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A linearization of the Einstein-Maxwell field equations

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0. Introduction

Our principal objective in this paper is to solve the Belinskii ansatz for the Einstein-Maxwell field equations [1]. For a space-time metric $(ds)^2 = g_{ij}\,dx^i\,dx^j$ and an electromagnetic potential $A_i\,dx^i$, the field equations are given as follows.

$$R_{ij} = -F_{im}F_j{}^m/2 + g_{ij}F_{mn}F^{mn}/8\,, \qquad \nabla_m F^{im} = 0\,, \tag{0.0}$$

where R_{ij} is the Ricci tensor, $F_{ij} = \partial_i A_j - \partial_j A_i$ is the electromagnetic field and ∇_m denotes the covariant differential operator.

To explain the ansatz, we introduce a 2-dimensional reduction. For a symmetric matrix $g = (g_{ij})_{1 \le i,j \le 2} \in \mathrm{gl}\,(2, R[[t, z]])$ with $\det g = t^2$ and $A = (A_1, A_2) \in R^2[[t, z]]$, we set

$$-(ds)^2 = e^{2\sigma}(-(dt)^2 + (dz)^2) + \textstyle\sum_{1 \le i,j \le 2} g_{ij}\,dx^i\,dx^j\,, \qquad x^0 = t\,,\ x^3 = z,$$

$$\sigma \in R[[t, z]],\ A_0 = A_3 = 0 \text{ and } h = \begin{bmatrix} g + {}^tAA & {}^tA \\ A & 1 \end{bmatrix} \in \mathrm{gl}(3, R[[t, z]])\,.$$

Then the following two systems of equations are equivalent [1].

(0.1) $((ds)^2, A_i)$ satisfies (0.0) for some σ and $F_{mn}F^{mn} = 0$.

$$\partial_t(t\partial_t h \cdot h^{-1}) - \partial_z(t\partial_z h \cdot h^{-1}) = 0\,. \tag{0.2}$$

Moreover we assume some boundary conditions which are deduced from suitable physical assumptions.

(0.3)
$$g_{11}(0, z) = g_{12}(0, z) = A_1(0, z) = A_2(0, 0) = 0\,, \quad g_{22}(0, 0) > 0\,, \quad \partial_t g(0, z) = 0\,.$$

If (g, A) satisfies (0.2–3), then we call (g, A) *a solution of the Belinskii ansatz*.

For $u_{11}, u_{31} \in R[[x]]$ with $u_{11}(0) > 0$, we define $u = (u_{ij}) \in SL(3, R[[x]])$ as follows.

$$f = 1/u_{11}\,, \qquad a = \pm x u_{31} f,$$

$$u_{21} = \begin{cases} a u_{31}/2 + c u_{11} & \text{with } c \in R \text{ if } u_{31} \ne 0\,, \\ \text{an arbitrary element of } R[[x]] & \text{if } u_{31} = 0\,, \end{cases}$$

$$u_{12} = x^2 u_{21}, \quad u_{13} = x^2 u_{31}, \quad u_{22} = f + a^2 + u_{21} u_{12},$$
$$u_{23} = a + u_{21} u_{13} f, \quad u_{32} = a + u_{31} u_{12} f, \quad u_{33} = 1 + u_{31} u_{13} f.$$

Also we define $w_k \in \mathrm{gl}(3, R[[t, z]])$, $k \in Z$ as $\Sigma_{k \in Z} w_k \lambda^k = \exp(t^2 \partial_t / 2\lambda) \times \{u(\lambda + 2z) \operatorname{diag}(1 + 2z/\lambda, 1_2)\} \in \mathrm{gl}(3, R[[t, z, \lambda, \lambda^{-1}]])$, and we set $W_{ij} = w_{j-i}$, $W = (W_{ij})_{i \in Z, j < 0}$ and $W_- = (W_{ij})_{i,j<0}$. Then an argument in K. Nagatomo [3, §3] implies that the matrix W_- is invertible and that $Y = W \cdot W_-^{-1}$ is well-defined. From the explicit form of $Y_{0,-1}$, it follows that there exists a unique $h \in \mathrm{gl}(3, R[[t, z]])$ such that

$$h(0, z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & f + a^2 & a \\ 0 & a & 1 \end{bmatrix}_{x=2z}, \qquad \partial_t^2 h(0, z) = 2/f \begin{bmatrix} 1 & \partial_x(x u_{21} f) & \partial_x(x u_{31} f) \\ * & * & * \\ * & * & * \end{bmatrix}_{x=2z}$$

and

$$t \partial_t h = (\partial_z Y_{0,-1}) h .$$

We can now state our main

THEOREM. (i) *h is decomposed as* $\begin{bmatrix} g + {}^t A A & {}^t A \\ A & 1 \end{bmatrix}$ *and* (g, A) *is a solution of the Belinskii ansatz.*

(ii) *All solutions of the Belinskii ansatz are obtained through the above procedure.*

In §1, we study the solvability of the Belinskii ansatz. In §2, we consider some potentials which will be associated with solutions of the Belinskii ansatz. In §3, we prove the theorem. Then a crucial point is that our treating equations have regular singularities along $t = 0$. It enables us to control the solutions with their boundary values.

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1. The Belinskii ansatz

Let $g \in \mathrm{gl}(2, R[[t, z]])$ satisfy ${}^t g = g$ and $\det g = t^2$, and let $A \in R^2[[t, z]]$. As easily seen, (0.2) is equivalent to the following system:

(1.1) $$d(t * dg \cdot g^{-1}) + d{}^t A t * dA \cdot g^{-1} = 0, \qquad d(t * dA \cdot g^{-1}) = 0,$$

(1.2) $$*dA \cdot g^{-1} \, d{}^t A = 0 .$$

Here d denotes exterior differentiation and $*$ is the Hodge operator with respect to the metric $(dt)^2 - (dz)^2$. For $\varphi \in R^N[[t, z]]$ and $n \in Z_+$, we define $\varphi^{[n]} \in R^N[[z]]$ as $\varphi = \Sigma_{n \geq 0} \varphi^{[n]} t^n$.

LEMMA 1.1. *Let* (g, A) *satisfy* (0.3) *and* (1.1). *Then* (g, A) *is determined by* $g_{22}^{[0]}$, $A_2^{[0]}$, $A_1^{[2]}$ *and* $g_{12}^{[2]}$.

PROOF. Let f, a and γ stand for $g_{22}^{[0]}$, $A_2^{[0]}$ and $g_{12}^{[2]}$, respectively. Since $\det g = t^2$, we have $g_{11}^{[2]} = 1/f$. Letting $\tilde{g} = \det g \cdot g^{-1}$, we can rewrite the system (1.1) as follows:

$$((t\partial_t)^2 - 2t\partial_t)g = t^2\partial_z^2 g + (\partial_z g\, \partial_z \tilde{g} - \partial_t g\, \partial_t \tilde{g})g - t\partial_t{}^t A \cdot t\partial_t A + t^2 \partial_z{}^t A \cdot \partial_z A\,,$$

$$((t\partial_t)^2 - 2t\partial_t)A = t^2\partial_z^2 A + (\partial_z A\, \partial_z \tilde{g} - \partial_t A\, \partial_t \tilde{g})g\,.$$

Hence we obtain $A^{[1]} = 0$,

$$(n^2 - 2n)g^{[n]} = \begin{bmatrix} 0 & 0 \\ 0 & f'f\partial_z g_{11}^{[n]} + 2n(2g_{12}^{[n]}\gamma f - g_{22}^{[2]} g_{11}^{[n]} f - g_{22}^{[n]}) \end{bmatrix} + (\cdots)\,,$$

$$(n^2 - 2n)A^{[n]} = (0,\quad a'f\partial_z g_{11}^{[n]} + 2n(A_1^{[2]} g_{12}^{[n]} f - A_2^{[2]} g_{11}^{[n]} f + A_1^{[n]}\gamma f - A_2^{[n]})) + (\cdots)\,,$$

where $(\cdots)$ are terms including only $g^{[k]}$, $A^{[k]}$ with $k < n$, and the superscript $'$ denotes ∂_z. Therefore by induction we have the lemma. □

COROLLARY 1.2. *If* (g, A) *is a solution of* (0.3) *and* (1.1), *then we have* $g(t, z) = g(-t, z)$ *and* $A(t, z) = A(-t, z)$.

PROOF. Clearly the equations (0.3) and (1.1) are invariant under the transformation $t \to -t$. □

LEMMA 1.3. *If* (g, A) *satisfies* (0.3), (1.1) *and* (1.2), *then*

$$\text{(i)}\quad A_1^{[2]} = \pm\partial_z A_2^{[0]}/2g_{22}^{[0]}\,, \qquad \text{(ii)}\quad A_2^{[0]}\partial_z(g_{22}^{[0]} g_{12}^{[2]}) = 0\,.$$

PROOF. Since $d(t * dA \cdot g^{-1}) = 0$ and $dA_1|_{t=0} = 0$, there exists $B \in R^2[[t, z]]$ satisfying $dB = t * dA \cdot g^{-1}$ and $B(0, 0) = 0$. Then (1.2) means that $\partial_t B\, \partial_z{}^t A - \partial_z B\, \partial_t{}^t A = 0$.

We set $b = B_1^{[0]}$. Since $t\partial_t B = \partial_z A \cdot \tilde{g}$ and $t\partial_t A = \partial_z B \cdot g$, we obtain the following formulas:

$$B_2^{[0]} = 0\,, \qquad 2A_1^{[2]} = b'/f\,, \qquad 2B_2^{[2]} = a'/f\,,$$

$$2A_2^{[2]} = b'\gamma + (a'/2f)'f\,, \qquad 2B_1^{[2]} = (b'/2f)'f - a'\gamma\,,$$

$$4A_1^{[4]} = \partial_z B_1^{[2]}/f + b' g_{11}^{[4]} + (a'/2f)'\gamma\,,$$

$$4B_2^{[4]} = -(b'/2f)'\gamma + a' g_{11}^{[4]} + \partial_z A_2^{[2]}/f\,.$$

Therefore $(t\partial_t B\, \partial_z {}^t A)^{[2]} = (a')^2/f$ and $(\partial_z Bt\, \partial_t {}^t A)^{[2]} = (b')^2/f$. Thus we have (i). Also

$$(t\partial_t B\, \partial_z {}^t A)^{[4]} = -(b'/f)\gamma a' + (a')^2 g_{11}^{[4]} + ((b'/2f)')^2 f + (b'\gamma + (a'/2f)'f)'(a'/f)\,,$$

$$(\partial_z B\, t\partial_t {}^t A)^{[4]} = ((b'/2f)'f - a'\gamma)'(b'/f) + (a'/f)'b'\gamma + ((a'/2f)')^2 f + (b')^2 g_{11}^{[4]}\,.$$

Hence $(t\partial_t B\, \partial_z {}^t A - \partial_z B\, t\partial_t {}^t A)^{[4]} = 2(a'/f)(b'/f)(f\gamma)'$. □

Theorem 1.4. *Let $f, a, \gamma \in R[[z]]$ satisfy $f(0) > 0$, $a(0) = 0$ and $a(f\gamma)' = 0$. Then there exists a unique solution (g, A) of the Belinskii ansatz satisfying $g_{22}^{[0]} = f$, $g_{12}^{[2]} = \gamma$, $A_2^{[0]} = a$ and $A_1^{[2]} = \pm\partial_z a/2f$.*

Proof. First, we assume that $a = 0$. Then the proof of Lemma 1.1 implies that $A = 0$. Because $\mathrm{tr}\,(d(t * dg\cdot g^{-1})) = d(t * d \det g\cdot \det g^{-1})$, the system (1.1) is equivalent to the following:

$$g_{11}g_{22} - (g_{12})^2 = t^2\,,$$

$$d(t^{-1} * dg_{1i})\cdot g_{22} - d(t^{-1} * dg_{22})\cdot g_{1i} = 0\,, \qquad i = 1, 2\,.$$

Hence we obtain the following formulas:

$$g_{11}^{[n]}g_{22}^{[0]} + g_{11}^{[2]}g_{22}^{[n-2]} = \langle\langle g_{11}^{[k]}, g_{12}^{[k]}, k \le n-2,\quad g_{22}^{[j]}, j \le n-4\rangle\rangle\,,$$

$$n(n-2)g_{11}^{[n]}g_{22}^{[0]} - (n-2)(n-4)g_{22}^{[n-2]}g_{11}^{[2]} = \langle\langle g_{11}^{[k]}, k \le n-2,\quad g_{22}^{[j]}, j \le n-4\rangle\rangle\,,$$

$$n(n-2)g_{12}^{[n]}g_{22}^{[0]} = \langle\langle g_{12}^{[k]}, g_{22}^{[k]}, k \le n-2\rangle\rangle\,,$$

where $\langle\langle g_{\alpha\beta}^{[m]}, \cdots\rangle\rangle$ denotes term including only $g_{\alpha\beta}^{[m]}, \cdots$. Hence $g_{11}^{[n]}, g_{12}^{[n]}$ and $g_{22}^{[n-2]}$ are determined inductively.

Second, we consider the case $f\gamma = c \in R$. Set $s = \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix}$. Because $g_{11}^{[2]} = 1/f$ and $({}^t sgs, As)$ satisfies (0.3) and (1.1), we may assume that $\gamma = 0$. It is sufficient to prove now that there exists (g, A) satisfying (0.3), (1.1–2) and $g_{12} = 0$.

When $g_{12} = 0$, the equations (1.1–2) are rewritten as follows:

(1.1)′ $$g_{11}g_{22} = t^2\,, \qquad d(t^{-1} * dg_{22}\cdot g_{11}) + dA_2 t^{-1} * dA_2\cdot g_{11} = 0\,,$$

$$d(t^{-1} * dA_1\cdot g_{22}) = d(t^{-1} * dA_2\cdot g_{11}) = 0\,,$$

(1.1)″ $$g_{22}\, dA_1 * dA_1 + g_{11}\, dA_2 * dA_2 = 0\,,$$

(1.2)′ $$dA_1 * dA_2 = 0\,.$$

We see easily that there exists a solution (g, A) of (1.1)′ and (0.3). Let $B \in R^2[[t, z]]$ satisfy $dB = t^{-1} * dA\cdot\tilde{g}$ and $B(0, 0) = 0$. Then $d(t^{-1} * dB_1\cdot g_{22}) = 0$.

Therefore $A_2^{[0]} = \pm B_1^{[0]}$ implies that $A_2 = \pm B_1$. Since $dA_1 = t^{-1} * dB_1 \cdot g_{11}$, we have

$$g_{22}\, dA_1 * dA_1 = g_{11}(* dB_1)\, dB_1 = -g_{11}\, dA_2 * dA_2 \qquad \text{and}$$
$$dA_2 * dA_1 = dA_2\, dB_1 t^{-1} g_{11} = 0\,. \quad \square$$

2. Potentials

Let $D_1 = -\lambda\partial_z + t\partial_t + 2\lambda\partial_\lambda$ and $D_2 = -\lambda\partial_t + t\partial_z$. For U and $V \in \mathrm{gl}(3, R[[t, z]])$, the compatibility condition of

(2.0) $D_1\Psi = U\Psi$ and $D_2\Psi = V\Psi$ with $\Psi \in \mathrm{gl}(3, R[[t, z, \lambda^{-1}]])$

is

(2.1) $\partial_t U - \partial_z V = 0$ and $t(\partial_z U - \partial_t V) + V + [U, V] = 0\,.$

For $\Psi = 1 + \Sigma_{n>0} \Psi_n \lambda^{-n}$, if $\partial_\lambda(D_1\Psi \cdot \Psi^{-1}) = \partial_\lambda(D_2\Psi \cdot \Psi^{-1}) = 0$, then we have $D_1\Psi \cdot \Psi^{-1} = -\partial_z\Psi_1$ and $D_2\Psi \cdot \Psi^{-1} = -\partial_t\Psi_1$.

LEMMA 2.1. *Let U and $V \in \mathrm{gl}(3, R[[t, z]])$ be a solution of* (2.1). *Then there exists a unique $\Psi = 1 + \Sigma_{n>0} \Psi_n \lambda^{-n}$ satisfying* (2.0) *and* $\Psi(0, 0, \lambda) = 1$.

PROOF. The system (2.0) means that $\partial_z \Psi_{n+1} = t\partial_t \Psi_n - 2n\Psi_n - U\Psi_n$ and $\partial_t \Psi_{n+1} = t\partial_z \Psi_n - V\Psi_n$. By induction we have $\partial_t(t\partial_t\Psi_n - 2n\Psi_n - U\Psi_n) = \partial_z(t\partial_t\Psi_n - V\Psi_n)$. $\square$

LEMMA 2.2. *For $u \in GL(3, R[[x]])$ and $\Psi = 1 + \Sigma_{n>0} \Psi_n \lambda^{-n} \in \mathrm{gl}(3, R[[t, z, \lambda^{-1}]])$, if $X = \Psi \exp(t^2\partial_z/2\lambda)\{u(\lambda + 2z)\, \mathrm{diag}(1 + 2z/\lambda, 1_2)\}$ belongs to $\mathrm{gl}(3, R[[t, z, \lambda]])$, then $\partial_\lambda(D_1\Psi \cdot \Psi^{-1}) = \partial_\lambda(D_2\Psi \cdot \Psi^{-1}) = 0$.*

PROOF. Let $T = \exp(t^2\partial_z/2\lambda)$ and $w = T\{u(\lambda + 2z)\, \mathrm{diag}(1 + 2z/\lambda, 1_2)\}$. Then $[D_1, T] = 0$, $[D_2, T] = -T \cdot t\partial_z$, $D_1 w = w\, \mathrm{diag}(-2, 0, 0)$ and $D_2 w = 0$. Therefore we have $D_1 X = (D_1\Psi \cdot \Psi^{-1})X + X\, \mathrm{diag}(-2, 0, 0)$ and $D_2 X = (D_2\Psi \cdot \Psi^{-1})X$. Because $X \in GL(3, R[[t, z, \lambda]])$, we see that $D_1\Psi \cdot \Psi^{-1}$ and $D_2\Psi \cdot \Psi^{-1}$ belong to $\mathrm{gl}(3, R[[t, z, \lambda]] \cap R[[t, z, \lambda^{-1}]])$. $\square$

We set $w = \Sigma_{k\in Z} w_k \lambda^k$, $W_{ij} = w_{j-i}$, $W = (W_{ij})_{j\in Z, j<0}$ and $W_- = (W_{ij})_{i,j<0}$. Then the following result is due to K. Nagatomo [3, §3].

LEMMA 2.3. *The matrix W_- is invertible and $W \cdot W_-^{-1}$ is well-defined.*

PROOF. First we show that $W_-(0, z)$ is invertible. Since $w(0, z) = \Sigma_{k\geq 0} \partial_x^k u(2z)\lambda^k(1 + \mathrm{diag}(2z, 0, 0)/\lambda)/k!$, we have $w_k(0, z) = 0$ for $k \leq -2$,

$w_{-1}(0, z) = u(2z)\,\mathrm{diag}\,(2z, 0, 0)$ and $w_k(0, z) = \partial_x^k u(2z)/k! + \partial_x^{k+1} u(2z)\,\mathrm{diag}\,(2z, 0, 0)/(k+1)!$ for $k \geq 0$. Let $K = 1 - w_0(0, z)^{-1} W_-(0, z)$. For $p \in R^N[[z]]$, we set $\mathrm{ord}\, p = \sup\{m \in Z; p \in z^m R^N[[z]]\}$. By induction we see that $K^n{}_{ij} = 0$ if $i - j > n$ and that $\mathrm{ord}\, K^n{}_{ij} \geq (n + i - j)/2$ for any $i, j < 0$. Hence $\Sigma_{n \geq 0} K^n$ is well-defined, and $W_-(0, z)^{-1} = \Sigma_{n \geq 0} K^n w_0(0, z)^{-1}$.

We set $W = \Sigma_{m \geq 0} W(m) t^{2m}$. Since $W_{ij}(m) = (\partial_z/2)^m w_{j-i+m}(0, z)/m!$, we see that $W_{ij}(m) = 0$ if $i - j \geq m + 2$. Let $H(n, m)_{ij}$ stand for $-(K^n w_0(0, z)^{-1} W_-(m))_{ij} = -\Sigma_{i-n \leq p \leq j+m+1} K^n{}_{ip} w_0(0, z)^{-1} W_{pj}(m)$, and let $H(n, m) = (H(n, m)_{ij})_{i,j<0}$. Note that $\mathrm{ord}\, H(n, m)_{ij} \geq (n + i - j - m - 1)/2$. Therefore $H = \Sigma_{n \geq 0, m \geq 1} H(n, m) t^{2m}$ is well-defined and $W_- = W_-(0, z)(1 - H)$.

By definition, $H(n, m)_{ij} = 0$ if $i - j > n + m + 1$. Hence if

$$(*) = H(n_1, m_1)_{ij_2} H(n_2, m_2)_{j_2 j_3} \cdots H(n_N, m_N)_{j_N j}$$

is nonzero, then $i - j_2 \leq n_1 + m_1 + 1, j_2 - j_3 \leq n_2 + m_2 + 1, \cdots, j_N - j \leq n_N + m_N + 1$. Therefore, fixing i, j, (n_k) and (m_k), we have $(*) = 0$ for almost all indices $j_2, \cdots, j_N$. Also $\mathrm{ord}\,(*) \geq (\Sigma n_k - \Sigma m_k + i - j - N)/2$. Hence, fixing i, j, (m_k) and $r > 0$, we see that $\mathrm{ord}\,(*) > r$ except a finite number of indices (n_k). Thus $\Sigma_{n_i \geq 0} H(n_1, m_1) \cdots H(n_N, m_N)$ is well-defined, and so is H^N.

For $(**) = (H(n_1, m_1) \cdots H(n_N, m_N))_{ip} W_-(0, z)^{-1}{}_{pj}$, we have $\mathrm{ord}\,(**) \geq (\Sigma n_k - \Sigma m_k + i - p - N)/2$. Hence, fixing i, j, (m_k) and r, we see that $\mathrm{ord}\,(**) \geq r$ for almost all indices (n_k) and $p < 0$. Thus $H^N \cdot W_-(0, z)^{-1}$ is well-defined. By the definition, if $W(m)_{ij} \neq 0$, then $i - m - 1 \leq j < 0$. Therefore $W \cdot (\Sigma_{N \geq 0} H^N W_-(0, z)^{-1})$ is well-defined, and $W_-^{-1} = \Sigma_{N \geq 0} H^N W_-(0, z)^{-1}$. □

Applying Lemma 2.2, we have the Birkhoff decomposition of w. Letting $\Psi = \Sigma_{j \in Z} \Psi_{-j} \lambda^j = 1 + \Sigma_{j > 0} - (W \cdot W_-^{-1})_{0,-j} \lambda^{-j}$, we see that $\Psi w \in \mathrm{gl}\,(3, R[[t, z, \lambda]])$ because $(\Psi_{-j})_{j<0} W_- + (W_{0j})_{j<0} = (\Psi_{-j})_{j \in Z} W = 0$. Also we notice that if $\tilde{\Psi} w \in \mathrm{gl}\,(3, R[[t, z, \lambda]])$ for $\tilde{\Psi} \in \mathrm{gl}\,(3, R[[t, z, \lambda^{-1}]])$ with $\tilde{\Psi}(0, 0, \lambda) = 1$, then $\tilde{\Psi} = \Psi$.

LEMMA 2.4. *Let* $\varphi \in \mathrm{gl}\,(3, R[[t, z]])$ *satisfy*

$$(t\partial_t)^2 \varphi - 2t\partial_t \varphi - t^2 \partial_z^2 \varphi + [t\partial_t \varphi, \partial_z \varphi] = 0\,, \tag{2.2}$$

$$\varphi_{11}^{[0]} = 2z\,, \qquad \varphi_{ij}^{[0]} = 0 \qquad i = 1, 2, 3\,, \qquad j = 2, 3 \qquad \textit{and} \qquad \varphi^{[1]} = 0\,.$$

Then φ *is determined by* $\varphi_{11}^{[2]}$, $\alpha_i = \varphi_{i1}^{[0]}$, $\varphi_{ij}^{[2]}$ *and* $\varphi_{1i}^{[4]}$ $i, j = 2, 3$. *Moreover if* $\mathrm{tr}\,\varphi^{[2]} = 0$, *then* $\mathrm{tr}\,\varphi = 2z$.

PROOF. We note that $(n^2 - 2n)\varphi^{[n]} - \partial_z^2 \varphi^{[n-2]} + \Sigma_{k+m=n} [k\varphi^{[k]}, \partial_z \varphi^{[m]}] = 0$. Let Φ stand for $[n\varphi^{[n]}, \partial_z \varphi^{[0]}]$. Then we have $\Phi_{1i} = -2n\varphi_{1i}^{[n]}$ $i = 2, 3$, $\Phi_{11} = n\varphi_{12}^{[n]}\alpha_2' + n\varphi_{13}^{[n]}\alpha_3'$, $\Phi_{ij} = -\alpha_i' \varphi_{1j}^{[n]}$ $2 \leq i, j \leq 3$ and $\Phi_{i1} = 2\varphi_{i1}^{[n]} + n\varphi_{12}^{[n]}\alpha_2' +$

$n\varphi_{i3}^{[n]}\alpha_3' - \alpha_i' n\varphi_{11}^{[n]}$ $i=2, 3$. Also we see that $n(n-2)\operatorname{tr}\varphi^{[n]} = \partial_z^2 \operatorname{tr}\varphi^{[n-2]}$. Therefore, by induction we have the lemma. □

3. A linearization of the Belinskii ansatz

Letting (g, A) be a solution of the Belinskii ansatz, we set $h = \begin{bmatrix} g + {}^tAA & {}^tA \\ A & 1 \end{bmatrix}$, $U = t\partial_t h \cdot h^{-1}$ and $V = t\partial_z h \cdot h^{-1}$. Since U and V satisfy (2.1), we have a solution $\Psi = 1 + \Sigma_{n>0} \Psi_n \lambda^{-n}$ with $\Psi(0, 0, \lambda) = 1$ of (2.0). Because $U(0, z) = \begin{bmatrix} 2 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$, we can set

$$-\Psi_1(0, z) = \begin{bmatrix} 2z & 0 & 0 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{bmatrix}, \qquad \alpha_i \in R[[z]] \qquad i = 2, 3\,.$$

For $u \in GL(3, R[[x]])$, we set $w = \exp(t^2\partial_z/2\lambda)\{u(\lambda + 2z)\operatorname{diag}(1 + 2z/\lambda, 1_2)\}$ and $X = \Psi w$.

LEMMA 3.1. *$X(0, z, \lambda) \in \mathfrak{gl}(3, R[[z, \lambda]])$ if and only if*

(3.1) $$\alpha_i u_{11}(2z) = 2zu_{i1}(2z) \qquad \textit{and} \qquad u_{1i}(0) = 0 \qquad i = 2, 3\,.$$

PROOF. Putting $t = 0$, we have $(-\lambda\partial_z + 2\lambda\partial_\lambda)\Psi = \begin{bmatrix} 2 & 0 & 0 \\ \alpha_2' & 0 & 0 \\ \alpha_3' & 0 & 0 \end{bmatrix}\Psi$. Hence $\Psi(0, z, \lambda) = \begin{bmatrix} \lambda/(\lambda + 2z) & 0 & 0 \\ -\alpha_2/(\lambda + 2z) & 1 & 0 \\ -\alpha_3/(\lambda + 2z) & 0 & 1 \end{bmatrix}$ and

$$X_{11}(0, z, \lambda) = u_{11}(\lambda + 2z)\,, \qquad X_{i1}(0, z, \lambda) = \lambda u_{i1}(\lambda + 2z)/(\lambda + 2z)\,,$$

$$X_{i1}(0, z, \lambda) = (-\alpha_i u_{11}(\lambda + 2z) + 2zu_{i1}(\lambda + 2z))/\lambda + u_{i1}(\lambda + 2z)\,,$$

$$X_{ij}(0, z, \lambda) = -\alpha_i u_{1j}(\lambda + 2z)/(\lambda + 2z) + u_{ij}(\lambda + 2z)\,, \qquad 2 \le i\,, j \le 3\,.$$

The lemma is now clear. □

We set $f = g_{22}^{[0]}$, $a = A_2^{[0]}$, $v = A_1^{[2]}$ *and* $\Delta_{ij} = u_{11}u_{ij} - u_{i1}u_{1j}$, $2 \le i, j \le 3$.

LEMMA 3.2. *Let u satisfy* (3.1). *Then $\partial_t^2 X(0, z, \lambda) \in \mathfrak{gl}(3, R[[z, \lambda]])$ if and only if*

(3.2) $$(fu_{11})' = 0\,, \qquad (a\Delta_{3i} - \Delta_{2i})' = 0 \qquad \textit{and}$$

$$a'\Delta_{2i} - aa'\Delta_{3i} = (f\Delta_{3i})'\,, \qquad i = 2, 3\,.$$

PROOF. Note that $\partial_t w = t\partial_z w/\lambda$, $\partial_t^2 X = \partial_t^2 \Psi \cdot w + 2\partial_t \Psi \cdot t\partial_z w/\lambda + \Psi \cdot \partial_t(t\partial_z w)/\lambda$ and $\Psi(t, z) = \Psi(-t, z)$. Since $\partial_t D_2 \Psi(0, z) = -\partial_t^2 \Psi_1(0, z) \cdot \Psi(0, z)$, we have $\partial_t^2 \Psi(0, z) = (\partial_z \Psi(0, z) + \partial_t^2 \Psi_1(0, z) \cdot \Psi(0, z))/\lambda$. Therefore

$$\partial_t^2 X = (\partial_z X + \partial_t^2 \Psi_1 \cdot X)/\lambda \qquad \text{on } t = 0\,.$$

Putting $t = \lambda = 0$, we have $X_{11} = u_{11}$, $X_{1i} = 0$, $X_{ij} = -\alpha_i \tilde{u}_{1j} + u_{ij}$ $2 \leq i, j \leq 3$, where $\tilde{u}_{1j} = u_{1j}(2z)/2z$. Also we see that

$$-\partial_t^2 \Psi_1 = V^{[1]} = \begin{bmatrix} -f'/f & 0 & 0 \\ \gamma' f - f'\gamma + av'f - aa'\gamma & f'/f + aa'/f & -af'/f - a^2a'/f + a' \\ v'f - a'\gamma & a'/f & -aa'/f \end{bmatrix}.$$

If $\partial_z X + \partial_t^2 \Psi_1 \cdot X = 0$ on $t = \lambda = 0$, we have

$$-(f'/f)u_{11} = \partial_z u_{11}\,,$$
$$(f' + a'a)\Delta_{2i} + (-af' - a^2a' + a'f)\Delta_{3i} = (\Delta_{2i}f)'\,,$$
$$a'\Delta_{2i} - aa'\Delta_{3i} = (\Delta_{3i}f)'\,.$$

These imply (3.2). Also we note that if $(fu_{11})' = 0$, then $(\partial_z X + \partial_t^2 \Psi_1 \cdot X)_{i1} = 0$, $i = 2, 3$ on $t = \lambda = 0$. Hence (3.2) implies $\partial_t^2 X(0, z, \lambda) \in \mathrm{gl}\,(3, R[[z, \lambda]])$. □

LEMMA 3.3. *Assume* (3.1–2). *If* $\partial_t^4 X_{1i}(0, z, \lambda) \in R[[z, \lambda]]$, $i = 2, 3$, *then*

(3.3) $$fu_{11}(f\tilde{u}_{1i})'' = \{(\alpha_2 - \alpha_3 a)\Delta_{2i} + (\alpha_3 a^2 + \alpha_3 f - \alpha_2 a)\Delta_{3i}\}''\,.$$

PROOF. Note that $\partial_t^2 w = \partial_z w/\lambda$, $\partial_t^4 w = 3\partial_z^2 w/\lambda^2$ and $\partial_t^3 D_2 \Psi = -\partial_t^4 \Psi_1 \cdot \Psi - 3\partial_t^2 \Psi_1 \cdot \partial_t^2 \Psi$ on $t = 0$. Therefore $\partial_t^4 X = \partial_t^4 \Psi \cdot w + 6\partial_t^2 \Psi \cdot \partial_z w/\lambda + 3\Psi \partial_z^2 w/\lambda^2$ and $\lambda \partial_t^4 \Psi = 3\partial_t^2 \partial_z \Psi + \partial_t^4 \Psi_1 \cdot \Psi + 3\partial_t^2 \Psi_1 \cdot \partial_t^2 \Psi$ on $t = 0$. After a calculation, we have

$$\partial_t^4 X = 3\{\partial_z^2 X + \partial_z(\partial_t^2 \Psi_1 \cdot X) + \partial_t^2 \Psi_1 \cdot \partial_z X + (\partial_t^2 \Psi_1)^2 X\}/\lambda^2 + \partial_t^4 \Psi_1 \cdot X/\lambda \text{ on } t = 0\,.$$

Also it is easily seen that $\partial_\lambda X_{1i}(0, z, 0) = \tilde{u}_{1i}$, $V_{12}^{[3]} = (f'/f^2)\gamma + \gamma' f + va'/f$, $V_{13}^{[3]} = -V_{12}^{[3]}a + vf'/f + v'$,

$$\mathrm{Res}_{\lambda=0}\, \partial_t^4 X_{1i}(0, z, \lambda)/3 = (f\tilde{u}_{1i})''/f - (2V^{[3]}X)_{1i}\,, \qquad i = 2, 3 \qquad \text{and}$$
$$f^2 u_{11}(V^{[3]}X)_{1i} = \{(f\gamma)' + va'f\}\Delta_{2i} + \{-(f\gamma)'a - va'fa + (fv)'f\}\Delta_{3i}$$
$$= \{(f\gamma)(\Delta_{2i} - a\Delta_{3i}) + vf^2\Delta_{3i}\}'\,.$$

Using $2f\gamma = \alpha_2' - \alpha_3' a$, $2fv = \alpha_3'$ and (3.2), we have $2\{(f\gamma)(\Delta_{2i} - a\Delta_{3i}) + vf^2\Delta_{3i}\} = \{\alpha_2(\Delta_{2i} - a\Delta_{3i}) + \alpha_3(-a\Delta_{2i} + (a^2 + f)\Delta_{3i})\}'$. □

PROPOSITION 3.4. *Let* $u \in GL(3, R[[x]])$ *satisfy* (3.1–2–3). *Then* $\Psi w \in \mathfrak{gl}(3, R[[t, z, \lambda]])$.

PROOF. By Lemma 2.3, we have $\Psi^u \in \mathfrak{gl}(3, R[[t, z, \lambda^{-1}]])$ satisfying $\Psi^u w \in \mathfrak{gl}(3, R[[t, z, \lambda]])$ and $\Psi^u(0, 0, \lambda) = 1$. Then the uniqueness of the Birkhoff decomposition implies that $\Psi^u(0, z, \lambda) = \Psi(0, z, \lambda)$. Therefore $\partial_t^2 \Psi_1(0, z) = -\partial_z X(0, z, 0) \cdot X(0, z, 0)^{-1} = \partial_t^2 \Psi_1^u(0, z)$ and $(\partial_t^4 \Psi_1(0, z) \cdot X(0, z, 0))_{1i} = -3 \operatorname{Res}_{\lambda=0} \{\partial_z^2 X + \partial_z(\partial_t^2 \Psi_1 \cdot X) + \partial_t^2 \Psi_1 \cdot \partial_z X + (\partial_t^2 \Psi_1)^2 X\}_{1i}/\lambda^2 = (\partial_t^4 \Psi_1^u(0, z) \cdot X(0, z, 0))_{1i}$. Since $X(0, z, 0) = \begin{bmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}$, we have $\det (X_{ij}(0, z, 0))_{2 \le i, j \le 3} \in R[[z]]^\times$. Thus $\partial_t^4 \Psi_1(0, z)_{1i} = \partial_t \Psi_1^u(0, z)_{1i}$, $i = 2, 3$. Applying Lemmas 2.4 and 2.1, we see that $\Psi_1 = \Psi_1^u$ and $\Psi = \Psi^u$. □

PROOF OF THEOREM. Let u be defined as in §0. We set $f(z) = 1/u_{11}(2z)$, $\alpha_i(z) = 2zu_{i1}(2z)f$ $i = 2, 3$, $a(z) = \pm\alpha_3(z)$ and $\gamma = (\alpha_2' - a\alpha_3')/2$. Using Theorem 1.4, we have a solution (g, A) of the Belinskii ansatz. Since u satisfies (3.1–2–3), Proposition 3.4 implies that $\Psi_1 = -(W \cdot W_-^{-1})_{0,-1}$.

If $t\partial_t h = -\partial_z \Psi_1 \cdot h$ for $h \in \mathfrak{gl}(3, R[[t, z]])$, then $nh^{[n]} = -\partial_z \Psi_1^{[0]} h^{[n]} - \Sigma_{k<n} \partial_z \Psi_1^{[n-k]} h^{[k]}$. Note that $-\partial_z \Psi_1(0, z) = \begin{bmatrix} 2 & 0 & 0 \\ \alpha_2' & 0 & 0 \\ \alpha_3' & 0 & 0 \end{bmatrix}$. Therefore h is determined by $h^{[0]}$ and $h_{1i}^{[2]}$ $i = 1, 2, 3$. Thus we see that $h = \begin{bmatrix} g + {}^tAA & {}^tA \\ A & 1 \end{bmatrix}$. □

EXAMPLE. Let $u_{11} = 1$, $u_{31} = 2\beta$ and $u_{21} = \pm 2\beta^2 x + \gamma$, with $\beta \neq 0$, $\gamma \in R$. Then we have

$$g = (1 + 2\beta^2 t^2)^{-2} \begin{bmatrix} t^2 & \gamma t^2 \\ \gamma t^2 & (1 + 2\beta^2 t^2)^4 + \gamma^2 t^2 \end{bmatrix},$$

$$A = 2\beta(1 + 2\beta^2 t^2)^{-1}(t^2, \pm 2z(1 + 2\beta^2 t^2) + \gamma t^2).$$

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