* r).

(2′) $[l, 2^{k+1}+1]$ *for* $l = 2^{k+2}+1$ *or* $2^{k+3}+2$, $[2^{k+2}+7, 2^{k+2}+1]$ *or* $[36, 5]$ (*odd* r);
$[2^{k+2}+1, 2^k]$, $[2^{k+2}+2^{k+1}+1, 2^{k+1}+2]$, $[2^{k+3}+2, 2^{k+2}+2]$ *or* $[20, 12]$ (*even* r).

As examples for $n < 0$, we have the following by Theorems 2 and 3:

Lemma 5.2. *Put* $cd_2(HP_n^{n+r}) = m_2(4r) - \varepsilon(n, r)$ *and* $cd_2(Q_{n+1}^{n+1+r}) = m_2(4r) - \varepsilon'(n, r)$.

(i) *Then,* $\varepsilon(-1, r) = \varepsilon(-2, 2r) = \varepsilon'(-1, r) = \varepsilon'(-2, 2r) = 0$ *for any* $r \geqq 1$.

(ii) $\varepsilon(-2, r) = 1$ (*resp.* 2) *if* $r \equiv 1 \bmod 4$ (*resp.* $r \equiv 3 \bmod 8$), *and* $\varepsilon'(-2, r) = 1$ (*resp.* 2) *if* $r \equiv 3 \bmod 4$ (*resp.* $r \equiv 5 \bmod 8$).

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