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Invariant measures and entropies of random dynamical systems and the variational principle for random Bernoulli shifts

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§ 1. Introduction

We consider a dynamical system in a compact metric space (M, d) in which continuous maps operated on points are successively chosen randomly according to a fixed probability law. Such random dynamical systems were studied, for example, by [3], [4], [7] and [8]. More precisely, let Φ be a set of maps with a measurable structure $\mathscr{F}$ and $\{\varphi_n(\omega): n \in \mathbf{N}\}$ be $(\Phi, \mathscr{F})$-valued stochastic process on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. The corresponding trajectories on M are $\{\varphi_n(\omega) \circ \cdots \circ \varphi_1(\omega)x: n \in \mathbf{N}\}$, $x \in M$, for $\omega \in \Omega$. Here the underlying probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ can be taken to be $(\Phi^{\mathbf{N}}, \mathscr{F}^{\mathbf{N}}, P)$ for some P. To avoid the dependence of the law for the choice of maps on time, we impose on P the stationality (i.e. the shift invariance). We also assume that P is ergodic for simplicity.

Define a map τ on $M \times \Phi^{\mathbf{N}}$ by $\tau(x, \tilde{\varphi}) = (\varphi_1 x, \sigma\tilde{\varphi})$, $x \in M$, $\tilde{\varphi} = (\varphi_1, \varphi_2, \ldots) \in \Phi^{\mathbf{N}}$, where σ is the shift transformation on $\Phi^{\mathbf{N}}$. This map τ is called the skew product transformation. In most articles, a probability measure of the following type was considered as an invariant measure of τ: $Q = \mu \times P$, where $P = \rho^{\mathbf{N}}$, ρ is a probability measure on $(\Phi, \mathscr{F})$ and μ is a stationary (i.e. invariant) probability measure of the transition probability $P(x, B) = \int_{\Phi} 1_B(\varphi x) d\rho(\varphi)$. Tsujii [10] treated a slightly different measure in connection with the theory of random fractals. He gave a τ-invariant measure which has the non-trivial decomposition with respect to the partition $\{M \times \{\tilde{\varphi}\}: \tilde{\varphi} \in \Phi^{\mathbf{N}}\}$. Even in his system P turns out to be of the type $\rho_\gamma^{\mathbf{N}}$ for some probability measure ρ_γ on Φ.

In this paper we consider the random dynamical sysytems in more general situation. We assume only that an ergodic shift invariant probability measure P on $(\Phi^{\mathbf{N}}, \mathscr{F}^{\mathbf{N}})$ is given and we are concerned with a τ-invariant Q on $(M \times \Phi^{\mathbf{N}}, \mathscr{B}_M \times \mathscr{F}^{\mathbf{N}})$ which is required to satisfy only the condition $Q(M \times F) = P(F)$, $F \in \tilde{\mathscr{F}}$. In § 2 we prove the existence of such a τ-invariant probability measure Q. When maps are homeomorphisms, as a natural extention of $\Phi^{\mathbf{N}}$, we can take the underlying probability space to be $(\Phi^{\mathbf{Z}}, \mathscr{F}^{\mathbf{Z}}, P)$ for a given ergodic shift invariant probability measure P on $(\Phi^{\mathbf{Z}}, \mathscr{F}^{\mathbf{Z}})$. Also in this case the existence of an invariant measure is assured. (See [2].)

In §3 we study the ergodic decomposition of (τ, Q) on $M \times \Phi^{\mathbf{N}}$. For a random dynamical system such as Tsujii's one, it has simple form. That is, the decomposition is essentially determined by a decomposition of M. We deduce this from the ergodic decomposition of a certain Markov operator. This result obtained here corresponds to a special case of deterministic version lemma in an i.i.d. random dynamical system (i.e. $P = \rho^{\mathbf{N}}$ and $Q = \mu \times P$ stated above) ([7]).

In §4 both topological and metrical entropies $h_{top}(P)$ and $h_Q(P)$ of a random dynamical system are defined as in a detrministic case or an i.i.d. case ([4]). The notion of the metrical entropy is closely related to that of the conditional entropy and here we follow [4] to obtain the Kolmogorov-Sinai type lemma. We also briefly consider the variational principle and the Shannon-McMillan-Breiman type theorem in connection with the entropy.

In §5 we consider the maximal measure in a random Bernoulli shift. Similarly to the deterministic case, we obtain the result that there exists a unique invariant measure which maximizes the metrical entropy. Techniques used are similar to those in the deterministic case. But the τ-invariance condition of measures is somewhat different so that we have to make some modifications.

In §6 we take Tsujii random dynamical systems and apply the results obtained in the previous sections. We review the measure given in [10] and check the ergodicity as an application of §3. The entropy of a certain special system is also treated.

§2. Random dynamical systems and invariant measures

Let $(M, \mathscr{B})$ be a pair of a compact metric space (M, d) and its topological σ-algebra $\mathscr{B}$, and $C_{sur}(M, M)$ be the set of surjective continuous maps from M to itself. The set $C_{sur}(M, M)$ is endowed with the topology generated by the metric $r(f, g) = \sup_{x \in M} d(fx, gx)$ (i.e. the uniform topology) and the measurable structure $\mathscr{F}_0$ generated by this topology. Let Φ be a measurable subset of $C_{sur}(M, M)$ and $\mathscr{F} = \mathscr{F}_0 \cap \Phi$. We denote the product space $(\Phi^{\mathbf{N}}, \mathscr{F}^{\mathbf{N}})$ by $(\tilde{\Phi}, \tilde{\mathscr{F}})$, an element of $\tilde{\Phi}$ by $\tilde{\varphi} = (\varphi_n)_{n \in \mathbf{N}}$ and the shift transformation on $\tilde{\Phi}$ by $\sigma: \sigma\tilde{\varphi} = (\varphi_{n+1})_{n \in \mathbf{N}}$. We fix an *ergodic* σ-invariant probability measure P on $(\tilde{\Phi}, \tilde{\mathscr{F}})$. Then the coordinate functions $\{\varphi_n : n \in \mathbf{N}\}$ becomes an ergodic stationary process on $(\tilde{\Phi}, \tilde{\mathscr{F}}, P)$.

DEFINITION 2.1. The pair $\{(M, d), (\tilde{\Phi}, \tilde{\mathscr{F}}, P)\}$ is called a *topological random dynamical system.* Its trajectory is $\{{}^n\tilde{\varphi}x : n \in \mathbf{Z}_+\}$ $(\mathbf{Z}_+ = \{0, 1, 2, \dots,\})$

for $x \in M$, $\tilde{\varphi} = (\varphi_n)_{n \in \mathbf{N}} \in \tilde{\Phi}$, where

$$^n\tilde{\varphi} = \begin{cases} id & \text{if } n = 0 \\ \varphi_n \circ \cdots \circ \varphi_1 & \text{if } n \geq 1. \end{cases}$$

For a topological random dynamical system, we define the skew product transformation τ on $M \times \tilde{\Phi}$ by

$$\tau(x, \tilde{\varphi}) = (\varphi_1 x, \sigma\tilde{\varphi}), \ x \in M, \ \tilde{\varphi} \in \tilde{\Phi}.$$

Our first task is to find an invariant measure in the following sense.

DEFINITION 2.2. A measure $Q \in \mathscr{P}(M \times \tilde{\Phi})$ is called *invariant* if it satisfies

$$\tau^* Q = Q \tag{2.1}$$

$$\pi_{\tilde{\Phi}}^* Q = P, \tag{2.2}$$

where $\pi_{\tilde{\Phi}}$: $M \times \tilde{\Phi} \to \tilde{\Phi}$ is the natural projection. The set of invariant measures is denoted by $\mathscr{I}_P(M \times \tilde{\Phi})$.

REMARK 2.3. Throughout this paper $\mathscr{P}(X)$ or $\mathscr{P}(X, \mathscr{B})$ denotes the set of probability measures on a measurable space $(X, \mathscr{B})$ and the image measure of a measure ν by a map f is denoted by $f^*\nu$ (or sometimes $\nu \circ f^{-1}$).

We define $\Phi^{p,q} = \prod_{n=p}^q \Phi_n$, $\Phi_n = \Phi$ and $\mathscr{F}^{p,q} = \prod_{n=p}^q \mathscr{F}_n$, $\mathscr{F}_n = \mathscr{F}$ for $1 \leq p \leq q \leq \infty$. For $\varphi^{(1)} = (\varphi_n)_{n=p}^q \in \Phi^{p,q}$ and $\varphi^{(2)} = (\varphi_n)_{n=q+1}^r \in \Phi^{q+1,r}$, we denote $\varphi^{(1)}\varphi^{(2)} = (\varphi_n)_{n=p}^r \in \Phi^{p,r}$. Let π_k: $\tilde{\Phi} \to \Phi^{k+1,\infty}$ be the natural projection, i.e. $\pi_k \tilde{\varphi} = (\varphi_n)_{n=k+1}^\infty \in \Phi^{k+1,\infty}$ for $\tilde{\varphi} = (\varphi_n)_{n=1}^\infty \in \tilde{\Phi}$. Under these notations we have $\tilde{\varphi} = (\varphi_1, \cdots. \varphi_k)\pi_k\tilde{\varphi}$ or simply $\tilde{\varphi} = \varphi_1 \cdots \varphi_k \pi_k \tilde{\varphi}$ for all $k \in \mathbf{N}$.

For $n \geq 1$, define a bijection θ_n: $\tilde{\Phi} \to \Phi^{n+1,\infty}$ by $(\theta_n\tilde{\varphi})_k = \varphi_{k-n}$, $k \geq n + 1$. We see that $\pi_n = \theta_n \circ \sigma^n$. Since P is σ-invariant, we have $\pi_n^* P(\theta_n F) = P(\sigma^{-n} F) = P(F)$ for $F \in \tilde{\mathscr{F}}$.

Take $Q \in \mathscr{P}(M \times \tilde{\Phi})$ with $\pi_{\tilde{\Phi}}^* Q =$ P. Since $(M, \mathscr{B})$ is a standard measurable space, there exists the family $\{Q_{\tilde{\varphi}}: \tilde{\varphi} \in \tilde{\Phi}\}$ of regular conditional probability measures of Q with respect to the partition $\{M \times \{\tilde{\varphi}\}: \tilde{\varphi} \in \tilde{\Phi}\}$ of $M \times \tilde{\Phi}$:

$$Q(B \times F) = \int_F Q_{\tilde{\varphi}}(B) dP(\tilde{\varphi}), \ B \in \mathscr{B}, \ F \in \tilde{\mathscr{F}},$$

where we regard $Q_{\tilde{\varphi}}$ as a measure on $(M, \mathscr{B})$. Similarly we have the family $\{P_{\theta_n\tilde{\varphi}}: \tilde{\varphi} \in \tilde{\Phi}\}$ of regular conditional probability measures of P with respect to the partition $\{\Phi^{1,n} \times \{\theta_n\tilde{\varphi}\}: \tilde{\varphi} \in \tilde{\Phi}\}$ of $\tilde{\Phi}$:

$$P(F_1 \times \theta_n F) = \int_{\theta_n F} P_{\theta_n\tilde{\varphi}}(F_1) d\pi_n^* P(\theta_n\tilde{\varphi}) = \int_F P_{\theta_n\tilde{\varphi}}(F_1) dP(\tilde{\varphi})$$

for $F_1 \in \mathscr{F}^{1,n}$ and $F \in \tilde{\mathscr{F}}$, where we regard $P_{\theta_n \tilde{\varphi}}$ as a measure on $(\Phi^{1,n}, \mathscr{F}^{1,n})$.

LEMMA 2.4. *For $Q \in \mathscr{P}(M \times \tilde{\Phi})$ with $\pi^*_{\tilde{\Phi}} Q = P$, $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$ if and only if*

$$\int Q_{\varphi\theta_1\tilde{\varphi}}(\varphi^{-1}B)dP_{\theta_1\tilde{\varphi}}(\tilde{\varphi}) = Q_{\tilde{\varphi}}(B) \text{ for } P\text{-a.e. } \tilde{\varphi} \text{ and all } B \in \mathscr{B}. \tag{2.3}$$

PROOF. Here we denote θ_1 by θ for simplicity. First let us assume $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$. For $B \in \mathscr{B}$, consider the function

$$f_B(\tilde{\varphi}) = \int Q_{\varphi\theta\tilde{\varphi}}(\varphi^{-1}B)dP_{\theta\tilde{\varphi}}(\varphi), \quad \tilde{\varphi} \in \tilde{\Phi}.$$

By the definition of $\{Q_{\tilde{\varphi}}\}$ and $\{P_{\theta\tilde{\varphi}}\}$, we have, for $F \in \mathscr{F}$,

$$\begin{aligned}
\int_F f_B(\tilde{\varphi})dP(\tilde{\varphi}) &= \int_F \int Q_{\varphi\theta\tilde{\varphi}}(\varphi^{-1}B)dP_{\theta\tilde{\varphi}}(\varphi)dP(\tilde{\varphi}) \\
&= \iint 1_B(\varphi x)1_{\Phi\times\theta F}(\varphi\theta\tilde{\varphi})dQ_{\varphi\theta\tilde{\varphi}}(x)dP(\varphi\theta\tilde{\varphi}) \\
&= \int 1_{B\times F}(\psi_1 x, \sigma\tilde{\psi})dQ(x, \tilde{\psi}) \quad (\tilde{\psi} = \varphi\theta\tilde{\varphi}) \\
&= Q(\tau^{-1}(B \times F)) \\
&= Q(B \times F).
\end{aligned}$$

By the uniqueness of $\{Q_{\tilde{\varphi}}\}$ we get

$$f_B(\tilde{\varphi}) = Q_{\tilde{\varphi}}(B) \qquad \text{for P-a.e. } \tilde{\varphi}.$$

Thus we obtain (2.3).

Next assume (2.3). Then we have as above

$$\begin{aligned}
Q(\tau^{-1}(B \times F)) &= \int_F \int Q_{\varphi\theta\tilde{\varphi}}(\varphi^{-1}B)dP_{\theta\tilde{\varphi}}(\varphi)dP(\tilde{\varphi}) \\
&= \int_F Q_{\tilde{\varphi}}(B)dP(\tilde{\varphi}) \\
&= Q(B \times F)
\end{aligned}$$

for $B \in \mathscr{B}$ and $F \in \tilde{\mathscr{F}}$.

REMARK 2.5. Under above notations, the τ^n-invariance condition of $Q \in \mathscr{P}(M \times \tilde{\Phi})$ with $\pi^*_{\tilde{\Phi}} Q = P$ is

$$\int_{\Phi^{1,n}} Q_{\varphi_1\cdots\varphi_n\theta_n\tilde{\varphi}}({}^n\varphi^{-1}B)dP_{\theta_n\tilde{\varphi}}(\varphi_1,\cdots,\varphi_n)=Q_{\tilde{\varphi}}(B) \tag{2.4}$$

for P-a.e. $\tilde{\varphi}$ and for all $B\in\mathscr{B}$, where ${}^n\varphi=\varphi_n\circ\cdots\circ\varphi_1$ for $(\varphi_1,\cdots,\varphi_n)$.

To deduce the existence theorem of invariant measure we prepare the following results from [1]. Let $C(M)$ be the set of continuous functions on M and $\mathfrak{M}(M)$ be the set of finite signed measures on M. Consider the Banach space of integrable random continuous functions,

$$L^1(\tilde{\Phi},C(M))=\{f:\ f:\tilde{\Phi}\longrightarrow C(M)\text{ measurable, } |||f|||=\int\|f(\tilde{\varphi})\|dP(\tilde{\varphi})<\infty\},$$

where $\|f(\tilde{\varphi})\|=\sup\{|f(\tilde{\varphi})|:x\in M\}$ for $f(\tilde{\varphi})\in C(M)$, with the norm $|||\ |||$. The linear space of bounded random signed measures,

$$L^\infty(\tilde{\Phi},\mathfrak{M}(M))=\{\mu:\ \mu:\tilde{\Phi}\longrightarrow\mathfrak{M}(M)\text{ measurable,}$$
$$(P\text{-})\ \text{ess.sup}\{|\mu(\tilde{\varphi})|(M):\tilde{\varphi}\in\tilde{\Phi}\}<\infty\},$$

can be regarded as the dual space of $L^1(\tilde{\Phi},C(M))$ by the duality

$$(f,\mu)=\int_{\tilde{\Phi}}\int_M f(\tilde{\varphi})(x)d\mu(\tilde{\varphi})(x)dP(\tilde{\varphi})$$

for $f\in L^1(\tilde{\Phi},C(M))$ and $\mu\in L^\infty(\tilde{\Phi},\mathfrak{M}(M))$.

THEOREM 2.6. $\mathscr{I}_P(M\times\tilde{\Phi})\neq\phi$.

PROOF. Define a linear operator T on $L^\infty(\tilde{\Phi},\mathfrak{M}(M))$ by

$$(T\mu)(\tilde{\varphi})(B)=\int\mu(\varphi\theta_1\tilde{\varphi})(\varphi^{-1}B)dP_{\theta_1\tilde{\varphi}}(\varphi),\ B\in\mathscr{B}.$$

Then T is continuous with respect to the weak-$*$ topology. In fact,

$$\begin{aligned}(f,T\mu)&=\iint f(\tilde{\varphi})(\varphi x)d\mu(\varphi\theta_1\tilde{\varphi})(x)dP_{\theta_1\tilde{\varphi}}(\varphi)dP(\tilde{\varphi})\\&=\iint f(\sigma\tilde{\psi})(\psi_1x)d\mu(\tilde{\psi})(x)dP(\tilde{\psi})\\&=(\tau^*f,\mu)\end{aligned}$$

where τ^* is the linear operator on $L^1(\tilde{\Phi},C(M))$ defined by $(\tau^*f)(\tilde{\varphi})=f(\sigma\tilde{\varphi})(\varphi_1x)$. Since τ^* is clearly continuous, so is its dual T. Since $L(\tilde{\Phi},\mathscr{P}(M))=\{\mu\in L^\infty(\tilde{\Phi},\mathfrak{M}(M)):\mu(\tilde{\varphi})\in\mathscr{P}(M)\ P\text{-a.e. }\tilde{\varphi}\}$ is a T-invariant weak-$*$ compact convex subset of $L^\infty(\tilde{\Phi},\mathfrak{M}(M))$, by Schauder-Tychonoff's fixed point theorem there exists a $\mu \in L(\tilde{\Phi}, \mathscr{P}(M))$ with $T\mu = \mu$. Define $Q \in \mathscr{P}(M \times \tilde{\Phi})$ by

$$Q(B \times F) = \int_F \mu(\tilde{\varphi})(B)\, dP(\tilde{\varphi}), \ B \in \mathscr{B}, \ F \in \mathscr{F}.$$

It is clear that Q defined above satisfies (2.3) in Lemma 2.4. So we obtain an element $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$.

DEFINITION 2.7. Let $(M, \mathscr{B})$, $(\Phi, \mathscr{F})$ and P be the same as above. For $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$, the triplet $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ is callled a *metrical random dynamical system*, or simply a random dynamical system.

In the following, we briefly refer to invertible random dynamical systems. (See the first part of [2].) Let (M, d) be the same as in the previous case. We asuume that each $\varphi \in \Phi$ is a homeomorphism on M. The measurable structure $\mathscr{F}$ is associated with Φ in the same way as before. We denote $(\Phi^{\mathbf{Z}}, \mathscr{F}^{\mathbf{Z}})$ by $(\hat{\Phi}, \hat{\mathscr{F}})$, an elment of $\hat{\Phi}$ by $\hat{\varphi} = (\hat{\varphi}_n)_{n \in \mathbf{Z}}$ and the shift transformation on $\hat{\Phi}$ by σ. We fix an ergodic σ-invariant probability measure P on $(\hat{\Phi}, \hat{\mathscr{F}})$. Corresponding to Definition 2.1, the pair $\{(M, d), (\hat{\Phi}, \hat{\mathscr{F}}, P)\}$ is called an *invertible* topological random dynamical system. The skew product transformation, which is denoted by τ in this case too, is given by

$$\tau(x, \hat{\varphi}) = (\varphi_1 x, \sigma\hat{\varphi}), \ (x, \hat{\varphi}) \in M \times \hat{\Phi}.$$

Note that in this case τ is invertible and the inverse is given by

$$\tau^{-1}(x, \hat{\varphi}) = (\varphi_0^{-1} x, \sigma^{-1}\hat{\varphi}), \ (x, \hat{\varphi}) \in M \times \hat{\Phi}.$$

The measure $Q \in \mathscr{P}(M \times \hat{\Phi})$ is called invariant if it satisfies $\pi_{\hat{\Phi}}^* Q = P$ and $\tau^* Q = Q$, where $\pi_{\hat{\Phi}}: M \times \hat{\Phi} \to \hat{\Phi}$ is the natural projection. The set of invariant measures is denoted by $\mathscr{I}_P(M \times \hat{\Phi})$. The triplet $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$ is called an *invertible* metrical random dynamical system. The existence of $Q \in \mathscr{I}_P(M \times \hat{\Phi})$ is assured by the same arguments as in the proof of Theorem 2.6. Note that if $\{Q_{\hat{\varphi}} : \hat{\varphi} \in \hat{\Phi}\}$ is the family of regular conditional probability measures of Q with respect to the partition $\{M \times \{\hat{\varphi}\} : \hat{\varphi} \in \hat{\Phi}\}$, τ-invariance condition of Q is given by the following simple form:

$$\varphi_1^* Q_{\hat{\varphi}} = Q_{\sigma\hat{\varphi}} \quad \text{for} \quad P\text{-a.e. } \hat{\varphi}. \tag{2.5}$$

Of course we assume $\pi_{\hat{\Phi}}^* Q = P$ and regard $Q_{\hat{\varphi}}$ as a probability measure on $(M, \mathscr{B})$. To prove (2.5) compare

$$Q(B \times F) = \int_F Q_{\hat{\varphi}}(B)\, dP(\hat{\varphi})$$

with

$$\tau^*Q(B \times F) = Q(\{(x, \hat{\varphi}) \in M \times \hat{\Phi}: \varphi_1 x \in B, \sigma\hat{\varphi} \in F\})$$
$$= \int_{\sigma^{-1}F} Q_{\hat{\varphi}}(\varphi_1^{-1}B)\, dP(\hat{\varphi}) = \int_F Q_{\sigma^{-1}\hat{\varphi}}(\varphi_0^{-1}B)\, dP(\hat{\varphi})$$

for $B \in \mathfrak{B}$ and $F \in \hat{\mathscr{F}}$, where $\mathfrak{B}$ is a countable basis for $(M, \mathscr{B})$.

REMARK 2.8. The condition (2.5) is derived from $\tau^*Q = Q$. From $(\tau^n)^*Q = Q\,(n \in \mathbf{Z})$, we obtain

$$({}^n\hat{\varphi})^* Q_{\hat{\varphi}} = Q_{\sigma^n\hat{\varphi}} \qquad \text{for} \quad P\text{-a.e. } \hat{\varphi}, \tag{2.6}$$

where $\{{}^n\hat{\varphi}: n \in \mathbf{Z}\}$ is the cocycle defined by

$${}^n\hat{\varphi} = \begin{cases} \varphi_n \circ \cdots \circ \varphi_1 & \text{if } n \geq 1, \\ \text{id} & \text{if } n = 0, \\ \varphi_{n+1}^{-1} \circ \cdots \circ \varphi_0^{-1} & \text{if } n \leq -1. \end{cases}$$

REMARK 2.9. For simplicity we will abuse the underlying P and invariant Q both in a non-invertible and in an invertible random dynamical system. We will make use of different notations only for the infinite products $\tilde{\Phi}$ and $\hat{\Phi}$, elements $\tilde{\varphi} \in \tilde{\Phi}$ and $\hat{\varphi} \in \hat{\Phi}$. But the corresponding shifts will be both denoted by σ. Thus throughout this paper $\{(M, d), (\tilde{\Phi}, \tilde{\mathscr{F}}, P)\}$ or $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ means a non-invertible random dynamical system which is considered in the former part of this section and $\{(M, d), (\hat{\Phi}, \hat{\mathscr{F}}, P)\}$ or $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$ an invertible random dynamical system considered in the above.

§3. Ergodic decomposition of a Markov random dynamical system

For a metrical random dynamical system $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$, let $\{Q^x: x \in M\}$ and $\{Q^{x,\varphi_1,\ldots,\varphi_n}: (x, \varphi_1, \cdots, \varphi_n) \in M \times \Phi^{1,n}\}\,(n \in \mathbf{N})$ be the families of regular conditinal probability measures of Q with respect to the partitions $\{\{x\} \times \tilde{\Phi}: x \in M\}$ and $\{\{(x, \varphi_1, \cdots, \varphi_n)\} \times \Phi^{n+1,\infty}: (x, \varphi_1, \cdots, \varphi_n) \in M \times \Phi^{1,n}\}$ respectively. Here we regard Q^x and $Q^{x,\varphi_1,\ldots,\varphi_n}$ as probability measures on $\tilde{\Phi}$ and $\Phi^{n+1,\infty}$ respectively.

DEFINITION 3.1. Let $\{Q^x: x \in M\}$ and $\{Q^{x,\varphi_1,\ldots,\varphi_n}: (x, \varphi_1, \cdots, \varphi_n) \in M \times \Phi^{1,n}\}$ $(n \in \mathbf{N})$ be the same as above for a random dynamical system $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$. We say that Q is *Markov* if satisfies

$$Q^{x,\varphi_1,\ldots,\varphi_n} = \theta_n^* Q^{{}^n\tilde{\varphi}x} \qquad \text{for all} \quad n \in \mathbf{N},\ Q\text{-a.e. } (x, \tilde{\varphi}), \tag{3.1}$$

where $\theta_n: \tilde{\Phi} \to \Phi^{n+1,\infty}$ is the map defined in §2.

Let $\pi: \tilde{\Phi} \to M^{\mathbf{Z}_+}$ be the map defined by

$$\pi(x, \tilde{\varphi}) = ({}^n\tilde{\varphi}x)_{n\in\mathbf{Z}_+} \quad \text{for} \quad (x, \tilde{\varphi}) \in M \times \tilde{\Phi},$$

$\bar{\sigma}$ be the shift transformation on $M^{\mathbf{Z}_+}$ and $\mathscr{B}_n = \prod_{i=0}^{\infty} \mathscr{B}'_i \, (n \in \mathbf{Z}_+)$, where

$$\mathscr{B}'_i = \begin{cases} \mathscr{B} & \text{if } i = 0, \dots, n, \\ \text{the trivial } \sigma\text{-algebra of } M & \text{otherwise.} \end{cases}$$

We also define $\bar{Q} = \pi^* Q$ and $\mu = \pi_M^* Q$, where $\pi_M: M \times \tilde{\Phi} \to M$ is the natural projection.

LEMMA 3.2. *If Q is Markov, then $\bar{Q} \in \mathscr{P}(M^{\mathbf{N}_+}, \mathscr{B}^{\mathbf{Z}_+})$ is a Markov measure, that is,*

$$\bar{E}(f \circ \bar{\sigma}^n | \mathscr{B}_n)(\bar{y}) = \bar{E}(f | \mathscr{B}_0) \circ \bar{\sigma}^n(\bar{y}), \tag{3.2}$$

for all $n \in \mathbf{Z}$, $\bar{Q}$-a.e. $\bar{y} \in M^{\mathbf{Z}_+}$, $f \in L^\infty(M^{\mathbf{Z}_+}, \mathscr{B}^{\mathbf{Z}_+})$, where $\bar{E}(\ |\ )$ (resp. $\bar{E}$) denotes the conditional expectation (resp. the expectation) with respect to $\bar{Q}$ and $L^\infty(X, \mathscr{A})$ is the set of bounded measurable functions on a measurable space $(X, \mathscr{A})$.

PROOF. Let $\{\bar{Q}^x : x \in M\}$ be the family of regular conditional probability measures of $\bar{Q}$ with respect to the partition $\{\{x\} \times M^{1,\infty}\}$ ($M^{1,\infty} = \prod_{i=1}^{\infty} M_i$, $M_i = M$) of $M^{\mathbf{Z}_+}$. We regard $\bar{Q}^x$ as a probability measure on $(M^{\mathbf{Z}_+}, \mathscr{B}^{\mathbf{Z}_+})$. To prove (3.2), it suffices to show

$$\bar{E}^x(f \circ \bar{\sigma}^n | \mathscr{B}_n)(\bar{y}) = \bar{E}^{y_n}(f), \tag{3.3}$$

for all $n \in \mathbf{N}$, $\bar{Q}^x$-a.e. $\bar{y} = (y_n)_{n\in\mathbf{Z}_+} \in M^{\mathbf{Z}_+}$, μ-a.e. x, $f \in L^\infty(M^{\mathbf{Z}_+}, \mathscr{B}^{\mathbf{Z}_+})$, where $\bar{E}^x(\ |\ )$ (resp. $\bar{E}^x$) denotes the conditional expectation (resp. the expection) with rspect to $\bar{Q}^x$.

To prove (3.3), we first see that the following equality holds:

$$E^x(g \circ \sigma^n | \mathscr{F}_n)(\tilde{\varphi}) = E^{{}^n\tilde{\varphi}x}(g), \tag{3.4}$$

for all $n \in \mathbf{N}$, Q^x-a.e. $\tilde{\varphi}$, μ-a.e. x, $g \in L^\infty(\tilde{\Phi}, \tilde{\mathscr{F}})$, where $E^x(\ |\ )$ (resp. E^x) is the conditional expectation (resp. the expectation) with respect to Q^x and $\mathscr{F}_n$ is the sub σ-algebra of $\tilde{\mathscr{F}}$ defined by $\mathscr{F}_n = \prod_{i=1}^{\infty} \mathscr{F}'_i$ with

$$\mathscr{F}'_i = \begin{cases} \mathscr{F} & \text{if } 1 \le i \le n, \\ \text{the trivial } \sigma\text{-algebra of } \Phi & \text{otherwise.} \end{cases}$$

For, if $F \in \mathscr{F}_n$, we have from (3.1),

$$\begin{aligned}\int_F E^x(g\circ\sigma^n|\mathscr{F}_n)\,dQ^x &= \int_F g\circ\sigma^n\,dQ^x\\ &= \iint_F g(\sigma^n\tilde{\varphi})\,dQ^{x,\varphi_1,\cdots,\varphi_n}(\theta_n\sigma^n\tilde{\varphi})\,d\pi_{1,n}^*Q^x(\varphi_1,\cdots,\varphi_n)\\ &= \iint_F g(\sigma^n\tilde{\varphi})\,dQ^{n_{\tilde{\varphi}}x}(\sigma^n\tilde{\varphi})\,d\pi_{1,n}^*Q^x(\varphi_1,\cdots,\varphi_n)\\ &= \int_F E^{n_{\tilde{\varphi}}x}(g)\,dQ^x(\tilde{\varphi}),\end{aligned}$$

where $\pi_{1,n}\colon\tilde{\Phi}\to\Phi^{1,n}$ is the natural projection. This implies (3.4). If $\tilde{Q}^x=\delta_x\times Q^x$ (δ_x is the point mass at $x\in M$), it is easy to see from (3.4) that

$$\tilde{E}^x(h\circ\tau^n|\pi^{-1}(\mathscr{B}_n))(y,\tilde{\varphi})=\tilde{E}^{n_{\tilde{\varphi}}x}(h), \tag{3.5}$$

for all $n\in\mathbf{Z}_+$, $\tilde{Q}^x$-a.e. $(y,\tilde{\varphi})$, μ-a.e. x, $h\in L^\infty(M\times\tilde{\Phi},\mathscr{B}\times\tilde{\mathscr{F}})$, where $\tilde{E}^x(\ |\ )$(resp. $\tilde{E}^x$) is the conditional expectation (resp. the expectation) with respect to $\tilde{Q}^x$. Therefore we have

$$\begin{aligned}\bar{E}^x(f\circ\bar{\sigma}^n|\mathscr{B}_n)\circ\pi(y,\tilde{\varphi}) &= \tilde{E}^x(f\circ\bar{\sigma}^n|\pi^{-1}(\mathscr{B}_n))(y,\tilde{\varphi})\\ &= \tilde{E}^x(f\circ\pi\circ\sigma^n|\pi^{-1}(\mathscr{B}_n))(y,\tilde{\varphi})\\ &= \tilde{E}^{n_{\tilde{\varphi}}x}(f\circ\pi)=\bar{E}^{n_{\tilde{\varphi}}x}(f),\end{aligned}$$

for all $n\in\mathbf{Z}_+$, $\tilde{Q}$-a.e. $(y,\tilde{\varphi})$, μ-a.e. x, $f\in L^\infty(M^{\mathbf{Z}_+},\mathscr{B}^{\mathbf{Z}_+})$. Since π is surjective mod. $\bar{Q}^x$ for μ-a.e. x, (3.3) follows.

REMARK 3.3. Since $\{M^{\mathbf{Z}_+},\mathscr{B}^{\mathbf{Z}_+},\bar{\sigma},\bar{Q}\}$ is a factor of $\{M\times\tilde{\Phi},\mathscr{B}\times\tilde{\mathscr{F}},\tau,Q\}$, in the case of (3.1) (in which $\bar{Q}$ is Markov by the above lemma) $\bar{Q}$ is a stationary Markov measure. The transition probability $P(x,B)$ for $x\in M$ and $B\in\mathscr{B}$ is given by $P(x,B)=\bar{Q}^x(B)=\int 1_B(\varphi_1 x)\,dQ^x(\tilde{\varphi})$ with the initial stationary distribution μ. In this case the corresponding Markov operator $\mathscr{Q}$ on $L^1(M,\mu)$ is defined by $\mathscr{Q}f(x)=\int f(\varphi_1 x)\,dQ^x(\tilde{\varphi})$ for $f\in L^1(M,\mu)$. We denote the dual operator on $\mathscr{P}(M)$ by $\mathscr{Q}^*\colon\mathscr{Q}^*\nu(B)=\iint 1_B(\varphi_1 x)\,dQ^x(\tilde{\varphi})\,d\nu(x)$ for $B\in\mathscr{B}$ and $\nu\in\mathscr{P}(M)$.

DEFINITION 3.4. Suppose that Q is Markov. We say that Q is *M-ergodic* if $\mathscr{Q}f=f$ μ-a.e. implies $f=$ const. μ-a.e. for $f\in L^1(M,\mu)$.

THEOREM 3.5. *Let Q be Markov. Then Q is M-ergodic if and only if (τ,Q) is ergodic.*

PROOF. Suppose that (τ,Q) is ergodic. Take $f\in L^1(M,\mu)$ such that $\mathscr{Q}f=f$

μ-a.e. For $c \in \mathbf{R}$, we define $B_c = \{x \in M : f(x) > c\}$. Since 1_{B_c} is $\mathscr{Q}$-invariant (see [4], p.19), that is,

$$\int 1_{B_c}(\varphi_1 x)\, dQ^x(\tilde{\varphi}) = 1_{B_c}(x) \qquad \text{for} \quad \mu\text{-a.e.}\, x,$$

we have $1_{B_c}(\varphi_1 x) = 1_{B_c}(x)$ for Q-a.e. $(x, \tilde{\varphi})$. This equality shows that 1_{B_c}, viewed as an element of $L^1(M \times \tilde{\Phi}, Q)$, is τ-invariant. By the ergodicity of τ, we have that $1_{B_c} = \text{const.}$ μ-a.e. From this, we obtain $\mu(B_c) = 0$ or 1 for all $c \in \mathbf{R}$, which implies that $f = c_0$ μ-a.e. for some $c_0 \in \mathbf{R}$.

Next suppose that Q is M-ergodic. Take $g \in L^1(M \times \tilde{\Phi}, Q)$ such that $g \circ \tau = g$ Q-a.e. and define g_0 by

$$g_0(x) = \int g(x, \tilde{\varphi})\, dQ^x(\tilde{\varphi}).$$

Clearly $g_0 \in L^1(M, \mu)$. By the τ-invariance of g and (3.1), we have

$$\begin{aligned}
\mathscr{Q} g_0(x) &= \int g_0(\varphi_1 x)\, dQ^x(\tilde{\varphi}) \\
&= \int\!\!\int g(\varphi_1 x, \tilde{\psi})\, dQ^{\varphi_1 x}(\tilde{\psi})\, d\pi^*_{1,1} Q^x(\varphi_1) \\
&= \int\!\!\int g(x, \varphi_1 \theta_1 \tilde{\psi})\, dQ^{x, \varphi_1}(\theta_1 \tilde{\psi})\, d\pi^*_{1,1} Q^x(\varphi_1) \\
&= \int g(x, \tilde{\varphi})\, dQ^x(\tilde{\varphi}) = g_0(x) \qquad \text{for} \quad \mu\text{-a.e.}\, x.
\end{aligned}$$

Then by the M-ergodicity of Q, we have that $g_0 = c$ μ-a.e. for some $c \in \mathbf{R}$. Next if we define $g_n = E(g | \mathscr{B} \times \mathscr{F}_n)$, we have

$$\begin{aligned}
g_n(x, \tilde{\varphi}) &= \int g(x, \varphi_1, \cdots, \varphi_n \theta_n \tilde{\psi})\, dQ^{x, \varphi_1, \ldots, \varphi_n}(\theta_n \tilde{\psi}) \\
&= \int g({}^n\tilde{\varphi} x, \tilde{\psi})\, dQ^{x, \varphi_1, \ldots, \varphi_n}(\theta_n \tilde{\psi}) \\
&= \int g({}^n\tilde{\varphi} x, \tilde{\psi})\, dQ^{{}^n\tilde{\varphi} x}(\tilde{\psi}) \\
&= g_0({}^n\tilde{\varphi} x) = c \qquad \text{for} \quad Q\text{-a.e.}\, (x, \tilde{\varphi}),
\end{aligned}$$

where we used the condition $g \circ \tau^n = g$ Q-a.e. Therefore letting $n \to \infty$, by Doob's theorem we obtain $g(x, \tilde{\varphi}) = c$ Q-a.e. This implies the ergodicity of (τ, Q).

Let Q be Markov and put $\mathscr{E}_P(M) = \{\nu \in \mathscr{P}(M): \mathscr{Q}^*\nu = \nu$ and $\mathscr{Q}f = f$ ν-a.e. implies $f = \text{const.}$ ν-a.e. for $f \in L^1(M, \nu)\}$. Then $\mu = \pi_M^* Q$ is expressed in the following way by a probability measure λ on $\mathscr{E}_P(M)$:

$$\mu = \int_{\mathscr{E}_P(M)} \nu d\lambda(\nu).$$

The ergodic decomposition of M with respect to $\mathscr{Q}$ is $\zeta = \{\chi^{-1}(\nu): \nu \in \mathscr{E}_P(M)\}$, where the map $\chi: M \to \mathscr{E}_P(M)$ is defined by

$$\chi(x) = \lim_{n\to\infty} n^{-1} \textstyle\sum_{i=0}^{n-1} (\mathscr{Q}^*)^i \delta_x,$$

χ is defined μ-a.e. by virtue of the Ornstein-Chacon theorem. (For details of ergodic decomposition of Markov opertors, see [4] and [5].)

COROLLARY 3.6. *Let ζ be the same as above. Then the ergodic decomposition of* (τ, Q) *is given by* $\zeta \times \tilde{\Phi} = \{C \times \tilde{\Phi}: C \in \zeta\}$.

PROOF. Let $\{\mu_C: C \in \zeta\}$ and $\{Q_C: C \in \zeta\}$ be the families of regular conditional probability measures of μ *and* Q with respect to the partitions ζ and $\zeta \times \tilde{\Phi}$ respectively. Clearly

$$Q_C(B \times F) = \int_B Q^x(F) d\mu_C(x) \quad \text{for} \quad B \in \mathscr{B} \quad \text{and} \quad F \in \tilde{\mathscr{F}}.$$

Since $\mathscr{Q}1_{C^c} = 1_{C^c}$ μ_C-a.e. and a.e. $C \in \zeta$ (C^c is the complement of C), we have

$$\begin{aligned} Q_C(C \times \tilde{\Phi} \backslash \tau^{-1}(C \times \tilde{\Phi})) &= \iint 1_C(x) 1_{C^c}(\varphi_1 x) dQ^x(\tilde{\varphi}) d\mu_C(x) \\ &= \int 1_C(x) \mathscr{Q} 1_{C^c}(x) d\mu_C(x) \\ &= \int 1_C(x) 1_{C^c}(x) d\mu_C(x) = 0 \text{ for a.e. } C \in \zeta \end{aligned}$$

which implies $\tau^{-1}(C \times \tilde{\Phi}) = C \times \tilde{\Phi} \bmod. Q_C$, a.e. $C \in \zeta$. By the same arguments as in the latter part of the proof of Theorem 3.5, we consequently have $(\tau|_{C\times\tilde{\Phi}}, Q_C)$ is ergodic a.e. $C \in \zeta$. This implies the statement in the corollary.

§4. Topological entropy and metrical entropy

Let $\{(M, d), (\tilde{\Phi}, \tilde{\mathscr{F}}, P)\}$ be a topological random dynamical system. For a finite open convering α of M, the minimal cardinality of subconverings of α is denoted by $\mathscr{N}(\alpha)$. Define the open covering $\alpha_n(\tilde{\varphi})$ by $\alpha_n(\tilde{\varphi}) = \bigvee_{i=0}^{n-1} {}^i\tilde{\varphi}^{-1}\alpha$ (the refinement of ${}^{i}\tilde{\varphi}^{-1}\alpha$, $i = 0, \cdots, n-1$). Under this notation clearly $\alpha_{m+n}(\tilde{\varphi}) = \alpha_n(\tilde{\varphi}) \vee {}^{n}\tilde{\varphi}\alpha_m(\sigma^n\tilde{\varphi})$ and therefore $\mathcal{N}(\alpha_{m+n}(\tilde{\varphi})) \leq \mathcal{N}(\alpha_n(\tilde{\varphi}))$ $\mathcal{N}({}^{n}\tilde{\varphi}^{-1}\alpha_m(\sigma^n\tilde{\varphi})) = \mathcal{N}(\alpha_n(\tilde{\varphi}))\mathcal{N}(\alpha_m(\sigma^n\tilde{\varphi}))$. Hence by Kingman's subadditive ergodic theorem

$$h(P, \alpha) = \lim_{n\to\infty} \log \mathcal{N}(\alpha_n(\tilde{\varphi}))/n$$

exists P-a.e.$\tilde{\varphi}$. (Recall that (σ, P) is ergodic.) By the same arguments as in the deterministic case, we know the existence of the limit

$$h_{top}(P) = \lim_{\text{diam}\alpha\to 0} h(P, \alpha),$$

which is called the *topological entropy* of $\{(M, d), (\tilde{\Phi}, \tilde{\mathcal{F}}, P)\}$.

We say that for $\tilde{\varphi} \in \tilde{\Phi}$, $n \in \mathbf{N}$ and $\varepsilon > 0$, $E \subset M$ is $(\tilde{\varphi}, n, \varepsilon)$-separated if all x, $y \in E$, $x \neq y$, satisfy $d({}^{i}\tilde{\varphi}x, {}^{i}\tilde{\varphi}y) > \varepsilon$ for some $0 \leq i \leq n-1$. The maximal cardinality of $(\tilde{\varphi}, n, \varepsilon)$-separated sets is denoted by $s(\tilde{\varphi}, n, \varepsilon)$. We say that for $\tilde{\varphi} \in \tilde{\Phi}$, $n \in \mathbf{N}$ and $\varepsilon > 0$, $F \subset M$ is $(\tilde{\varphi}, n, \varepsilon)$-spanning if for any $x \in M$, there exists $y \in F$ such that $d({}^{i}\tilde{\varphi}x, {}^{i}\tilde{\varphi}y) \leq \varepsilon$ for all $0 \leq i \leq n-1$. The minimal cardinality of $(\tilde{\varphi}, n, \varepsilon)$-spanning sets is denoted by $r(\tilde{\varphi}, n, \varepsilon)$.

LEMMA 4.1. *Let* $\{(M, d), (\tilde{\Phi}, \tilde{\mathcal{F}}, P)\}$ *be the same as above. Then*

$$\begin{aligned} h_{top}(P) &= \lim_{\varepsilon\to 0} \operatorname{limsup}_n \log s(\tilde{\varphi}, n, \varepsilon)/n \\ &= \lim_{\varepsilon\to 0} \operatorname{liminf}_n \log s(\tilde{\varphi}, n, \varepsilon)/n \\ &= \lim_{\varepsilon\to 0} \operatorname{limsup}_n \log r(\tilde{\varphi}, n, \varepsilon)/n \\ &= \lim_{\varepsilon\to 0} \operatorname{liminf}_n \log r(\tilde{\varphi}, n, \varepsilon)/n \qquad \text{for } P\text{-a.e.}\tilde{\varphi}. \end{aligned}$$

The proof is essentially the same as in the deterministic case. See [4] and [11].

For a finite measurable partition ξ of a probability space $(X, \mathcal{A}, \nu)$ and a sub σ-algebra $\mathcal{C}$ of $\mathcal{A}$, we define the conditional entropy $H_\nu(\xi|\mathcal{C})$ of ξ given $\mathcal{C}$ by

$$H_\nu(\xi|\mathcal{C}) = \int I_\nu(\xi|\mathcal{C})d\nu, \; I_\nu(\xi|\mathcal{C}) = -\sum_{A\in\xi} 1_A \log \nu(A|\mathcal{C}).$$

When $\mathcal{C}$ is the trivial σ-algebra of X constructed from X and ϕ, $H_\nu(\xi|\mathcal{C})$ and $I_\nu(\xi|\mathcal{C})$ are simply denoted by $H_\nu(\xi)$ and $I_\nu(\xi)$ respectively. In this case $H_\nu(\xi) = \sum_{A\in\xi} \kappa(\nu(A))$ where $\kappa(x) = -x\log x$ if $0 < x \leq 1$ and $\kappa(0) = 0$. $H_\nu(\xi|\mathcal{F}(\eta))$ and $I_\nu(\xi|\mathcal{F}(\eta))$ will be sometimes denoted by $H_\nu(\xi|\eta)$ and $I_\nu(\xi|\eta)$ respectively, where $\mathcal{F}(\eta)$ is the σ-algebra corresponding to the partition η.

Let $\{(M, \mathcal{B}), (\tilde{\Phi}, \tilde{\mathcal{F}}, P), Q\}$ be a metrical random dynamical system. For a finite measurable partition ξ of M, we define $H_n(\xi)$ by

$$H_n(\xi)=\int H_{Q_{\tilde{\varphi}}}(\xi_n(\tilde{\varphi}))\,dP(\tilde{\varphi}),$$

where $\xi_n(\tilde{\varphi})=\bigvee_{i=0}^{n-1}{}^i\tilde{\varphi}^{-1}\xi$ (the refinement of partitions ${}^i\tilde{\varphi}^{-1}\xi$, $0\le i\le n-1$).

LEMMA 4.2. *Let $H_n(\xi)$ be the same as above. Then*

$$H_{n+m}(\xi)\le H_n(\xi)+H_m(\xi)\qquad \textit{for all}\quad n,\ m\in\mathbf{N}.$$

PROOF. Since $\xi_{n+m}(\tilde{\varphi})=\xi_n(\tilde{\varphi})\vee{}^n\tilde{\varphi}^{-1}\xi_m(\sigma^n\tilde{\varphi})$, we have

$$\begin{aligned}H_{n+m}(\xi)&\le\int\{H_{Q_{\tilde{\varphi}}}(\xi_n(\tilde{\varphi}))+H_{Q_{\tilde{\varphi}}}({}^n\tilde{\varphi}^{-1}\xi_m(\sigma^n\tilde{\varphi}))\}\,dP(\tilde{\varphi})\\&=H_n(\xi)+\int H_{Q_{\tilde{\varphi}}}({}^n\tilde{\varphi}^{-1}\xi_m(\sigma^n\tilde{\varphi}))\,dP(\tilde{\varphi})\end{aligned}$$

So it suffices to show

$$\int H_{Q_{\tilde{\varphi}}}({}^n\tilde{\varphi}^{-1}\xi_m(\sigma^n\tilde{\varphi}))\,dP(\tilde{\varphi})\le H_m(\xi). \tag{4.1}$$

In the following we use the same notations as in §2. We see that

the left hand side of (4.1)

$$\begin{aligned}&=\iint[\textstyle\sum_{A\in\xi_m(\sigma^n\tilde{\varphi})}\kappa(Q_{\varphi_1,\dots,\varphi_n\theta_n\sigma^n\tilde{\varphi}}({}^n\tilde{\varphi}^{-1}A))]\,dP_{\theta_n\sigma^n\tilde{\varphi}}(\varphi_1,\cdots,\varphi_n)\,dP(\sigma^n\tilde{\varphi})\\&\le\int\left[\textstyle\sum_{A\in\xi_m(\sigma^n\tilde{\varphi})}\kappa\left(\displaystyle\int Q_{\varphi_1,\dots,\varphi_n\theta_n\sigma^n\tilde{\varphi}}({}^n\tilde{\varphi}^{-1}A)\,dP_{\theta_n\sigma^n\tilde{\varphi}}(\varphi_1,\cdots,\varphi_n)\right)\right]dP(\sigma^n\tilde{\varphi})\\&=\int[\textstyle\sum_{A\in\xi_m(\sigma^n\tilde{\varphi})}\kappa(Q_{\sigma^n\tilde{\varphi}}(A))]\,dP(\sigma^n\tilde{\varphi})\\&=\int H_{Q_{\sigma^n\tilde{\varphi}}}(\xi_m(\sigma^n\tilde{\varphi}))\,dP(\sigma^n\tilde{\varphi})=H_m(\xi).\end{aligned}$$

Here we used Jensen's inequality and τ^n-invariance condition (2.4) of Q. Thus we obtain (4.1).

From Lemma 4.2 we know the existence of the limit

$$h_Q(P,\xi)=\lim_{n\to\infty}H_n(\xi)/n.$$

The value $h_Q(P)$ defined below is called the metrical entropy of $\{(M,\mathscr{B}),(\tilde{\Phi},\tilde{\mathscr{F}},P),Q\}$:

$$h_Q(P)=\sup\{h_Q(P,\xi):\ \xi\text{ is a finite measurable partition of }M\}.$$

For an invertible random dynamical system, we can define entropies similarly. That is, for $\{(M, d), (\hat{\Phi}, \hat{\mathscr{F}}, P)\}$,

$$h_{top}(P) = \lim_{\substack{\mathrm{diam}\alpha \to 0 \\ \alpha:\text{open covering}}} \lim_{n\to\infty} \log \mathscr{N}(\alpha_n(\hat{\varphi}))/n \quad P\text{-a.e.}\hat{\varphi}$$

where for a finite open convering α of M, $\alpha_n(\tilde{\varphi}) = \vee_{i=0}^{n-1} {}^i\hat{\varphi}^{-1}\alpha$ and $\mathscr{N}(\alpha_n(\tilde{\varphi}))$ is the minimal cardinality of subcoverings of $\alpha_n(\hat{\varphi})$, and for $\{(M, \mathscr{B}), (\hat{\Phi}, \tilde{\hat{\mathscr{F}}}, P), Q\}$ with $Q \in \mathscr{I}_P(M \times \hat{\Phi})$,

$$h_Q(P) = \sup\{h_Q(P, \xi) \colon \xi \text{ is a finite measurable partition of } M\},$$

$$h_Q(P, \xi) = \lim_{n\to\infty} H_{Q_{\tilde{\varphi}}}(\xi_n(\hat{\varphi}))/n \qquad \text{for} \quad P\text{-a.e.}\hat{\varphi},$$

$$\xi_n(\hat{\varphi}) = \vee_{i=0}^{n-1} {}^i\hat{\varphi}^{-1}\xi.$$

As for the P-a.e. existence of $h_Q(P, \xi)$ in the above, we note the following estimate and then apply Kingman's subadditive ergodic theorem to $\{H_{Q_{\hat{\varphi}}}(\xi_n(\hat{\varphi})) \colon n \in \mathbf{N}\}$ as functions of $\hat{\varphi}$:

$$\begin{aligned} H_{Q_{\hat{\varphi}}}(\xi_{n+m}(\hat{\varphi})) &= H_{Q_{\hat{\varphi}}}(\xi_n(\hat{\varphi}) \vee {}^n\hat{\varphi}^{-1}\xi_m(\sigma^n\hat{\varphi})) \\ &\le H_{Q_{\hat{\varphi}}}(\xi_n(\hat{\varphi})) + H_{Q_{\hat{\varphi}}}({}^n\hat{\varphi}^{-1}\xi_m(\sigma^n\hat{\varphi})) \\ &= H_{Q_{\hat{\varphi}}}(\xi_n(\hat{\varphi})) + H_{Q_{\sigma^n\hat{\varphi}}}(\xi_m(\sigma^n\hat{\varphi})), \end{aligned}$$

where we use the τ^n-invariance condition (2.6).

The following propositions are the random version of the equality $h_\mu(f, \xi) = H_\mu(\xi \mid \vee_{i=1}^{\infty} f^{-1}\xi)$ for a deterministic (f, μ).

PROPOSITION 4.3. *Let* $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ *be a random dynamical system. Then for a finite measurable partition* ξ *of* M,

$$h_Q(P, \xi) \le \int H_{Q_{\tilde{\varphi}}}(\xi \mid \vee_{n=1}^{\infty} {}^n\tilde{\varphi}^{-1}\xi)\, dP(\tilde{\varphi}).$$

PROOF. Clarly for $n \in \mathbf{N}$.

$$\begin{aligned} H_n(\xi) &= \int H_{Q_{\tilde{\varphi}}}(\xi_n(\tilde{\varphi}))\, dP(\tilde{\varphi}) \\ &= \int H_{Q_{\tilde{\varphi}}}(\xi \mid \vee_{i=1}^{n-1} {}^i\tilde{\varphi}^{-1}\xi)\, dP(\tilde{\varphi}) + \int H_{Q_{\tilde{\varphi}}}(\vee_{i=1}^{n-1} {}^i\tilde{\varphi}^{-1}\xi)\, dP(\tilde{\varphi}). \end{aligned}$$

But

$$\int H_{Q_{\tilde{\varphi}}}(\vee_{i=1}^{n-1} {}^i\tilde{\varphi}^{-1}\xi)\, dP(\tilde{\varphi}) = \int H_{Q_{\tilde{\varphi}}}(\varphi_1^{-1}\xi_{n-1}(\sigma\tilde{\varphi}))\, dP(\tilde{\varphi}) \le H_{n-1}(\xi).$$

(Replace n and m by 1 and $n-1$ respectively in (4.1).) Therefore

$$H_n(\xi) - H_{n-1}(\xi) \le \int H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=0}^{n-1}\, {}^i\tilde\varphi^{-1}\xi)\, dP(\tilde\varphi)$$

and from this

$$H_n(\xi) \le \sum_{k=1}^{n} \int H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=1}^{k-1}\, {}^i\tilde\varphi^{-1}\xi)\, dP(\tilde\varphi),$$

where we put $H_{Q_{\tilde\varphi}}(\xi)$ for the term corresponding to $k=1$. Thus, considering

$$H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=1}^{n-1}\, {}^i\tilde\varphi^{-1}\xi) \downarrow H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=1}^{\infty}\, {}^i\tilde\varphi^{-1}\xi) \qquad \text{as} \quad n \uparrow \infty,$$

we obtain

$$\begin{aligned} h_Q(P, \xi) &= \lim_{n\to\infty} H_n(\xi)/n \\ &\le \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=1}^{k-1}\, {}^i\tilde\varphi^{-1}\xi)\, dP(\tilde\varphi) \\ &= \int H_{Q_{\tilde\varphi}}(\xi \mid \vee_{i=1}^{\infty}\, {}^i\tilde\varphi^{-1}\xi)\, dP(\tilde\varphi). \end{aligned}$$

For an invertible random dynamical system, we obtain the following stronger results.

PRPOSITION 4.4. *Let $\{(M, \mathscr{B}), (\hat\Phi, \hat{\mathscr{F}}, P), Q\}$ be an invertible random dynamical system. Then for a finite measurable partition ξ of M,*

$$h_Q(P, \xi) = \int H_{Q_{\hat\varphi}}(\xi \mid \vee_{i=1}^{\infty}\, {}^i\tilde\varphi^{-1}\xi)\, dP(\hat\varphi). \tag{4.2}$$

PROOF. Clearly

$$\begin{aligned} H_{Q_{\hat\varphi}}(\xi_n(\hat\varphi)) &= H_{Q_{\hat\varphi}}(\xi \mid \vee_{i=1}^{n-1}\, {}^i\hat\varphi^{-1}\xi) + H_{Q_{\hat\varphi}}(\varphi_1^{-1} \vee_{i=0}^{n-2}\, {}^i(\sigma\hat\varphi)^{-1}\xi) \\ &= H_{Q_{\hat\varphi}}(\xi \mid \vee_{i=1}^{n-1}\, {}^i\hat\varphi^{-1}\xi) + H_{Q_{\sigma\hat\varphi}}(\vee_{i=0}^{n-2}\, {}^i(\sigma\hat\varphi)^{-1}\xi) \\ &= H_{Q_{\hat\varphi}}(\xi \mid \vee_{i=1}^{n-1}\, {}^i\hat\varphi^{-1}\xi) + H_{Q_{\sigma\hat\varphi}}(\xi_{n-1}(\sigma\hat\varphi)). \end{aligned}$$

Therefore putting

$$G(n, \hat\varphi) = \begin{cases} H_{Q_{\hat\varphi}}(\xi \mid \vee_{i=1}^{n-1}\, {}^i\hat\varphi^{-1}\xi), & n \ge 2, \\ H_{Q_{\hat\varphi}}(\xi), & n = 1, \end{cases}$$

and $H_n(\xi, \hat\varphi) = H_{Q_{\hat\varphi}}(\xi_n(\hat\varphi))$ for $n \in \mathbf{N}$, we have

$$H_n(\xi, \hat\varphi) = G(n, \hat\varphi) + H_{n-1}(\xi, \sigma\hat\varphi)$$

and so

$$H_n(\xi, \hat{\varphi}) = \textstyle\sum_{i=0}^{n-1} G(n-i, \sigma^i\hat{\varphi}). \tag{4.3}$$

Since

$$G(n, \hat{\varphi}) \downarrow g(\hat{\varphi}) \equiv H_{Q_{\hat{\varphi}}}(\xi | \vee_{i=1}^{\infty} {}^i\hat{\varphi}^{-1}\xi),$$

$$n^{-1}\textstyle\sum_{i=0}^{n-1} g(\sigma^i\hat{\varphi}) \longrightarrow \displaystyle\int g dP \text{ (note that } 0 \le g \le \log(\#\xi)),$$

and

$$H_n(\xi, \hat{\varphi})/n \longrightarrow h_Q(P, \xi) \qquad \text{for} \quad P\text{-a.e.}\varphi,$$

for arbitrary $\varepsilon > 0$, there exist $K \in \hat{\mathscr{F}}$ and $N \in \mathbf{N}$ such that

$$\begin{cases} P(K) > 1 - \varepsilon, \\ |G(n, \hat{\varphi}) - g(\hat{\varphi})| < \varepsilon, \\ |n^{-1}\sum_{i=0}^{n-1} g(\sigma^i\hat{\varphi}) - \int g dP| < \varepsilon, \\ |H_n(\xi, \hat{\varphi})/n - h_Q(P, \xi)| < \varepsilon, \end{cases} \tag{4.4}$$

for all $\hat{\varphi} \in K$ and for $n \ge N$, $n \in \mathbf{N}$. If we put

$$J_n^{\hat{\varphi}} = \{0 \le i \le n-1 : \sigma^i\hat{\varphi} \in K, \ n - i \ge N\},$$

there exists $N_{\hat{\varphi}} \in \mathbf{N}$ such that

$$1 - \# J_n^{\hat{\varphi}}/n < \varepsilon \text{ for all } n \ge N_{\hat{\varphi}}, \text{ for } P\text{-a.e. } \hat{\varphi}. \tag{4.5}$$

Then we have, from (4.3), (4.4) and (4.5),

$$\begin{aligned} \left| h_Q(P, \xi) - \int g dP \right| &\le |h_Q(P, \xi) - H_n(\xi, \hat{\varphi})/n| \\ &\quad + n^{-1}\textstyle\sum_{i \in J_n^{\hat{\varphi}}} |G(n-i, \sigma^i\hat{\varphi}) - g(\sigma^i\hat{\varphi})| + n^{-1}\sum_{i \notin J_n^{\hat{\varphi}}} |G(n-i, \sigma^i\hat{\varphi}) - g(\sigma^i\hat{\varphi})| \\ &\quad + \left| n^{-1}\textstyle\sum_{i=0}^{n-1} g(\sigma^i\hat{\varphi}) - \displaystyle\int g dP \right| \\ &< \varepsilon + \varepsilon + \varepsilon\log(\#\xi) + \varepsilon \\ &= (3 + \log(\#\xi))\varepsilon \end{aligned}$$

for some $\hat{\varphi} \in K$ and $n \ge N_{\hat{\varphi}}$. Since $\varepsilon > 0$ is arbitrary, we obtain (4.2).

In what follows, we will summarize some results on conditional entropies for later use. The arguments developed here are almost the same as in [4]. Let $(X, \mathscr{A}, \nu)$ be a probability space and $\varphi: X \to X$ be an endomorphism, i.e. a measurable ν-preserving transformation (resp. an automorphism i.e. an invertible ν-preserving transformation). We assume that a sub σ-algebra $\mathscr{C}$ satisfies $\varphi^{-1}\mathscr{C} \subset \mathscr{C}$ (resp. $\varphi^{-1}\mathscr{C} = \mathscr{C}$). Then for a finite measurable partition ξ of X,

$$(4.6) \qquad h_\nu^{\mathscr{C}}(\varphi, \xi) = \lim_{n\to\infty} n^{-1} H_\nu(\vee_{i=0}^{n-1} \varphi^{-1}\xi | \mathscr{C})$$

exists, which is called the entropy of φ with respect to ξ given $\mathscr{C}$. We also define $h_\nu^{\mathscr{C}}(\varphi)$ by

$$h_\nu^{\mathscr{C}}(\varphi) = \sup\{h_\nu^{\mathscr{C}}(\varphi, \xi): \xi \text{ is a finite measurable partition of } X\}.$$

The following properties hold:

$$(4.7) \qquad h_\nu^{\mathscr{C}}(\varphi, \xi) \le h_\nu^{\mathscr{C}}(\varphi, \eta) + H_\nu(\xi | \mathscr{F}(\eta) \vee \mathscr{C}),$$

$$(4.8) \qquad h_\nu^{\mathscr{C}}(\varphi, \xi) = h_\nu^{\mathscr{C}}(\varphi, \vee_{i=0}^{k} \varphi^{-i}\xi) \text{ (resp. } h_\nu^{\mathscr{C}}(\varphi, \vee_{i=-k}^{k} \varphi^{-i}\xi)) \text{ for } k \in \mathbf{N},$$

$$(4.9) \qquad h_\nu^{\mathscr{C}}(\varphi) = \lim_{n\to\infty} h_\nu(\varphi, \eta_n),$$

where ξ and η are finite measurable partitions in (4.7) and (4.8) and $\{\eta_n\}$ is an increasing sequence of finite measurable partitions of X such that $\mathscr{F}(\vee_n \eta_n) \vee \mathscr{C} = \mathscr{B}$ in (4.9).

Returning back to random dynamical systems, we can apply the above results.

LEMMA 4.5. *If ξ and ζ are finite measurable partitions of M and $\tilde{\Phi}$ (resp. $\hat{\Phi}$) respectively, then for $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ (resp. $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$),*

$$(4.10) \qquad h_Q(P, \xi) = h_Q^{\mathfrak{N}\times\tilde{\mathscr{F}}}(\tau, \xi \times \zeta) \quad \text{(resp. } h_Q(P, \xi) = h_Q^{\mathfrak{N}\times\hat{\mathscr{F}}}(\tau, \xi \times \zeta)),$$

where $\mathfrak{N}$ is trivial σ-algera of M constructed from M and ϕ.

PROOF. Since clearly

$$Q_{\tilde{\varphi}}(\cap_{i=0}^{n-1} \tau^{-i}(A_i \times B_i) | \mathfrak{N} \times \tilde{\mathscr{F}})(x, \tilde{\varphi}) = Q_{\tilde{\varphi}}(\cap_{i=0}^{n-1} {}^i\tilde{\varphi}^{-1} A_i) \prod_{i=0}^{n-1} 1_{B_i}(\sigma^i \tilde{\varphi})$$

for $A_i \in \xi$ and $B_i \in \zeta$, we have

$$H_Q(\vee_{i=0}^{n-1} \tau^{-i}(\xi \times \zeta) | \mathfrak{N} \times \tilde{\mathscr{F}}) = \int H_{Q_{\tilde{\varphi}}}(\xi_n(\tilde{\varphi}))\, dP(\tilde{\varphi}).$$

Replacing $\mathscr{C}$, ν, φ and ξ by $\mathfrak{N} \times \tilde{\mathscr{F}}$, Q, τ and $\xi \times \zeta$ in (4.6), we obtain (4.10). In the invertible case, the equality can be shown just in the same way.

LEMMA 4.6. *For $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$* (resp. $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$) *sup-*

pose that a finite measurable partition ξ *of* M *satisfies* $\mathscr{F}(\vee_{i=0}^{\infty}{}^{i}\varphi^{-1}\xi)=\mathscr{B}$ *for* P*-a.e.*$\tilde{\varphi}$ *(resp.* $\mathscr{F}(\vee_{i=-\infty}^{\infty}{}^{i}\hat{\varphi}^{-1}\xi)=\mathscr{B}$ *for* P*-a.e.*$\hat{\varphi}$*). Then*

$$h_Q(P)=h_Q(P,\xi)\ (\textit{in both cases}).$$

PROOF. Replace $\mathscr{C}$, ν, φ, ξ and η by $\mathfrak{N}\times\tilde{\mathscr{F}}$, Q, τ, $\eta\times\zeta_0$ and $\vee_{i=0}^{k}\tau^{-i}(\xi\times\zeta_0)$ respectively in (4.7), where ζ_0 is the trivial partition of $\tilde{\Phi}$ and η is an arbitrary finite measurable partition of M. Then in view of (4.8) and Lemma 4.5, we have

$$h_Q(P,\eta)\le h_Q(P,\xi)+H_Q(\eta\times\zeta_0|\mathscr{F}(\vee_{i=0}^{k}\tau^{-i}(\xi\times\zeta_0))\vee(\mathfrak{N}\times\tilde{\mathscr{F}})). \tag{4.11}$$

But clearly

$$H_Q(\eta\times\zeta_0|\mathscr{F}(\vee_{i=0}^{k}\tau^{-i}(\xi\times\zeta_0))\vee(\mathfrak{N}\times\tilde{\mathscr{F}}))$$
$$=\int H_{Q_{\tilde{\varphi}}}(\eta|\mathscr{F}(\vee_{i=0}^{k}{}^{i}\tilde{\varphi}^{-1}\xi))\,dP(\tilde{\varphi}).$$

Letting $k\to\infty$, we obtain

$$h_Q(P,\eta)\le h_Q(P,\xi)$$

which implies $h_Q(P,\xi)=h_Q(P)$. In the invertible case the equality can be shown just in the same way.

To derive the variational principle for a random dynamical system, we quote from [6] the relativised variational principle for continuous maps. Let X and Y be compact mteric spaces and $T: X\to X$, $S: Y\to Y$, $\pi: X\to Y$ be continuous maps such that π is surjective and $\pi\circ T=S\circ\pi$. The relativised variational principle for T, S are given by

THEOREM 4.7 ([6]). *Fix a* $\nu\in\mathscr{P}(Y)$ *such that* $S^*\nu=\nu$. *Then*

$$\sup\{h_\mu^{\mathscr{C}}(T):\mu\in\mathscr{P}(X),\ T^*\mu=\mu,\ \pi^*\mu=\nu\}=\int h(T,\pi^{-1}(y))d\nu(y).$$

where $\mathscr{C}=\pi^{-1}(\mathscr{B}_Y)$.

$$h(T,K)=\lim_{\delta\to 0}\limsup_n\log s_n(T,K,\delta),\quad \textit{and}$$

$$s_n(T,K,\delta)=\max\{\#E: E\subset K,\ E\ \textit{is}\ (T,n,\delta)\textit{-separated}\}$$

for a compact $K\subset X$. *(*$E\subset K$ *is called* (T,n,δ)*-separated if* $d(T^ix,T^iy)>\delta$ *for some* $0\le i\le n-1$, *for all* x, $y\in E$, $x\neq y$.*)*

To apply this theorem we see the next lemma in connection with Lemma 4.5.

LEMMA 4.8. *Let* $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ *be a random dynamical system (resp.* $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$ *be an invertible random dynamical system). Then*

$$h_Q(P) = h_Q^{\mathfrak{N}\times\tilde{\mathscr{F}}}(\tau) \quad (resp.\ h_Q(P) = h_Q^{\mathfrak{N}\times\hat{\mathscr{F}}}(\tau)).$$

PROOF. From Lemma 4.5, it is easy to see $h_Q(P) \le h_Q^{\mathfrak{N}\times\tilde{\mathscr{F}}}(\tau)$. We prove the converse inequality. Since M and Φ are seprarable metric spaces, we can choose increasing sequences $\{\xi_n\}$ and $\{\zeta_n\}$ of finite measurable partitions of M and $\tilde{\Phi}$ respectively such that $\mathscr{F}(\vee_n \xi_n) = \mathscr{B}$ and $\mathscr{F}(\vee_n \zeta_n) = \tilde{\mathscr{F}}$. Then $\mathscr{F}(\vee_n(\xi_n \times \zeta_n)) = \mathscr{B} \times \tilde{\mathscr{F}}$ and by (4.9) and Lemma 4.5,

$$h_Q(P) \ge \limsup_n h_Q(P, \xi_n) = \lim_{n\to\infty} h_Q^{\mathfrak{N}\times\tilde{\mathscr{F}}}(\tau, \xi_n \times \zeta_n) = h_Q^{\mathfrak{N}\times\tilde{\mathscr{F}}}(\tau).$$

In the invertible case the proof is the same as above.

Applying Theorem 4.7 to a random dynamical system (resp. to an invertible random dynamical system), in view of Lemma 4.1 and Lemma 4.8, we immediately have the following theorem.

THEOREM 4.9. *Let* Φ *be compact with respect to the uniform topology and* $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ *be a topological random dynamical system (resp.* $\{(M, \mathscr{B}), (\hat{\Phi}, \hat{\mathscr{F}}, P), Q\}$ *be an invertible topological random dynamical system). Then*

$$h_{top}(P) = \sup\{h_Q(P) \colon Q \in \mathscr{I}_P(M \times \tilde{\Phi})\} \text{ (resp.}$$

$$h_{top}(P) = \sup\{h_Q(P) \colon Q \in \mathscr{I}_P(M \times \hat{\Phi})\}.)$$

Lastly in this section we briefly treat the random version of the Shannon-McMillan-Breiman theorem for an invertible random dynamical system.

THEOREM 4.10. *Let* $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P)\}$ *be an invertible random dynamical system. Then for a finite partition* ξ *of* M,

$$I_{Q_{\tilde{\varphi}}}(\xi, x, \hat{\varphi}) = \lim_{n\to\infty} I_{Q_{\hat{\varphi}}}(\xi_n(\tilde{\varphi}))(x)/n$$

exists for Q*-a.e.*$(x, \tilde{\varphi})$ *and*

$$\int I_{Q_{\hat{\varphi}}}(\xi, x, \hat{\varphi}) dQ(x, \varphi) = h_Q(P, \xi),$$

PROOF. Since proof is analogous to that of the deterministic case, we show a rough sketch of it. First note that the following equality holds:

$$I_{Q_{\hat{\varphi}}}(\xi_n(\hat{\varphi}))(x) = \textstyle\sum_{i=0}^{n-1} f_{n-i} \circ \tau^i(x, \hat{\varphi}) \tag{4.12}$$

where

$$f_j(x, \hat{\varphi}) = \begin{cases} I_{Q_{\hat{\varphi}}}(\xi \mid \vee_{k=0}^{j-1} {}^k\hat{\varphi}^{-1}\xi)(x), & j \ge 2, \\ I_{Q_{\hat{\varphi}}}(\xi), & j = 1. \end{cases}$$

Indeed, we have

$$\begin{aligned} I_{Q_{\hat\varphi}}(\xi_n(\tilde\varphi))(x) &= I_{Q_{\hat\varphi}}(\xi \vee {}^1\hat\varphi^{-1}\xi \vee \cdots \vee {}^{n-1}\hat\varphi^{-1}\xi)(x) \\ &= I_{Q_{\hat\varphi}}(\xi | \vee_{i=1}^{n-1} {}^i\hat\varphi^{-1}\xi) + I_{Q_{\hat\varphi}}(\vee_{i=1}^{n-1} {}^i\hat\varphi^{-1}\xi)(x) \\ &= f_n(x, \hat\varphi) + I_{Q_{\hat\varphi}}(\varphi_1^{-1}(\vee_{i=0}^{n-2} {}^i(\sigma\hat\varphi)^{-1}\xi))(x) \\ &= f_n(x, \hat\varphi) + I_{Q_{\sigma\hat\varphi}}(\vee_{i=0}^{n-2} {}^i(\sigma\hat\varphi)^{-1}\xi)(\varphi_1 x) \\ &= f_n(x, \hat\varphi) + I_{Q_{\sigma\hat\varphi}}(\xi_{n-1}(\sigma\hat\varphi))(\varphi_1 x), \end{aligned}$$

where we used the τ-invariance condition (2.5) of Q. Using this equation repeatedly we obtain (4.12).

On the other hand by Doob's theorem,

$$f(x, \hat\varphi) = \lim_{n\to\infty} f_n(x, \hat\varphi) \tag{4.13}$$

exists both for $Q_{\hat\varphi}$-a.e. and in $L^1(M, Q_{\hat\varphi})$ for P-a.e.$\hat\varphi$. (Note that $f_n \leq \log \# \xi$.) But

$$\begin{aligned} \int f(x, \hat\varphi) dQ(x, \hat\varphi) &= \int\int f(x, \hat\varphi) dQ_{\hat\varphi}(x)\, dP(\hat\varphi) \\ &= \int \lim_{n\to\infty} \int f_n(x, \hat\varphi)\, dQ_{\hat\varphi}(\hat\varphi)\, dP(\hat\varphi) \\ &= \int \lim_{n\to\infty} H_{Q_{\hat\varphi}}(\xi | \vee_{i=1}^{n} {}^i\hat\varphi^{-1}\xi)\, dP(\hat\varphi) \\ &= \int H_{Q_{\hat\varphi}}(\xi | \vee_{n=1}^{\infty} {}^n\hat\varphi^{-1}\xi)\, dP(\hat\varphi) \\ &= h_Q(P, \xi), \end{aligned}$$

where we used Proposition 4.4 in the last eqality. Applying Birkhoff's ergodic theorem, we know that

$$\hat f(x, \hat\varphi) = \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} f \circ \tau^i(x, \hat\varphi)$$

exists for Q-a.e.$(x, \hat\varphi)$ and

$$\int \hat f dQ = \int f dQ = h_Q(P, \xi).$$

Then follwing the arguments in the deterministic case we can show the statements in the theorem is true with $I_{Q_{\hat\varphi}}(\xi, x, \hat\varphi) = \hat f$.

§5. Unique maximal measure of a random Bernoulli shift

Let $S=\{1,\cdots,s\}(s\in\mathbf{N})$, $M=S^{\mathbf{Z}}$ and ψ be the shift transformation on $M:(\psi\hat{x})_n=x_{n+1}$, $n\in\mathbf{Z}$ for $\hat{x}=(x_n)_{n\in\mathbf{Z}}\in M$. It is well known that each $\psi^i(i\in\mathbf{N})$ has the unique maximal measure $\mu=\{1/s,\cdots,1/s\}^{\mathbf{Z}}$(see e.g.[7]). This maximal measure is common throughout $i\in\mathbf{N}$. (The maximal measure means the measure with espect to which its metrical entropy coincides with the topological entropy.) We are interested in what occurrs when the shifts operated are randomly chosen. The following theorem is a slight generalization of the unique maximality in the deterministic shift.

THEOREM 5.1. *Let M, ψ and μ be the same as above. Suppose that $\Phi=\{\psi,\psi^2\}$ and P is an ergodic σ-invariant probability measure on Φ. Then*

$$h_Q(P)\le h_{top}(P)\qquad \text{for all}\quad Q\in\mathscr{I}_P(M\times\hat{\Phi}). \tag{5.1}$$

The equality in (5.1) *holds if and only if $Q=\mu\times P$.*

PROOF. Though inequality in (5.1) is deduced from Theorem 4.9, here we try to prove (5.1) derectly. The following notations are used.

$$_i[x_1\cdots x_j]_j=\{\hat{y}=(y_n)_{n\in\mathbf{N}}\in M: y_k=x_k,\ i\le k\le j\}$$

$$\mathscr{C}^{i,j}=\{_i[x_i\cdots x_j]_j: x_i,\cdots,x_j\in S\}$$

for $i,j\in\mathbf{Z}$ with $i\le j$ and

$$(\psi^i)=\{\hat{\varphi}\in\hat{\Phi}: \varphi_1=\psi^i\},\ i=1,2.$$

Let us define the partition $\alpha=\{_0[x_0x_1]_1: x_0,x_1\in S\}$ of M. Then we have

$$I_\mu(\alpha|\vee_{i=1}^{\infty}{}^i\hat{\varphi}^{-1}\alpha)=\begin{cases}\text{lod } s & \text{if}\quad \hat{\varphi}\in(\psi)\\ 2\log s & \text{if}\quad \hat{\varphi}\in(\psi^2),\end{cases} \tag{5.2}$$

and so

$$h_{\mu\times P}(P)=h_{\mu\times P}(P,\alpha)=P((\psi))\log s+2P((\psi^2))\log s, \tag{5.3}$$

using Proposition 4.4 and Lemma 4.6. We can easily check that $h_{top}(P)$ coincides with the right hand side of (5.3) taking the sequence of open coverings $\beta_n=\{C: C\in\mathscr{C}^{-n,n}\}$.

Suppose that $Q\in\mathscr{I}_P(M\times\hat{\Phi})$. Then

$$H_{Q_{\hat{\varphi}}}(\alpha|\vee_{i=1}^{\infty}{}^i\hat{\varphi}^{-1}\alpha)\le H_{Q_{\hat{\varphi}}}(\alpha|\vee_{i=1}^{n-1}{}^i\hat{\varphi}^{-1}\alpha) \tag{5.4}$$

for all $n\in\mathbf{N}$, and for $\hat{x}=(x_k)_{k\in\mathbf{Z}}\in M$,

$$(5.5)\qquad I_{Q_{\hat\varphi}}(\alpha|\vee_{i=1}^{n}{}^{i}\hat\varphi^{-1}\alpha)(\hat x)$$
$$=\begin{cases} -\log\dfrac{Q_{\hat\varphi}({}_0[x_0\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)})}{Q_{\hat\varphi}({}_1[x_1\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)})} & \text{if } \hat\varphi\in(\psi)\\[2ex] -\log\dfrac{Q_{\hat\varphi}({}_0[x_0\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)})}{Q_{\hat\varphi}({}_2[x_2\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)})} & \text{if } \hat\varphi\in(\psi^2).\end{cases}$$

where $N(\hat\varphi, n) = 1 + \#\{1\le i\le n: \varphi_i=\psi\} + 2\#\{1\le i\le n: \varphi_i=\psi^2\}$. Set for $\hat x=(x_k)_{k\in\mathbf{Z}}$,

$$a_{\hat x,\tilde\varphi,n} = Q_{\hat\varphi}({}_0[x_0\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)}),$$
$$b_{\hat x,\tilde\varphi,n} = Q_{\hat\varphi}({}_1[x_1\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)}),$$
$$c_{\hat x,\tilde\varphi,n} = Q_{\hat\varphi}({}_2[x_2\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)}).$$

Then from Proposition 4.4, Lemma 4.6, (5.2)–(5.5), we have

$$\begin{aligned} h_{\mu\times P}(P)-h_Q(P) &= h_{\mu\times P}(P,\alpha)-h_Q(P,\alpha)\\ &\ge \int_{(\psi)}\int\left(-\log\frac{b_{\hat x,\hat\varphi,n}}{s\,a_{\hat x,\hat\varphi,n}}\right)dQ_{\hat\varphi}(x)\,dP(\hat\varphi)\\ &\quad+\int_{(\psi^2)}\int\left(-\log\frac{c_{\hat x,\hat\varphi,n}}{s^2\,a_{\hat x,\hat\varphi,n}}\right)dQ_{\hat\varphi}(x)\,dP(\hat\varphi).\end{aligned}$$

Put $C_{\hat\varphi,n}\equiv\{\hat x\in C^{0,N(\hat\varphi,n)}: a_{\hat x,\hat\varphi,n}>0\}$ and take a maximal subset $E_{\hat\varphi,n}$ of $C_{\hat\varphi,n}$ such that for all $\hat x=(x_k)_{k\in\mathbf{N}}$, $\hat y=(y_k)_{k\in\mathbf{Z}}\in E_{\hat\varphi,n}$, $\hat x\ne\hat y$ satisfy ${}_0[x_0\cdots x_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)}\ne{}_0[y_0\cdots y_{N(\hat\varphi,n)}]_{N(\hat\varphi,n)}$. Since $-\log y\ge 1-y$, $y>0$ with the equality only when $y=1$, we have

$$\begin{aligned} &h_{\mu\times P}(P)-h_Q(P)\\ &\ge\int_{(\psi)}\left\{\textstyle\sum_{\hat x\in E_{\hat\varphi,n}}\left(1-\dfrac{b_{\hat x,\hat\varphi,n}}{s\,a_{\hat x,\hat\varphi,n}}\right)a_{\hat x,\hat\varphi,n}\right\}dP(\hat\varphi)\\ &\quad+\int_{(\psi^2)}\left\{\textstyle\sum_{\hat x\in E_{\hat\varphi,n}}\left(1-\dfrac{c_{\hat x,\hat\varphi,n}}{s\,a_{\hat x,\hat\varphi,n}}\right)a_{\hat x,\hat\varphi,n}\right\}dP(\hat\varphi)\\ &=1-s^{-1}\int_{(\psi)}\left(\textstyle\sum_{\hat x\in E_{\hat\varphi,n}}b_{\hat x,\varphi,n}\right)dP(\hat\varphi)\\ &\quad-s^{-2}\int_{(\psi^2)}\left(\textstyle\sum_{\hat x\in E_{\hat\varphi,n}}c_{\hat x,\hat\varphi,n}\right)dP(\hat\varphi)\\ &\ge 0,\end{aligned}$$

where we used $\sum_{\hat{x}\in E_{\hat{\phi},n}} b_{\hat{x},\hat{\phi},n} \le s$ and $\sum_{\hat{x}\in E_{\hat{\phi},n}} c_{\hat{x},\hat{\phi},n} \le s^2$. Thus we obtain (5.1).

Now we prove that the equation in (5.1) implies $Q = \mu \times P$. In order that the inequalities used in the above estimate become equalities, the following conditions are necessary:

$$\frac{b_{\hat{x},\hat{\phi},n}}{s\, a_{\hat{x},\hat{\phi},n}} = 1 \qquad \text{for all} \quad \hat{x}\in C_{\hat{\phi},n},\ P\text{-a.e. } \hat{\phi}\in(\psi), \quad \text{and}$$

$$\frac{c_{\hat{x},\hat{\phi},n}}{s^2\, a_{\hat{x},\hat{\phi},n}} = 1 \qquad \text{for all} \quad \hat{x}\in C_{\hat{\phi},n},\ P\text{-a.e. } \hat{\phi}\in(\psi^2),$$

for all $n\in\mathbf{N}$. We may replace a.e. $\hat{\phi}$ by all $\hat{\phi}$ in the above condition with an elimination of a σ-invariant null set N. Putting $a_{\hat{x},\hat{\phi},n}$ $b_{\hat{x},\hat{\phi},n}$ and $c_{\hat{x},\hat{\phi},n}$ back into measures of cylinders, we have for all $n\in\mathbf{N}$,

(5.6) $$Q_{\hat{\phi}}({}_1[x_1 \cdots x_{N(\hat{\phi},n)}]_{N(\hat{\phi},n)}) = s\, Q_{\hat{\phi}}({}_0[x_0 \cdots x_{N(\hat{\phi},n)}]_{N(\hat{\phi},n)})$$

for all $\hat{x}(=(x_k)_{k\in\mathbf{Z}})\in C_{\hat{\phi},n}$, and $\hat{\phi}\in(\psi)$, and

(5.7) $$Q_{\hat{\phi}}({}_2[x_1 \cdots x_{N(\hat{\phi},n)}]_{N(\hat{\phi},n)}) = s^2\, Q_{\hat{\phi}}({}_0[x_0 \cdots x_{N(\hat{\phi},n)}]_{N(\hat{\phi},n)})$$

for all $\hat{x}\in C_{\hat{\phi},n}$, and $\hat{\phi}\in(\psi^2)$.

We will show the following equality for all $n\in\mathbf{N}$ by induction.

(5.8) $$Q_{\hat{\phi}}({}_0[x_0 \cdots x_{N(\hat{\phi},n)}]_{N(\hat{\phi},n)}) = s^{-\{N(\hat{\phi},n)+1\}}$$

for all $x_0,\cdots,x_{N(\hat{\phi},n)}\in S$, $\hat{\phi}\in\hat{\Phi}$.

First we consider the case when $n=1$. Then from (5.6) and (5.7),

(5.9) $Q_{\hat{\phi}}({}_1[x_1x_2]_2) = s\, Q_{\hat{\phi}}({}_0[x_0x_1x_2]_2)$ for all $\hat{x}\in C_{\hat{\phi},1}$, $\hat{\phi}\in(\psi)$, and

(5.10) $Q_{\hat{\phi}}({}_2[x_2x_3]_3) = s^2\, Q_{\hat{\phi}}({}_0[x_0x_1x_2x_3]_3)$ for all $\hat{x}\in C_{\hat{\phi},1}$, $\hat{\phi}\in(\psi^2)$.

Summing over x_1, x_2 in (5.9) and over x_2, x_3 in (5.10), we have

$$Q_{\hat{\phi}}({}_0[x_0]_0) \le s^{-1} \quad \text{for all} \quad x_0\in S \text{ s.t. } Q_{\hat{\phi}}({}_0[x_0]_0)>0,\ \hat{\phi}\in(\psi), \text{ and}$$

$$Q_{\hat{\phi}}({}_0[x_0x_1]_1) \le s^{-2} \quad \text{for all} \quad x_0,\ x_1\in S \text{ s.t. } Q_{\hat{\phi}}({}_0[x_0x_1]_1)>0,\ \hat{\phi}\in(\psi^2).$$

Since

$$\textstyle\sum_{x_0\in S} Q_{\hat{\phi}}({}_0[x_0]_0) = 1 \quad \text{and} \quad \sum_{x_0,x_1\in S} Q_{\hat{\phi}}({}_0[x_0x_1]_1) = 1,$$

we have from above

(5.11) $$Q_{\hat{\phi}}({}_0[x_0]_0) = s^{-1} \qquad \text{for all} \quad x_0\in S,\ \hat{\phi}\in(\psi), \text{ and}$$

(5.12) $$Q_{\hat{\phi}}({}_0[x_0x_1]_1) = s^{-2} \qquad \text{for all} \quad x_0,\ x_1\in S,\ \hat{\phi}\in(\psi^2).$$

Next summing over x_2 in (5.9),

$$Q_{\hat{\varphi}}({}_0[x_0x_1]_1) \le s^{-1} Q_{\hat{\varphi}}({}_1[x_1]_1)$$

for all $x_0, x_1 \in S$ s.t. $Q_{\hat{\varphi}}({}_0[x_0x_1]_1) > 0$, $\hat{\varphi} \in (\psi)$. But from (5.11) and (5.12), if $\hat{\varphi} \in (\psi)$,

$$Q_{\hat{\varphi}}({}_1[x_1]_1) = \varphi_1^* Q_{\hat{\varphi}}({}_0[x_1]_0) = Q_{\sigma\hat{\varphi}}({}_0[x_1]_0) = s^{-1},$$

where we used the codition (2.5). Therefore we have

$$Q_{\hat{\varphi}}({}_0[x_0x_1]_1) \le s^{-2}$$

for above x_0, x_1 and $\hat{\varphi} \in (\psi)$, which by the same arguments as above together with (5.12) yields

$$Q_{\hat{\varphi}}({}_0[x_0x_1]_1) = s^{-2} \qquad \text{for all} \quad x_0,\ x_1 \in S,\ \hat{\varphi} \in \hat{\Phi}.$$

Then returning back to (5.9) and (5.10), we have

$$Q_{\hat{\varphi}}({}_0[x_0x_1x_2]_2) = s^{-1}Q_{\hat{\varphi}}({}_1[x_1x_2]_2) = s^{-1}Q_{\sigma\hat{\varphi}}({}_0[x_1x_2]_1) = s^{-3}$$

for all $\hat{x} \in C_{\hat{\varphi},1}$, $\hat{\varphi} \in (\psi)$, and

$$Q_{\hat{\varphi}}({}_0[x_0x_1x_2x_3]_3) = s^{-2}Q_{\hat{\varphi}}({}_2[x_2x_3]_3) = s^{-2}Q_{\sigma\hat{\varphi}}({}_0[x_2x_3]_1) = s^{-4}$$

for all $\hat{x} \in C_{\hat{\varphi},1}$, $\hat{\varphi} \in (\psi^2)$. Again by the similar arguments and combining two cases, we obtain (5.8) when $n = 1$.

Assume that (5.8) holds for $n \in \mathbf{N}$. Since $N(\hat{\varphi}, n+1) = N(\hat{\varphi}, n+1) = N(\sigma\hat{\varphi}, n) + i$ if $\hat{\varphi} \in (\psi^i)$, $i = 1, 2$, from (5.6) and (5.7) we have

$$\begin{aligned} Q_{\hat{\varphi}}({}_0[x_0 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)}) &= s^{-1}Q_{\hat{\varphi}}({}_1[x_1 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)}) \\ &= s^{-1}Q_{\sigma\hat{\varphi}}({}_0[x_1 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)-1}) \\ &= s^{-1}Q_{\sigma\hat{\varphi}}({}_0[x_1 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\sigma\hat{\varphi},n)}) = s^{-\{2+N(\sigma\hat{\varphi},n)\}} \\ &= s^{-\{N(\hat{\varphi},n+1)+1\}} \qquad \text{for all} \quad \hat{x} \in C_{\hat{\varphi},n+1},\ \hat{\varphi} \in (\psi), \text{ and} \end{aligned}$$

$$\begin{aligned} Q_{\hat{\varphi}}({}_0[x_0 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)}) &= s^{-2}Q_{\hat{\varphi}}({}_2[x_2 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)}) \\ &= s^{-2}Q_{\sigma\hat{\varphi}}({}_0[x_2 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\hat{\varphi},n+1)-2}) \\ &= s^{-2}Q_{\sigma\hat{\varphi}}({}_0[x_2 \cdots x_{N(\hat{\varphi},n+1)}]_{N(\sigma\hat{\varphi},n)}) = s^{-\{3+N(\sigma\hat{\varphi},n)\}} \\ &= s^{-\{N(\hat{\varphi},n+1)+1\}} \qquad \text{for all} \quad \hat{x} \in C_{\hat{\varphi},n+1},\ \hat{\varphi} \in (\psi^2). \end{aligned}$$

From these we obtain (5.8) for $n+1$ similarly as above. Therefore by induction we obtain (5.8) for all $n \in \mathbf{N}$.

Since $N(\hat{\varphi}, n) \uparrow \infty$ as $n \uparrow \infty$, taking (2.6) into consideration, we consequently obtain $Q_{\hat{\varphi}}({}_m[x_m \cdots x_n]_n) = s^{n-m+1}$ for all $x_m, \cdots, x_n \in S$, $m, n \in \mathbf{Z}$ such that $m \leq n$ and $\hat{\varphi} \in \hat{\Phi}$. This implies $Q_{\hat{\varphi}} = \mu$ for all $\hat{\varphi} \in \hat{\Phi}$ and $Q = \mu \times P$. Thus the theorem is proved.

REMARK 5.2. We can obtain the analogous results to the above theorem for $\Phi = \{\psi, \cdots, \psi^k\}$ $(k \in \mathbf{N})$ by applying the same arguments.

REMARK 5.3. We consider the case when $M = S^{\mathbf{N}}$, $\tilde{\Phi} = \{\psi, \psi^2\}^{\mathbf{N}}$, ($\psi: M \to M$ is the shift transformation) and P is σ-invariant and erodic. Suppose that $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$. In order that $h_Q(P) = h_{top}(P)$, we obtain the following condition by the same arguments as above and Proposition 4.3

$$Q_{\tilde{\varphi}}({}_1[x_1 \cdots x_{N(\tilde{\varphi},n)}]) = s\, Q_{\tilde{\varphi}}({}_0[x_0 \cdots x_{N(\tilde{\varphi},n)}]_{N(\tilde{\varphi},n)})$$

for all $\tilde{x}\,(=(x_k)_{k \in \mathbf{N}}) \in C_{\tilde{\varphi},n}$, $\tilde{\varphi} \in (\psi)$, and

$$Q_{\tilde{\varphi}}({}_2[x_2 \cdots x_{N(\tilde{\varphi},n)}]) = s^2\, Q_{\tilde{\varphi}}({}_0[x_0 \cdots x_{N(\tilde{\varphi},n)}]_{N(\tilde{\varphi},n)})$$

for all $\tilde{x} \in C_{\tilde{\varphi},n}$, $\tilde{\varphi} \in (\psi^2)$, for all $n \in \mathbf{N}$. In the above, notations are the same as in the proof of Theorem 5.1. Including the case $\tilde{x} \notin C_{\tilde{\varphi},n}$, we consequently have

$$Q_{\tilde{\varphi}}({}_1[x_1 \cdots x_{N(\tilde{\varphi},n)}]) = s\, Q_{\tilde{\varphi}}({}_0[x_0 \cdots x_{N(\tilde{\varphi},n)}]_{N(\tilde{\varphi},n)}) \tag{5.13}$$

for all $x_0, \cdots, x_{N(\tilde{\varphi},n)} \in S$, $\tilde{\varphi} \in (\psi)$, and

$$Q_{\tilde{\varphi}}({}_2[x_2 \cdots x_{N(\tilde{\varphi},n)}]) = s^2\, Q_{\tilde{\varphi}}({}_0[x_0 \cdots x_{N(\tilde{\varphi},n)}]_{N(\tilde{\varphi},n)}) \tag{5.14}$$

for all $x_0, \cdots, x_{N(\tilde{\varphi},n)} \in S$, $\tilde{\varphi} \in (\psi^2)$. Although we have not yet obtained the unique maximality in this system of random (non-invertible) shift, (5.13) and (5.14) will be a criterion for a certain measure to be maximal. (See Fact 6.4 in §6.)

§6. Application to Tsujii random dynamical systems

Let $(M, \mathscr{B})$ amd $(\Phi, \mathscr{F})$ be the same as in §2. We fix a $\rho \in \mathscr{P}(\Phi, \mathscr{F})$. Consider a $\mu \in \mathscr{P}(M)$, a sub σ-algebra $\mathscr{B}_0$ of $\mathscr{B}$ and a measurable function $\gamma: M \times \Phi \to \mathbf{R}_+$ which satisfy the following conditions

(6.1) $\varphi^* \mu = \mu$ for all $\varphi \in \Phi$,

(6.2) $\varphi^{-1}\mathscr{B}$ is independent of $\mathscr{B}_0$ with respect to μ for all $\varphi \in \Phi$,

(6.3) $\gamma(\cdot, \varphi): M \to \mathbf{R}_+$ is $\mathscr{B}_0$-measurable for all $\varphi \in \Phi$,

(6.4) $\int \gamma(x, \varphi)\, d\rho(\varphi) = 1$ for all $x \in M$.

Define $Q \in \mathscr{P}(M \times \tilde{\Phi})$ by

$$(6.5)\qquad Q(B \times F_1 \times \cdots \times F_n \times \Phi^{n+1,\infty}) = \int 1_B(x) \prod_{i=1}^n 1_{F_i}(\varphi_i)\gamma({}^{i-1}\varphi x,\ \varphi_i)\, d\rho(\varphi_1) \cdots d\rho(\varphi_n) d\mu(x)$$

for all $B \in \mathscr{B}$, $F_1, \cdots, F_n \in \mathscr{F}$ and all $n \in \mathbf{N}$, where we set ${}^i\varphi = \varphi_1 \circ \cdots \circ \varphi_2$ if $i \geq 1$ and ${}^0\varphi = id$ for $(\varphi_1, \cdots, \varphi_n)$. Put

$$(6.6)\qquad P = \pi_{\tilde{\Phi}}^* Q.$$

Then $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$ ([10]). We call the metrical random dynamical system $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ constructed above a Tsujii random dynamical system.

In the following we fix a Tsujii random dynamical system $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ constructed from μ, ρ, $\mathscr{B}_0$ and γ.

FACT 6.1. *$P = \rho_\gamma^{\mathbf{N}}$ for $\rho_\gamma \in \mathscr{P}(\Phi)$ defined by*

$$d\rho_\gamma(\varphi) = \bar{\gamma}(\varphi)\, d\rho(\varphi),\ \bar{\gamma}(\varphi) = \int \gamma(x, \varphi)\, d\mu(x).$$

PROOF. For $F_1, \cdots, F_n \in \mathscr{F}$, using (6.1)–(6.6), we have

$$P(F_1 \times \cdots \times F_n \times \Phi^{n+1,\infty})$$
$$= \int\int 1_{F_1}(\varphi_1)\gamma(x, \varphi_1) \prod_{i=2}^n 1_{F_i}(\varphi_i)\gamma({}^{i-1}\varphi x,\ \varphi_i) d\mu(x) d\rho(\varphi_1) \cdots d\rho(\varphi_n)$$
$$= \int 1_{F_1}(\varphi_1)\left[\int \gamma(x, \varphi_1) d\mu(x) \int \prod_{i=2}^n 1_{F_i}(\varphi_i)\gamma({}^{i-1}\varphi x,\ \varphi_i) d\mu(x)\right] d\rho(\varphi_1) \cdots d\rho(\varphi_n)$$

and

$$\int \prod_{i=2}^n \gamma({}^{i-1}\varphi x,\ \varphi_i) d\mu(x) = \int \prod_{i=2}^n \gamma({}^{i-1}_{\ \ 2}\varphi x,\ \varphi_i) d\mu(x)$$

where

$${}^{i-1}_{\ \ 2}\varphi = \begin{cases} id & \text{if } i = 2 \\ \varphi_{i-1} \circ \cdots \circ \varphi_2 & \text{if } i \geq 3, \end{cases} \qquad \text{for} \quad (\varphi_1, \cdots, \varphi_n).$$

Therefore we have

$$P(F_1 \times \cdots \times F_n \times \Phi^{n+1,\infty})$$
$$= \rho_\gamma(F_1) \int\int \prod_{i=2}^n 1_{F_i}(\varphi_i)\gamma({}^{i-1}_{\ \ 2}\varphi x,\ \varphi_i) d\mu(x) d\rho(\varphi_2) \cdots d\rho(\varphi_n)$$

Repeating this procedure, we obtain

$$P(F_1 \times \cdots \times F_n \times \Phi^{n+1,\infty}) = \rho_\gamma(F_1) \cdots \rho_\gamma(F_n)$$

which implies $P = \rho_\gamma^{\mathbf{N}}$ since $n \in \mathbf{N}$ and $F_1, \cdots, F_n \in \mathscr{F}$ are arbitrary.

A typical element of $\mathscr{I}_P(M \times \tilde{\Phi})$ is $\nu \times P$ with $\rho_\gamma^* \nu = \nu$, where $\rho_\gamma^*: \mathscr{P}(M) \to \mathscr{P}(M)$ is defined by

$$\rho_\gamma^* \lambda(B) = \int\int 1_B(\varphi x)\, d\rho_\gamma(\varphi)\, d\lambda(x) \qquad \text{for} \quad B \in \mathscr{B},\ \lambda \in \mathscr{P}(M).$$

These measures are often considered. But $Q \in \mathscr{I}_P(M \times \tilde{\Phi})$ defined by (6.5) is not equal to any of these measures except for the trivial case.

FACT 6.2. *Suppose that*

$$Q = \nu \times P,\ \rho_\gamma^* \nu = \nu. \tag{6.7}$$

Then $\nu = \mu$ *and*

$$\gamma(x, \varphi) = \bar{\gamma}(\varphi) \quad \mu \times \rho\text{-a.e.}(x, \varphi). \tag{6.8}$$

PROOF. If (6.7) holds, then $\pi_M^* Q = \nu$, where $\pi_M: M \times \tilde{\Phi} \to M$ is the natural projection. On the other hand, we have $\pi_M^* Q = \mu$. Therefore $\nu = \mu$. Let $\pi_1: M \times \tilde{\Phi} \to M \times \Phi^{1,1}$ be the natural projection. Then both $\pi_1^* Q$ and $\pi_1^*(\nu \times P)$ are absolutely continuous with respect to $\mu \times \rho$ and the Radon-Nikodym derivatives at (x, φ) are $\gamma(x, \varphi)$ and $\bar{\gamma}(\varphi)$ respectively. Hence if (6.7) holds, we obtain (6.8).

If (6.8) fails to hold, Q constructed by (6.5) is decomposed into the family of regular conditional probability measures $\{Q_{\tilde{\varphi}}: \tilde{\varphi} \in \tilde{\Phi}\}$, for which the mapping $\tilde{\varphi} \to Q_{\tilde{\varphi}}$ is not trivially measurable, namely, not constant. Therefore the invariant measure Q is one of examples whose decomposition $\{Q_{\tilde{\varphi}}: \tilde{\varphi} \in \tilde{\Phi}\}$ depends essentially on $\tilde{\varphi} \in \tilde{\Phi}$.

Next let us consider the ergodicity of this system $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$. The tools used here are the results obtained in §3.

FACT 6.3. *Let* $\{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$ *be a Tsujii random dynamical system constructed as above. Then* Q *is Markov in the sense of Definition* 3.1.

PROOF. It suffices to note that

$$Q^{x,\varphi_1,\dots,\varphi_n}(F_{n+1} \times \cdots \times F_m \times \Phi^{m+1,\infty})$$
$$= \int \cdots \int \prod_{i=n+1}^m 1_{F_i}(\varphi_i) \gamma({}^{i-1}\varphi x, \varphi_i) d\rho(\varphi_{n+1}) \cdots d\rho(\varphi_m)$$

$$= Q^{n\tilde{\varphi}x}(\theta_n^{-1}(F_{n+1} \times \cdots \times F_m \times \Phi^{m+1,\infty}))$$

for all $F_{n+1}, \cdots, F_m \in \mathscr{F}$ and $m, n \in \mathbf{N}$, $m > n$, Q-a.e. $(x, \tilde{\varphi})$.

From Fact 6.3 and Theorem 3.5, (τ, Q) is ergodic if and only if Q is M-ergodic and the M-ergodicity of Q is nothing but the ergodicity of the Markov operator

$$\mathscr{Q}f(x) = \int f(x, \varphi)\gamma(x, \varphi)\, d\rho(\varphi) \qquad \text{for } f \in L^1(M, \mu).$$

This coincides with the result obtained in Theorem 4 in [10].

Apart from general situations, we give a concrete example of Tsujii random dynamical system. We consider the following objects:

$S = \{1, \cdots, s\}$, $M = S^{\mathbf{N}}$,

$\mathscr{B}$: the σ-algebra of M generated by cylinders,

$\Phi = \{\psi, \psi^2\}$, $\psi: M \to M$ is the shift,

$\mathscr{F} = \{\phi, \{\psi\}, \{\psi\}, \{\psi, \psi^2\}\}$,

$\rho \in \mathscr{P}(\Phi, \mathscr{F})$ such that $\rho_1 = \rho(\{\psi^i\}) > 0$ for $i = 1, 2$,

$\mu = \{\underbrace{1/s, \cdots, 1/s}_{s}\}^{\mathbf{N}} \in \mathscr{P}(M, \mathscr{B})$,

$\mathscr{B}_0$: the σ-algebra generated by $\{{}_0[x]_0 : x \in S\}$,

$\gamma: M \times \Phi \to R_+$ such that

$\gamma(\tilde{x}, \psi^j) = \gamma_{ij}$ if $\tilde{x} \in {}_0[i]_0$, $1 \le i \le s$, $j = 1, 2$,

$\gamma_{i1}\rho_1 + \gamma_{i2}\rho_2 = 1$ for $1 \le i \le s$,

(6.9) $\gamma_{ij} \neq \gamma_{i'j}$ for some $1 \le i \neq i' \le s$, $j = 1$ or 2.

It is easy to see that $M, \mathscr{B}, \cdots, \gamma$ above satisfy the conditions (6.1)-(6.4). Hence we can construct Q and P by (6.5) and (6.6), from which we obtain a Tsujii random dynamical system $\mathscr{S} = \{(M, \mathscr{B}), (\tilde{\Phi}, \tilde{\mathscr{F}}, P), Q\}$. Next let us consider the entropies of $\mathscr{S}$. It is clear that $\mu \times P \in \mathscr{S}_P(M \times \tilde{\Phi})$ is one of the maximal measure, that is,

$$\begin{aligned} h_{\mu\times P}(P) &= h_{top}(P) \\ &= (\rho_\gamma(\{\psi\}) + 2\rho_\gamma(\{\psi^2\}))\log s. \end{aligned}$$

(See Theorem 5.1, (5.3) and Fact 6.1.) On the other hand, in view of Fact 6.2 and (6.9), we have $Q \neq \mu \times P$. As for $h_Q(P)$, we obtain the following result.

FACT 6.4. $h_Q(P) < h_{top}(P)$.

PROOF. Suppose that $h_Q(P) = h_{top}(Q)$. Then, putting $n = 1$ in (5.13) and (5.14), we have

$$Q_{\tilde{\varphi}}({}_1[x_1x_2]_2) \geq s\, Q_{\tilde{\varphi}}({}_0[x_0x_1x_2]_2) \tag{6.10}$$

for all x_0, x_1, $x_2 \in S$ and for all $\tilde{\varphi} \in (\psi)$, and

$$Q_{\tilde{\varphi}}({}_2[x_2x_3]_3) \geq s^2 Q_{\tilde{\varphi}}({}_0[x_0x_1x_2x_3]_3) \tag{6.11}$$

for all x_0, x_1, x_2, $x_3 \in S$ and for all $\tilde{\varphi} \in (\psi^2)$. But from the definition (6.5) of Q, we have

$$\int_{(\psi)} Q_{\tilde{\varphi}}({}_0[x_0x_1x_2]_2)\, dP(\tilde{\varphi}) = Q({}_0[x_0x_1x_2]_2 \times (\psi))$$
$$= \rho_1 \int_{{}_0[x_0x_1x_2]_2} \gamma(\tilde{x}, \psi)\, d\mu(\tilde{x}) = \rho_1 \gamma_{x_0 1} s^{-3}$$

and

$$\int_{(\psi)} Q_{\tilde{\varphi}}({}_0[x_1x_2]_2)\, dP(\tilde{\varphi}) = Q({}_1[x_1x_2]_2 \times (\psi))$$
$$= \rho_1 \int_{{}_1[x_1x_2]_2} \gamma(\tilde{x}, \psi)\, d\mu(\tilde{x}) = \rho_1 s^{-3} \textstyle\sum_{k=1}^{s} \gamma_{k1}.$$

Therefore (6.10) implies

$$\textstyle\sum_{k=1}^{s} \gamma_{k1} \geq s\gamma_{i1} \qquad \text{for all} \quad i \in S,$$

from which we consequently have

$$\gamma_{i1} = s^{-1} \textstyle\sum_{k=1}^{s} \gamma_{k1} \qquad \text{for all } i \in S.$$

This contradicts (6.9) if $j = 1$ in (6.9). If j in (6.9) is equal to 2, we can similarly deduce a contradiction from (6.11). Therefore we obtain the desired inequality.

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