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On some contractive properties for the heat equations

Tetsuya KOYAMA

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0. Introduction

This work is concerned with contractive properties for the solutions of the following initial boundary value problem (IBVP) for the heat equation:

$$\text{(IBVP)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u(x, t) = \Delta u(x, t)\,, & x \in \Omega\,,\ t > 0\,, \\ u(x, 0) = u_0\,, & x \in \Omega\,, \\ u(x, t) = 0\,, & x \in \partial\Omega\,,\ t > 0\,, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ and $\partial\Omega$ denotes its boundary.

For a solution $u(x, t)$ of (IBVP), consider the following type of contraction property:

$$(\mathrm{D}_p) \qquad \|\nabla u(\cdot, t)\|_{L^p(\Omega)} \le \|\nabla u(\cdot, s)\|_{L^p(\Omega)}\,, \qquad 0 < s \le t\,,$$

for $1 \le p \le \infty$. Here, ∇u is the gradient of u. In [1], H. Engler showed that (D_p) holds for any domain Ω, if p is close to 2 in some sense. It is well known that (D_2) holds for any domain because $\|\nabla u(\cdot, t)\|_{L^2(\Omega)}$ is the Dirichlet integral of $u(\cdot, t)$. Furthermore, if the mean curvature H of $\partial\Omega$ is nonnegative (in this case, Ω is said to be H-convex), it is known that (D_p) holds for any p. (See [1].) Engler generalized this result to the case of arbitrary domains. In this note, we consider three functionals OSC_ε, H_α and Lip which are equivalent to the functional

$$u \mapsto \max\,\{|u(x) - u(y)|\,|\,x, y \in \bar{\Omega},\ |x - y| \le \varepsilon\}\,,$$

the usual Hölder norm and Lipschitz norm, respectively. These functionals, as well as the functional $u \mapsto \|\nabla u\|_{L^p(\Omega)}$, represent the regularity of u. The aim of this note is to show that the above three functionals have the same type of contractive properties as in (D_p) under the assumption that Ω is convex.

1. Three kinds of Lyapunov functionals for Δ

Let Ω be a bounded convex domain in $\mathbf{R}^N$. In what follows, we consider the Banach space

$$C_0(\bar{\Omega}) = \{u \in C(\bar{\Omega}) | u(x) = 0 \text{ for } x \in \partial\Omega\}$$

and the contraction semigroup $\{S(t) | t \geq 0\}$ on $C_0(\bar{\Omega})$ generated by Δ. For any $u_0 \in C_0(\bar{\Omega})$, $S(t)u_0$ gives the generalized solution of (IBVP). A functional ϕ on $C_0(\bar{\Omega})$ is said to be *a Lyapunov functional for* Δ, if ϕ is lower semicontinuous and

$$\phi(S(t)u_0) \leq \phi(u_0), \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad u_0 \in C_0(\bar{\Omega}).$$

For the Lyapunov functionals for operator semigroups, we refer to Pazy [4].

We then consider the following three kinds of functionals on $C_0(\bar{\Omega})$.

Firstly, for each $\varepsilon > 0$, we define continuous seminorm $\mathrm{OSC}_{\varepsilon,1}$, $\mathrm{OSC}_{\varepsilon,2}$ and OSC_ε on $C_0(\bar{\Omega})$ by

$$\mathrm{OSC}_{\varepsilon,1}(u) = \max\{|u(x) - u(y)| \,|\, x, y \in \bar{\Omega}, |x - y| \leq \varepsilon\},$$

$$\mathrm{OSC}_{\varepsilon,2}(u) = \max\{|u(x) + u(y)| \,|\, x, y \in \bar{\Omega} \text{ and there exists } z \in \partial\Omega \text{ such that } |x - z| + |z - y| \leq \varepsilon\},$$

and

$$\mathrm{OSC}_\varepsilon(u) = \max\{\mathrm{OSC}_{\varepsilon,1}(u), \mathrm{OSC}_{\varepsilon,2}(u)\}, \qquad \text{for} \quad u \in C_0(\bar{\Omega}),$$

respectively. The functional $\mathrm{OSC}_{\varepsilon,2}$ represents the regularity near the boundary and is indispensable for OSC_ε to be a Lyapunov functional for Δ. (See Remark in Section 2.) Also, note that $\mathrm{OSC}_{\varepsilon,1} \leq \mathrm{OSC}_\varepsilon \leq 2\mathrm{OSC}_{\varepsilon,1}$.

Secondly, for any $\alpha \in (0, 1)$ and $u \in C_0(\bar{\Omega})$, we put

$$\mathrm{H}_\alpha(u) = \sup_{\varepsilon>0} \varepsilon^{-\alpha}\mathrm{OSC}_\varepsilon(u)$$

and define the associated space

$$C_0^\alpha(\bar{\Omega}) = \{u \in C_0(\bar{\Omega}) | \mathrm{H}_\alpha(u) < \infty\},$$

Note that H_α is equivalent to the usual Hölder norm

$$\|u\|_\alpha = \sup\{|u(x) - u(y)| |x - y|^{-\alpha} | x, y \in \bar{\Omega}, x \neq y\}.$$

In fact,

$$\|u\|_\alpha = \sup_{\varepsilon>0} \varepsilon^{-\alpha}\mathrm{OSC}_{\varepsilon,1}(u)$$

holds.

Finally, we define

$$\mathrm{Lip}\,(u) = \sup_{\varepsilon>0} \varepsilon^{-1}\mathrm{OSC}_\varepsilon(u),$$

and write the associated space as

$$Lip_0(\bar{\Omega}) = \{u \in C_0(\bar{\Omega}) | \mathrm{Lip}\,(u) < \infty\}.$$

Note that Lip coincides with the usual Lipschitz norm

$$\|u\|_{Lip} = \sup\{|u(x) - u(y)||x - y|^{-1} | x, y \in \bar{\Omega}, x \neq y\}.$$

In fact,

$$\|u\|_{Lip} = \sup_{\varepsilon > 0} \varepsilon^{-1} \mathrm{OSC}_{\varepsilon,1}(u)$$

and there is a number $0 < k \leq 1$ (which may depend on u and ε) such that

$$\mathrm{OSC}_\varepsilon(u) \leq k^{-1} \mathrm{OSC}_{k\varepsilon,1}(u)$$

for any $u \in C_0(\bar{\Omega})$.

Our main theorem is then stated as follows.

THEOREM 1. *Let Ω be a bounded convex domain in $\mathbf{R}^N$, and $\{S(t)\}$ be the contraction semigroup on $C_0(\bar{\Omega})$ associated with* (IBVP). *Then we have the following.*

(i) OSC_ε *is a Lyapunov functional for Δ.*

(ii) *Let $0 < \alpha < 1$. Then $C_0^\alpha(\bar{\Omega})$ is invariant under $\{S(t)\}$ and* H_α *is a Lyapunov functional for Δ.*

(iii) *$Lip_0(\Omega)$ is invariant under $\{S(t)\}$ and* Lip *is a Lyapunov functional for Δ.*

2. Lyapunov estimates of the resolvents of Δ

Fix $\tau > 0$ and consider the following boundary value problem:

$$\text{(E)} \qquad \begin{cases} u(x) - \tau \Delta u(x) = v(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

The aim of this section is to show the following theorem which plays a crucial role in proving Theorem 1.

THEOREM 2. *For any $v \in C_0(\bar{\Omega})$ and $\tau > 0$, there exists a unique solution $u(x) \in C_0(\bar{\Omega})$ of* (E) *and, for any $\varepsilon > 0$,*

$$\text{(D)} \qquad \mathrm{OSC}_\varepsilon(u) \leq \mathrm{OSC}_\varepsilon(v)$$

holds.

REMARK. There is a case in which $u(x)$, $v(x)$, $\tau > 0$ and $\varepsilon > 0$ are as in Theorem 2, but

$$\mathrm{OSC}_{\varepsilon,1}(u) \leq \mathrm{OSC}_{\varepsilon,1}(v)$$

fails for any sufficiently small $\tau > 0$, and for some $\varepsilon > 0$. In fact, put

$\Omega = (-1, 1) \subset \mathbf{R}$,

$$v(x) = \begin{cases} (2/3) - |x|, & \text{if } 0 \le |x| < 1/3, \\ 1/3, & \text{if } 1/3 \le |x| < 2/3, \\ 1 - |x|, & \text{if } 2/3 \le |x| < 1, \end{cases}$$

and $\varepsilon = 2/3$. Then $u(\pm 1/3) > v(\pm 1/3) = \mathrm{OSC}_{(2/3),1}(v)$ for any sufficiently small $\tau > 0$.

Proof of Theorem 2. Fix any $\eta > 0$. Let Ω_η be a bounded convex domain in $\mathbf{R}^N$ which satisfies the following conditions:

$$\bar{\Omega} \subset \Omega_\eta, \quad \eta/2 \le \operatorname{dist}(z, \bar{\Omega}) < \eta \qquad \text{for any } z \in \partial\Omega_\eta, \quad \text{and } \partial\Omega_\eta \text{ is of class } C^2.$$

Put $v_\eta = v * \rho_{(\eta/2)} \in C_0(\bar{\Omega}_\eta) \cap C^\infty(\Omega_\eta)$, where $\rho_{(\eta/2)}$ is the Friedrichs mollifier. Then there exists a unique solution $u_\eta \in C_0(\bar{\Omega}_\eta) \cap C^\infty(\Omega_\eta)$ to the following problem (E_η):

$$(\mathrm{E}_\eta) \qquad \begin{cases} u_\eta(x) - \tau \Delta u_\eta(x) = v_\eta(x), & x \in \Omega_\eta, \\ u_\eta(x) = 0, & x \in \partial\Omega_\eta. \end{cases}$$

See for instance Mizohata [2], Chapter 3.

For each $\varepsilon > 0$ and $\eta > 0$, we define seminorms $\mathrm{OSC}_{\varepsilon,\eta,1}$, $\mathrm{OSC}_{\varepsilon,\eta,2}$ and $\mathrm{OSC}_{\varepsilon,\eta}$ on $C_0(\bar{\Omega}_\eta)$ by

$$\mathrm{OSC}_{\varepsilon,\eta,1}(u) = \max\{u(x) - u(y) \mid x, y \in \bar{\Omega}_\eta, |x - y| \le \varepsilon\},$$

$$\mathrm{OSC}_{\varepsilon,\eta,2}(u) = \max\{|u(x) + u(y)| \mid x, y \in \bar{\Omega}_\eta \text{ and there exists } z \in \partial\Omega_\eta \text{ such that } |x - z| + |z - y| \le \varepsilon\},$$

$$\mathrm{OSC}_{\varepsilon,\eta}(u) = \max\{\mathrm{OSC}_{\varepsilon,\eta,1}(u), \mathrm{OSC}_{\varepsilon,\eta,2}(u)\},$$

for $u \in C_0(\bar{\Omega}_\eta)$, respectively.

In what follows, we demonstrate the following key estimate

$$(\mathrm{D}_\eta) \qquad \mathrm{OSC}_{\varepsilon,\eta}(u_\eta) \le \mathrm{OSC}_{\varepsilon,\eta}(v_\eta).$$

The idea of the proof is illustrated as follows. Let Ω'_η be a copy of Ω_η, identify $\partial\Omega_\eta$ and $\partial\Omega'_\eta$ and put $\tilde{\Omega}_\eta = \bar{\Omega}_\eta \cup \bar{\Omega}'_\eta$. Then $\tilde{\Omega}_\eta$ is a C^2-manifold without boundary. Let $x' \in \bar{\Omega}'_\eta$ (or $x' \in \bar{\Omega}_\eta$) denote the corresponding point of $x \in \bar{\Omega}_\eta$ (or $x \in \bar{\Omega}'_\eta$ respectively). Put

$$\tilde{v}_\eta(x) = \begin{cases} v_\eta(x), & \text{if } x \in \bar{\Omega}_\eta, \\ -v_\eta(x'), & \text{if } x \in \bar{\Omega}'_\eta, \end{cases}$$

and consider the problem

$$(\tilde{\mathrm{E}}_\eta) \qquad \tilde{u}_\eta(x) - \tau \Delta \tilde{u}_\eta(x) = \tilde{v}_\eta(x), \qquad x \in \tilde{\Omega}_\eta .$$

Let $\tilde{u}_\eta(x)$ be a solution of $(\tilde{\mathrm{E}}_\eta)$ and define $u_\eta \in C_0(\bar{\Omega}_\eta)$ by $u_\eta(x) = \tilde{u}_\eta(x)$ for $x \in \bar{\Omega}_\eta$. Then u_η satisfies the original problem (E_η) and $\tilde{u}_\eta(x) = -u_\eta(x')$ for $x \in \Omega'_\eta$. Moreover $\mathrm{OSC}_{\varepsilon,\eta}(u_\eta)$ coincides with

$$\mathrm{OSC}^{\sim}_{\varepsilon,\eta}(\tilde{u}_\eta) = \max \{ |\tilde{u}_\eta(x) - \tilde{u}_\eta(y)| \,|\, x, y \in \tilde{\Omega}_\eta, d(x, y) \le \varepsilon \} ,$$

where d denotes a metric on $\tilde{\Omega}_\eta$ defined by

$$d(x, y) = \begin{cases} |x - y| , & \text{if } x, y \in \bar{\Omega}_\eta \text{ or } x', y' \in \bar{\Omega}_\eta , \\ \min \{ |x - z| + |z - y| \,|\, z \in \partial\Omega_\eta = \partial\Omega'_\eta \} , & \text{if } x, y' \in \bar{\Omega}_\eta \text{ or } x', y \in \bar{\Omega}_\eta . \end{cases}$$

Thus it is sufficient to show the contractive property (D_η) for $\mathrm{OSC}^{\sim}_{\varepsilon,\eta}$. The proof will be divided into essentially three cases as follows.

Case 1. Suppose that there exist x_0 and $y_0 \in \Omega_\eta$ such that $|x_0 - y_0| \le \varepsilon$ and $\mathrm{OSC}^{\sim}_{\varepsilon,\eta}(\tilde{u}_\eta) = u_\eta(x_0) - u_\eta(y_0)$.

For sufficiently small $h > 0$, we have

$$\begin{aligned} h^{-N-1} & \left\{ \int_{|\xi|=h} [u_\eta(x_0 + \xi) - u_\eta(x_0)] \, dS_\xi - \int_{|\xi|=h} [u_\eta(y_0 + \xi) - u_\eta(y_0)] \, dS_\xi \right\} \\ &= h^{-N-1} \int_{|\xi|=h} \{ [u_\eta(x_0 + \xi) - u_\eta(y_0 + \xi)] - [u_\eta(x_0) - u_\eta(y_0)] \} \, dS_\xi \le 0 . \end{aligned}$$

Here dS_ξ denotes the surface element on the sphere $\{\xi \in \mathbf{R}^N \,|\, |\xi| = h\}$. Letting $h \downarrow 0$, we have $\Delta u_\eta(x_0) \le \Delta u_\eta(y_0)$, and this yields

$$u_\eta(x_0) - u_\eta(y_0) \le v_\eta(x_0) - v_\eta(y_0)$$

and (D_η).

Case 2. Suppose that there exist $x_0 \in \Omega_\eta$, $y_0 \in \Omega'_\eta$ and $z_0 \in \partial\Omega_\eta$ such that $d(x_0, y_0) = |x_0 - z_0| + |z_0 - y_0| \le \varepsilon$ and $\mathrm{OSC}^{\sim}_{\varepsilon,\eta}(\tilde{u}_\eta) = \tilde{u}_\eta(x_0) - \tilde{u}_\eta(y_0) = u_\eta(x_0) + u_\eta(y'_0)$.

Let π denote the tangent hyperplane of $\partial\Omega_\eta$ at z_0, and let $r(x)$ denote the reflexion of a point $x \in \mathbf{R}^N$ with respect to π. Since

$$\min \{ |x_0 + \xi - z| + |z - r(r(y'_0) + \xi)| \,|\, z \in \partial\Omega_\eta \} \le \varepsilon$$

holds for $\xi \in \mathbf{R}^N$ with $|\xi|$ sufficiently small, we have

$$h^{-N-1} \int_{|\xi|=h} \{ [u_\eta(x_0 + \xi) - u_\eta(x_0)] + [u_\eta(r(r(y'_0) + \xi)) - u_\eta(y'_0)] \} \, dS_\xi \le 0$$

for sufficiently small $h > 0$. Letting $h \downarrow 0$, we have $\Delta u_\eta(x_0) + \Delta u_\eta(y'_0) \le 0$. This yields (D_η).

Case 3. Suppose that there exist $x_0 \in \Omega_\eta$ and $y_0 \in \partial\Omega_\eta$ such that $|x_0 - y_0| \le \varepsilon$ and $\mathrm{OSC}_{\varepsilon,\eta}^{\sim}(\tilde{u}_\eta) = u_\eta(x_0) - u_\eta(y_0) = u_\eta(x_0)$.

Let π, $r(x)$ and $h > 0$ be as in Case 2. Put

$$A = \{\xi \in \mathbf{R}^N \,|\, |\xi| = h,\ y_0 + \xi \in \Omega_\eta\},$$

$$B = \{\xi \in \mathbf{R}^N \,|\, |\xi| = h,\ r(y_0 + \xi) \in \Omega_\eta\},$$

$$C = \{\xi \in \mathbf{R}^N \,|\, |\xi| = h\} \backslash (A \cup B),$$

and note that

$$\begin{aligned}
&\int_{|\xi|=h} [u_\eta(x_0 + \xi) - u_\eta(x_0)]\, dS_\xi \\
&= \int_A \{[u_\eta(x_0 + \xi) - u_\eta(y_0 + \xi)] - [u_\eta(x_0) - u_\eta(y_0)]\}\, dS_\xi \\
&\quad + \int_B \{[u_\eta(x_0 + \xi) + u_\eta(r(y_0 + \xi))] - [u_\eta(x_0) + u_\eta(y_0)]\}\, dS_\xi \\
&\quad + \int_C [u_\eta(x_0 + \xi) - u_\eta(x_0)]\, dS_\xi .
\end{aligned}$$

It is not difficult to show that

$$d(x_0 + \xi, r(y_0 + \xi)') \le \varepsilon, \qquad \text{for} \quad \xi \in B$$

and that $\int_C = o(h^{N+1})$. From this we infer that $\Delta u_\eta(x_0) \le 0$. Thus (D_η) is obtained.

It is easy to show that $\mathrm{OSC}_{\varepsilon,\eta}(v_\eta) \le \mathrm{OSC}_\varepsilon(v)$ and $\lim_{\eta\downarrow 0} v_\eta(x) = v(x)$ uniformly on $\bar{\Omega}$. This and (D_η) together imply $\sup_{\eta>0} \mathrm{OSC}_{\varepsilon,\eta}(u_\eta) < \infty$. By the Ascoli-Arzela theorem and the closedness of Δ, we have $\lim_{\eta\downarrow 0} u_\eta(x) = u(x)$, where $u(x)$ is a solution of (E). Since

$$\mathrm{OSC}_\varepsilon(u_\eta) - 2\mathrm{OSC}_{\eta,\eta}(u_\eta) \le \mathrm{OSC}_{\varepsilon,\eta}(u_\eta) \le \mathrm{OSC}_\varepsilon(v),$$

letting $\eta \downarrow 0$ gives (D). The proof of Theorem 2 is thereby complete.

3. Proof of Theorem 1

Under the Dirichlet boundary condition, we see from the maximum principle and Theorem 2 that Δ is m-dissipative on $C_0(\bar{\Omega})$ and generates a (C_0)-contraction semigroup $\{S(t)\}$ on $C_0(\bar{\Omega})$ represented by

$$\lim_{\tau\downarrow 0} [1 - \tau\Delta]^{-[t/\tau]} u_0 = S(t) u_0$$

uniformly on $\bar{\Omega}$. (See Pazy [3].) On the other hand, Theorem 2 implies that

$$\mathrm{OSC}_\varepsilon([1 - \tau\Delta]^{-[t/\tau]}u_0) \leq \mathrm{OSC}_\varepsilon(u_0)\,.$$

Letting $\tau \downarrow 0$, we obtain (i).

Finally (ii) and (iii) can be easily seen from the definitions.

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Department of Mathematics,
Hiroshima Institute of Technology