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Asymptotic behavior of three Riemannian metrics on the moduli space of 1-instantons over a definite 4-manifold

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Introduction

The moduli space of instantons over a compact Riemannian 4-manifold carries three natural symmetric tensors γ_{I} (positive definite), $\gamma_{\mathrm{I-II}}$ and γ_{II} (positive semidefinite) [10] (also see §1).

These tensors have been explicitly computed for 1-instantons over S^4 [2], [5], [7], [10] and CP^2 [4], [8]; we know that γ_{I}, $\gamma_{\mathrm{I-II}}$ and γ_{II} are smooth and positive definite in these cases.

Let M be a compact oriented 1-connected Riemannian 4-manifold with positive definite intersection form, and $\mathscr{M}$ be the moduli space of 1-instantons over M. In [6] D. Groisser and T. H. Parker investigated the Riemannian geometry of $\mathscr{M}$. In particular they described the C^0-asymptotic behavior of γ_{I} on the collar of $\mathscr{M}$, using the collar map defined by S. K. Donaldson [1].

In this paper, we shall study the C^0-asymptotic behavior of the symmetric tensors $\gamma_{\mathrm{I-II}}$ and γ_{II} on a collar of $\mathscr{M}$. As a corollary of our theorem, we see that each of the symmetric tensors $\gamma_{\mathrm{I-II}}$ and γ_{II} defines a Riemannian metric on some collar of $\mathscr{M}$ with infinite volume.

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§1. Asymptotic behavior

We fix a smooth Riemannian metric g_M on M and a principal $Sp(1)$-bundle P over M with the second Chern number $c_2(P) = -1$. Also $\mathfrak{g}_P$ stands for the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{sp}(1)$.

Let A be a 1-instanton, that is, a self-dual connection on P. Assume that A represents a smooth point of $\mathscr{M}$. Then the tangent space $T_{[A]}\mathscr{M}$ is identified with $\{v \in \Gamma(M, T^*M \otimes \mathfrak{g}_P); D_A^* v = 0, p_- D_A v = 0\}$. Here D_A denotes the covariant derivative, D_A^* is its formal adjoint and $p_-: \bigwedge^2 T^*M \to \bigwedge^2_- T^*M$ denotes the projection onto anti-self-dual 2-covectors. We denote by (,) the inner product on $\bigwedge^2 T^*M \otimes \mathfrak{g}_P$ which is induced by g_M and twice the quaternionic norm on $\mathfrak{sp}(1) \subset H$. Let F_A be the curvature of A and let Q_A denote the orthogonal projection $\bigwedge^2 T^*M \otimes \mathfrak{g}_P \to \{\varphi \in \bigwedge^2 T^*M \otimes \mathfrak{g}_P; (\mathrm{ad}\, F_A)^* \varphi = 0\}$ where

$(\mathrm{ad}\, F_A)^*$ is the adjoint of $\mathrm{ad}\, F_A\colon \mathfrak{g}_P \to \bigwedge^2 T^*M \otimes \mathfrak{g}_P$ (with respect to the inner products on $\mathfrak{g}_P$ and $\bigwedge^2 T^*M \otimes \mathfrak{g}_P$). In [10], the three symmetric bilinear forms γ_J (J = I, II and I–II) on $T_{[A]}\mathcal{M}$ are defined as follows: for $v, w \in T_{[A]}\mathcal{M}$,

$$\gamma_{\mathrm{I}}(v, w) = \int_M (v, w)\omega_M\,, \qquad \gamma_{\mathrm{I-II}}(v, w) = \int_M (D_A v, D_A w)\omega_M\,,$$

$$\gamma_{\mathrm{II}}(v, w) = \int_M (Q_A D_A v, Q_A D_A w)\omega_M\,,$$

where ω_M is the Riemannian volume element with respect to g_M. Here we notice that γ_{II} has conformal invariance, and that T. Matumoto shows that the symmetric tensor γ_{II} on the moduli space of 1-instantons on S^4 gives a metric with constant sectional curvature $-5/32\pi^2$ (see [10]).

The symmetric tensors γ_{I} and $\gamma_{\mathrm{I-II}}$ are always smooth since g_M is smooth. On the other hand, we know only that γ_{II} is continuous if g_M is analytic on some neighborhood of any point of M. In fact, the measure of $\{x \in M;\, \mathrm{rank}\,(\mathrm{ad}\, F_A)_x \le 2\}$ is zero because any Yang-Mills connection is locally gauge equivalent to an analytic connection by the above assumption [11, Cor. 1.4]. We take a convergent sequence $\{A_n\}$ of irreducible self-dual connections. Then $\mathrm{Im}\,(\mathrm{ad}\, F_{A_n})$ is a subbundle of $\bigwedge^2 T^*M \otimes \mathfrak{g}_P$ over $M \backslash (\bigcup_n \{x \in M;\, \mathrm{rank}\,(\mathrm{ad}\, F_{A_n})_x \le 2\})$ for all n. Since $(Q_A D_A v, Q_A D_A w) = \{(D_A v, D_A w) - \Sigma_i (u_i, D_A v)(u_i, D_A w)\}$, where $\{u_i(x)$ with $x \in M\}$ is an orthonormal basis of $\mathrm{Im}\,(\mathrm{ad}\, F_A)_x \subset \bigwedge^2 T^*M \otimes \mathfrak{g}_P$, we see that γ_{II} is continuous by Lebesgue's dominated convergence theorem.

Let $\kappa\colon M \times (0, \lambda_0) \to \mathcal{M}$ be the collar map defined by S. K. Donaldson [1] (also see [3], [9]), and consider the following three Riemannian metrics μ_J (J = I, I–II and II) on $M \times (0, \lambda_0)$:

$$\mu_{\mathrm{I}} = 4\pi^2(g_M + 2(d\lambda)^2)\,, \qquad \mu_{\mathrm{I-II}} = (32\pi^2/5)(3g_M/2 + (d\lambda)^2)\,,$$

$$\mu_{\mathrm{II}} = (32\pi^2/5)(g_M + (d\lambda)^2)\,.$$

The symmetric tensors $\kappa^*\gamma_J$ can be compared with μ_J.

In case J = I, Groisser and Parker [6, Theorem II] proved that

$$\lim_{\lambda\to 0} \kappa^*\gamma_{\mathrm{I}} = \mu_{\mathrm{I}}\,.$$

The purpose of this paper is to prove the following.

THEOREM 1. *For* J = I − II *and* I, *we have* $\lim_{\lambda\to 0} \lambda^2\kappa^*\gamma_J = \mu_J$.

Hereafter in this paper J denotes I–II or II. By Theorem 1 we see that the metric $\lambda^2\kappa^*\gamma_J$ extends to $\partial\bar{\mathcal{M}} = M \times \{0\}$, and $\kappa^*\gamma_J$ is C^0-asymptotic to μ_J/λ^2. We can note that the sectional curvature of μ_J/λ^2 converges to $-5/32\pi^2$

as λ tends to zero, as so does that of γ_J when $M = S^4$ or CP^2 with standard Riemannian metric [8], [10]. But we do not know that C^1-asymptotic behavior of γ_J when M is a general one.

§2. Proof of Theorem 1

To begin with we prepare some notation. For $\varepsilon > 0$ let $B(\varepsilon) = \{x \in R^4; r = |x| < \varepsilon\}$. We fix a coordinate neighborhood $B = B(\varepsilon_0)$ around $m_0 \in M$ on which $g_M = \delta_{ij} + O(r^2)$ holds. Let β be a smooth function on M such that its support is contained in B and $\beta(x) = b_1 x_1 + \cdots + b_4 x_4 + b_0 r^2/2\lambda$ on a neighborhood of $m_0 = 0 \in B$. We may assume that β depends smoothly on the parameters $(b_1, b_2, b_3, b_4, b_0)$. Let X be the vector field on M defined by $d\beta = g_M(X, \cdot)$. Let D_λ, F_λ and Q_λ stand for D_A, F_A and Q_A with $[A] = \kappa(m_0, \lambda)$, respectively. Let τ_λ: $B(\rho) \to B(\lambda\rho)$ be the dilation by λ and put $g_\lambda = \tau_\lambda^* g_M/\lambda^2$. Then $\lim_{\lambda\to 0} g_\lambda = g_0 = (dx_1)^2 + \cdots + (dx_4)^2$. Let D_0 stand for the standard instanton $d + (1 + r^2)^{-1}\,\mathrm{Im}\,(x\, d\bar{x})$ on $H = R^4$. By virture of [3, Theorem 8.31], we may assume that $\lim_{\lambda\to 0} \tau_\lambda^* D_\lambda = D_0$ by rechoosing the representative of $[A_\lambda]$ if necessary.

Hereafter we take $\rho \gg 1$ and $0 < \lambda \ll 1$ such that $B(\lambda\rho) \subset B$, and all c_i, $i = 1, 2, \ldots$, appearing in the following denote constants independent of λ, b and ρ. Our estimates will rely on the following lemma.

LEMMA 2.

$$\text{(1)} \qquad \lim_{\lambda\to 0} \int_{M\setminus B(\lambda\rho)} |F_\lambda|^2 \omega_M = 8\pi^2(1 + 3\rho^2)/(1 + \rho^2)^3 .$$

(2) *Let* $|b|^2 = b_0^2 + \cdots + b_4^2$. *Then*

$$\limsup_{\lambda\to 0} \lambda^2 \int_{M\setminus B(\lambda\rho)} |X|^2(|F_\lambda|^2 + |F_\lambda|^3)\omega_M \le c_1 |b|^2/\rho .$$

PROOF. (1) The proof is carried out by the computation on the curvature form F_0 for the standard instanton in the following formula.

$$\begin{aligned}\lim_{\lambda\to 0} \int_{M\setminus B(\lambda\rho)} |F_\lambda|^2 \omega_M &= 8\pi^2 - \lim_{\lambda\to 0} \int_{B(\lambda\rho)} |F_\lambda|^2 \omega_M \\ &= 8\pi^2 - \int_{B(\rho)} |F_0|^2 \omega_0 .\end{aligned}$$

(2) First we consider the case that $b_0 = 0$, that is, $\beta(x) = b_1 x_1 + \cdots + b_4 x_4$ around 0. Then $|X|^2 \le c_2 |b|^2$. Also we know that $|F_\lambda| \le c_3 \lambda^{2-\delta}/r^{4-\delta}$ on

$B(r_0)\backslash B(\lambda\rho)$ for some $r_0 > 0$ and $0 < \delta < 1$ [6, §3 Fact B] (see also [1, Theorem 16] and [3, Theorem 9.8]). Since the support of X is compact, we have

$$\lambda^2 \int_{M\backslash B(\lambda\rho)} |X|^2(|F_\lambda|^2 + |F_\lambda|^3)\omega_M \le c_4|b|^2 \int_{\lambda\rho}^{\infty} \{(\lambda^{2-\delta}/r^{4-\delta})^2 + (\lambda^{2-\delta}/r^{4-\delta})^3\} r^3\, dr$$
$$\le c_5|b|^2(\lambda^2\rho^{-4+2\delta} + \rho^{-8+3\delta}).$$

Hence we have the required estimate in this case.

Second if $\beta(x) = b_0 r^2/2\lambda$ around 0, then we have

$$\lambda^2 \int_{M\backslash B(\lambda\rho)} |X|^2(|F_\lambda|^2 + |F_\lambda|^3)\omega_M \le c_6 b_0^2(\lambda^2\rho^{-2+2\delta} + \rho^{-6+3\delta}).$$

For the general case $\beta(x) = b_1x_1 + \cdots + b_4x_4 + b_0r^2/2\lambda$ around 0 we have the required estimate, applying Schwarz's inequality to the above estimates (cf. [6, (3.12)]). □

Proof of Theorem 1. Following [3, §9] and [6, §3], we describe the tangent vectors of $\mathcal{M}$ at $\kappa(m_0, \lambda)$ which is represented by D_λ. Since λ is sufficiently small, we can find $a_\lambda \in \Gamma(M, p_-(\bigwedge^2 T^*M) \otimes \mathfrak{g}_P)$ so that $p_-D_\lambda(p_-D_\lambda)^*a_\lambda = -p_-D_\lambda(\iota_X F_\lambda)$ [3, Theorem 7.19]. For this a_λ we set $u_\lambda = (p_-D_\lambda)^*a_\lambda$ and $v_\lambda = \iota_X F_\lambda + u_\lambda$. Then $p_-D_\lambda(p_-D_\lambda)^*a_\lambda = -p_-D_\lambda(\iota_X F_\lambda)$ means that $p_-D_\lambda v_\lambda = 0$. On the other hand $D_\lambda^* v_\lambda = D_\lambda^*(\iota_X F_\lambda + u_\lambda) = D_\lambda^*(\iota_X F_\lambda) = *D_\lambda(d\beta \wedge *F_\lambda) = 0$, since $a_\lambda \in \Gamma(M, p_-(\bigwedge^2 T^*M) \otimes \mathfrak{g}_P)$ and $d\beta = g_M(X, \cdot)$, where $*$ is the Hodge star operator. Thus $v_\lambda \in T_{\kappa(m_0,\lambda)}\mathcal{M}$. The parameters of v_λ are given by (b', b_0) through X with $b' = (b_1, b_2, b_3, b_4)$. Since the vector field X coincides with $X_{b_0,b'}$ defined in [3, (9.15)] in a neighborhood of m_0, we can show that Proposition 9.21 and Proposition 9.29 in [3] are valid also for X and a_λ instead of $X_{b_0,b'}$ and $\Phi_{b_0,b'}$. It follows that $\kappa_* b = (1 + O(\lambda))v_\lambda$ for $b = b_1\partial_1 + \cdots + b_4\partial_4 + b_0\partial_\lambda \in T_{(0,\lambda)}(B \times (0, \lambda_0))$ from this.

Let $P_\lambda = 1$ if $\mathrm{J} = \mathrm{I}$–II and $P_\lambda = Q_\lambda$ if $\mathrm{J} = \mathrm{II}$. In view of [3, Proof of Proposition 9.29], we have

$$\limsup_{\lambda\to 0} \int_M |D_\lambda u_\lambda|^2 \omega_M \le c_7.$$

Therefore

$$\lim_{\lambda\to 0} \lambda^2 \kappa^*\gamma_{\mathrm{J}}(b, b) = \lim_{\lambda\to 0} \lambda^2 \int_M |P_\lambda D_\lambda \iota_X F_\lambda|^2 \omega_M.$$

First we will estimate this integral on $B(\lambda\rho)$. Let Y be a vector field on $B(\rho)$ defined by $g_\lambda(Y, \cdot) = b_1\,dx_1 + \cdots + b_4 dx_4 + b_0 dr^2/2$. Then we have

$$\lim_{\lambda\to 0} \lambda^2 \int_{B(\lambda\rho)} |P_\lambda D_\lambda \iota_X F_\lambda|^2 \omega_M$$

$$= \lim_{\lambda\to 0} \int_{B(\rho)} |\tau_\lambda^* P_\lambda \tau_\lambda^* D_\lambda \iota_Y \tau_\lambda^* F_\lambda|_\lambda^2 \omega_\lambda$$

$$= \int_{B(\rho)} 48\{(4b_0^2(1-r^2)^2 + (|b|^2 - b_0^2)(4+2q)r^2)/(1+r^2)^6\}\omega_0$$

$$= (16\pi^2/5)(2b_0^2 + (|b|^2 - b_0^2)(q+2)) - (16\pi^2/5)\{2(15\rho^6 - 5\rho^4 + 5\rho^2 + 1)b_0^2 + (q+2)(|b|^2 - b_0^2)(10\rho^4 + 5\rho^2 + 1)\}/(1+\rho^2)^5 ,$$

where $q = 1$ if J = I–II and $q = 0$ if J = II. Hence Theorem 1 follows immediately from the next lemma.

LEMMA 3. $\limsup_{\lambda\to 0} \lambda^2 \int_{M\backslash B(\lambda\rho)} |D_\lambda \iota_X F_\lambda|^2 \omega_M \le c_8 |b|^2/\rho$.

PROOF. We denote by ∇_M the Levi–Civita connection with respect to g_M, and we set $\nabla = \nabla_M \otimes 1 + 1 \otimes D_\lambda$. Then $|D_\lambda \iota_X F_\lambda| \le |\nabla(X \otimes F)| \le c_9(|\nabla_M X||F_\lambda| + |X||\nabla F_\lambda|)$. The proof of Lemma 2 (2) implies that

$$\limsup_{\lambda\to 0} \lambda^2 \int_{M\backslash B(\lambda\rho)} |\nabla_M X|^2 |F_\lambda|^2 \omega_M \le c_{10} |b|^2/\rho .$$

Let Z be a vector field on M such that $g_M(Z, \cdot) = d|F_\lambda|^2/2 = (F_\lambda, \nabla F_\lambda)$. Then we have $|\nabla F_\lambda|^2 = -\operatorname{div} Z + (F_\lambda, \nabla^* \nabla F_\lambda)$. Using Bochner-Weitzenböck formula (cf. [9, Appendix II]), we see that $|(F_\lambda, \nabla^*\nabla F_\lambda)| \le c_{11}(|F_\lambda|^2 + |F_\lambda|^3)$ because D_λ is a Yang-Mills connection. In view of Lemma 2 (2), it is enough to show the following

LEMMA 4. $\limsup_{\lambda\to 0} |\lambda^2 \int_{M\backslash B(\lambda\rho)} |X|^2 \operatorname{div} Z\, \omega_M| \le c_{12} |b|^2/\rho$.

PROOF. Let $S(\varepsilon) = \{x \in R^4; |x| = \varepsilon\}$ for $\varepsilon > 0$. Using g_λ, we define, as usual, a norm $|\ |_\lambda$ on $\bigwedge^p T^*B(\rho) \otimes \tau_\lambda^* g_P$, a volume element ω_λ on $B(\rho)$ and a contraction $(\ ,\ )_\lambda$ with respect to g_λ.

If $\beta(x) = b_1 x_1 + \cdots + b_4 x_4$ around 0, then $|X|^2 \le c_2 |b|^2$. Applying Stokes' formula, we have

$$\lambda^2 \int_{M\backslash B(\lambda\rho)} \operatorname{div} Z\, \omega_M = \lambda^2 \int_{S(\lambda\rho)} \iota_Z \omega_M = \int_{S(\rho)} (d|\tau_\lambda^* F_\lambda|_\lambda^2/2, \omega_\lambda)_\lambda .$$

As $\lambda \to 0$, this integral converges to

$$\int_{S(\rho)} (d|F_0|_0^2/2, \omega_0)_0 = 768\pi^2\rho^4/(1+\rho^2)^5 .$$

Now we deal with the case $\beta(x) = r^2/2\lambda$. Let $\alpha = 8\lambda^2|X|^2$ and let a vector field W satisfy $g_M(W, \cdot) = d\alpha$. Since $\iota_Z d\alpha = L_W|F_\lambda|^2/2$, we have $(\iota_Z d\alpha)\omega_M = d(|F_\lambda|^2\iota_W\omega_M)/2 - |F_\lambda|^2 L_W\omega_M/2$. Also we see that $\alpha\,\mathrm{div}Z\,\omega_M = d(\alpha\iota_Z\omega_M) - (\iota_Z d\alpha)\omega_M$. Hence

$$\int_{M\backslash B(\lambda\rho)} \alpha\,\mathrm{div}Z\,\omega_M = \int_{S(\rho)} |dr^2|_\lambda^2 (d|\tau_\lambda^* F_\lambda|_\lambda^2, \omega_\lambda)_\lambda - \int_{S(\rho)} |\tau_\lambda^* F_\lambda|_\lambda^2 (d|dr^2|_\lambda^2, \omega_\lambda)_\lambda + \int_{M\backslash B(\lambda\rho)} |F_\lambda|^2 L_W\omega_M\,.$$

Now we note that

$$\int_{S(\rho)} |dr^2|_0^2 (d|F_0|_0^2, \omega_0)_0 = 3072\pi^2\rho^6/(1+\rho^2)^5\,,$$

$$\int_{S(\rho)} |F_0|_0^2 (d|dr^2|_0^2, \omega_0)_0 = 768\pi^2\rho^4/(1+\rho^2)^4\,.$$

Since $L_W\omega_M$ is bounded, we have the required estimate by Lemma 2 (1). □

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