

Martin boundary of a harmonic space with adjoint structure and its applications

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Introduction

Consider the parabolic operator

$$Lu(x, t) = a(x) \frac{\partial u}{\partial t} - \Delta_x u + \langle \mathbf{b}(x, t), \nabla_x u \rangle + c(x, t)u$$

and its adjoint operator

$$L^*u(x, t) = -a(x) \frac{\partial u}{\partial t} - \Delta_x u - \langle \mathbf{b}(x, t), \nabla_x u \rangle + c^*(x, t)u$$

with sufficiently smooth coefficients $a > 0$, \mathbf{b} (\mathbf{R}^n -valued), c and $c^* = c - \nabla_x \mathbf{b}$ on a domain D in $\mathbf{R}^n \times \mathbf{R}$. If we write $\hat{L} = (L + L^*)/2$, then, noting that $c = L1$ and $c^* = L^*1$, we have

$$\Delta_x u = -\hat{L}u + u\hat{L}1.$$

Therefore, if the “lateral” boundary $\partial_s D$ of D is sufficiently regular, then for $f, g \in C^2(\bar{D})$ such that g vanishes on $\partial_s D$, Green’s formula implies

$$(0.1) \quad \int_D \langle \nabla_x f, \nabla_x g \rangle dxdt + \int_D fg \hat{L}1 dxdt = \int_D g \hat{L}f dxdt,$$

provided that all the integrals exist.

The purpose of the present paper is to establish a formula corresponding to (0.1) on a harmonic space (X, \mathcal{H}) with an adjoint structure \mathcal{H}^* , as an application of the theory of Martin boundary of X with respect to the structures \mathcal{H} and \mathcal{H}^* .

In §2—§6, we develop a theory of Martin boundary of such a harmonic space (X, \mathcal{H}) . Theories of Martin boundary of general harmonic spaces have been discussed to some extent by M. Sieveking [8], K. Janssen [3] and C. Constantinescu–A. Cornea [1; Chapter 11]; and some results in §2—§6 of the present paper can be obtained from these general theories. However, in order to obtain some properties which we need in establishing the above mentioned formula, we rather follow the classical approaches by Martin–Brelot–Naim and

their “parabolic” counterparts discussed in J. L. Doob [2; Chap. XIX], and we give details of the theory for reader’s convenience.

We then introduce (in §7) a class \mathcal{F} of functions on X for which we can naturally define boundary values on the Martin boundary. In §7, we show that “energy finite” bounded functions belong to \mathcal{F} . Finally, in §8, we establish formulas of type (0.1) for an energy finite bounded function f and an energy finite function g of potential type. There, we make use of the boundary values of f on the Martin boundary of X and their relations with the minimal fine limits of f with respect to the adjoint structure \mathcal{H}^* (cf. [5; §5.6 and §6.7] for such relations).

§1. Preliminaries

Let X be a connected locally compact space with countable base and consider a pair $(\mathcal{H}, \mathcal{H}^*)$ of mutually adjoint harmonic sheaves on X as defined in [7]. By definition, (X, \mathcal{H}) and (X, \mathcal{H}^*) are P-harmonic spaces and there exists a Green function $G(x, y)$ satisfying the following conditions:

- (G0) $G(x, y)$ is lower semicontinuous on $X \times X$ and continuous off the diagonal;
- (G1) For each $y \in X$, $G(\cdot, y)$ is an \mathcal{H} -potential and is \mathcal{H} -harmonic on $X \setminus \{y\}$;
- (G*1) For each $x \in X$, $G(x, \cdot)$ is an \mathcal{H}^* -potential and is \mathcal{H}^* -harmonic on $X \setminus \{x\}$;
- (G2) Any continuous \mathcal{H} -potential p is uniquely expressed as $p = G\mu$ with a nonnegative measure μ on X , where $G\mu(x) = \int G(x, y) d\mu(y)$;
- (G*2) Any continuous \mathcal{H}^* -potential q is uniquely expressed as $q = G^*\nu$ with a nonnegative measure ν on X , where $G^*\nu(y) = \int G(x, y) d\nu(x)$.

We further assume that the constant function 1 is \mathcal{H} - and \mathcal{H}^* -superharmonic.

We denote by \mathcal{S}_+ the set of all nonnegative \mathcal{H} -superharmonic functions on X , \mathcal{P} the set of \mathcal{H} -potentials and \mathcal{P}_C the set of continuous \mathcal{H} -potentials. The corresponding sets with respect to \mathcal{H}^* are denoted by \mathcal{S}_+^* , \mathcal{P}^* and \mathcal{P}_C^* , respectively.

We recall ([7]) that, associated with $G(x, y)$, there exist measure representations $\sigma: \mathcal{R} \rightarrow \mathcal{M}$ and $\sigma^*: \mathcal{R}^* \rightarrow \mathcal{M}$, where \mathcal{R} (resp. \mathcal{R}^*) is the sheaf of functions which are locally expressible as the difference of two continuous \mathcal{H} - (resp. \mathcal{H}^* -) superharmonic functions and \mathcal{M} is the sheaf of signed measures on X . By definition, $\sigma(G\nu + u) = \nu$ if $u \in \mathcal{H}(X)$ and $G|_{\nu} \in \mathcal{P}_C$. Also, by assumption, $\sigma(1) \geq 0$ and $\sigma^*(1) \geq 0$.

The reduction operator with respect to \mathcal{H} (resp. \mathcal{H}^*) will be denoted by R

(resp. R^*) (cf. [7; p.3] and [1; pp.39–40]). For an open set U in X and $v \in \mathcal{S}_+$ (resp. $w \in \mathcal{S}_+^*$), let

$$R_U v = R(\chi_U v) \quad (\text{resp. } R_U^* w = R^*(\chi_U w)),$$

where χ_U is the characteristic function of U . (In [1], $R_U v$ is denoted by R_v^U .) Then $R_U v \in \mathcal{S}_+$ (resp. $R_U^* w \in \mathcal{S}_+^*$), and $R_U v \in \mathcal{P}$ (resp. $R_U^* w \in \mathcal{P}^*$) if U is relatively compact.

LEMMA 1.1. *Let U be a relatively compact open set in X . If $v \in \mathcal{S}_+$ is locally bounded on X and is continuous on U , then there exists a unique nonnegative measure μ on X such that $R_U v = G\mu$ and $\text{Supp } \mu \subset \bar{U}$. If, in addition, $v|_U \in \mathcal{H}(U)$, then $\text{Supp } \mu \subset \partial U$.*

PROOF. (i) Uniqueness: Let $\mathcal{R}_0^* = \{f \in \mathcal{R}^*(X) \mid \text{Supp } f \text{ is compact}\}$. Note that $f = G^*(\sigma^*(f))$ for $f \in \mathcal{R}_0^*$. Suppose $R_U v = G\mu$. Then for any $f \in \mathcal{R}_0^*$,

$$\int f d\mu = \int G^*(\sigma^*(f)) d\mu = \int G\mu d\sigma^*(f) = \int R_U v d\sigma^*(f).$$

Since \mathcal{R}_0^* is dense in $\mathcal{C}_0(X)$ (= the space of continuous functions with compact support in X ; cf. [1; Theorem 2.3.1]), μ is determined by $R_U v$.

(ii) Existence: Let $\{U_n\}$ be an exhaustion of U and choose $\varphi_n \in \mathcal{C}_0(X)$ such that $0 \leq \varphi_n \leq 1$ on X , $\varphi_n = 1$ on U_n and $\text{Supp } \varphi_n \subset U$ ($n = 1, 2, \dots$). Put $v_n = R(\varphi_n v)$. Then $v_n \in \mathcal{P}_C$ and $R_U v = \lim_{n \rightarrow \infty} v_n$ (cf. [1; Theorem 4.2.3]). Let $\mu_n = \sigma(v_n)$. Then $\text{Supp } \mu_n \subset U$ (cf. [7; Lemma 1.1]). Choose $\psi \in \mathcal{C}_0(X)$ such that $\psi = 1$ on U . Then

$$\mu_n(X) = \mu_n(U) \leq \int R^* \psi d\mu_n = \int v_n d\sigma^*(R^* \psi) \leq \int v d\sigma^*(R^* \psi) < \infty,$$

since $\text{Supp } \sigma^*(R^* \psi)$ is compact. Hence, a subsequence $\{\mu_{n_j}\}$ vaguely converges to a nonnegative measure μ with $\text{Supp } \mu \subset \bar{U}$. By the lower semicontinuity of $G(x, \cdot)$, $G\mu \leq \lim_{n \rightarrow \infty} v_n = R_U v$. On the other hand, since $v_n = v$ on U_n , $\mu_n|_{U_n} = \sigma(v)|_{U_n}$ for all n , so that $\mu|_{U_n} = \mu_n|_{U_n}$. Together with the continuity of $G(x, \cdot)$ on $\bar{U} \setminus \{x\}$, it follows that $G\mu_{n_j} \rightarrow G\mu$ on U . Hence $G\mu = R_U v = v$ on U , which implies $G\mu \geq R_U v$ on X . Thus, $G\mu = R_U v$. The above arguments also show that $\mu|_U = 0$ if $v|_U \in \mathcal{H}(U)$.

LEMMA 1.2 ([1; Proposition 7.1.2 and Corollary 7.1.2]). *Let U be a relatively compact open set in X . Then, for each $x \in X$, there exists a nonnegative measure δ_x^U on \bar{U} such that*

$$(R_U v)(x) = \int_{\bar{U}} v d\delta_x^U$$

for all $v \in \mathcal{S}_+$.

LEMMA 1.3. *Let U be an open set in X . Then*

$$[R_U^* G(x, \cdot)](y) = [R_U G(\cdot, y)](x)$$

for any $x, y \in X$.

PROOF. Since $R_U v = \lim_{n \rightarrow \infty} R_{U \cap X_n} v$ for an exhaustion $\{X_n\}$ of X and for any $v \in \mathcal{S}_+$ ([1; Corollary 4.2.2]), we may assume that U is relatively compact. By the above lemma,

$$[R_U G(\cdot, y)](x) = \int G(z, y) d\delta_x^U(z) = (G^* \delta_x^U)(y)$$

for any $x, y \in X$. We see easily that $R_U G(\cdot, y) = G(\cdot, y)$ if $y \in U$ (cf. [6; Proposition 2.5]). Hence $G^* \delta_x^U = G(x, \cdot)$ on U , so that $G^* \delta_x^U \geq R_U^* G(x, \cdot)$ on X for any $x \in X$, namely,

$$[R_U G(\cdot, y)](x) \geq [R_U^* G(x, \cdot)](y)$$

for all $x, y \in X$. By symmetry, we obtain the converse inequality.

§2. Martin boundary

A nonnegative measure λ on X will be called a *standard \mathcal{H} -reference measure* if $\lambda(X) < \infty$, $G^* \lambda$ is bounded continuous on X and is positive everywhere. In view of [4; Lemma 3.6], we see that a standard \mathcal{H} -reference measure is a reference measure with respect to \mathcal{H} in the sense of [3], namely X is the smallest absorbent set (with respect to \mathcal{H}) containing $\text{Supp } \lambda$. We fix such a measure λ throughout this and the next four sections.

Let

$$\mathcal{H}_\lambda = \{u \in \mathcal{H}(X) \mid u \geq 0, \int u d\lambda < \infty\} \quad \text{and}$$

$$\mathcal{H}_{\lambda,1} = \{u \in \mathcal{H}_\lambda \mid \int u d\lambda \leq 1\}.$$

By Harnack's inequality [1; Proposition 6.1.5], $\mathcal{H}_{\lambda,1}$ is locally uniformly bounded on X , and by [1; Theorem 11.1.1], we see that it is compact with respect to the locally uniform convergence topology.

We define the λ -Martin kernel $K_\lambda(x, y)$ by

$$K_\lambda(x, y) = \frac{G(x, y)}{(G^* \lambda)(y)} \quad \text{for } x, y \in X.$$

LEMMA 2.1. For a compact set E in X , $\{K_\lambda(x, \cdot)\}_{x \in E}$ is uniformly bounded outside a neighborhood of E .

PROOF. Let V be a relatively compact neighborhood of E . Then there is $\alpha > 0$ such that $G(x, y) \leq \alpha(G^*\lambda)(y)$ for all $x \in E$ and $y \in \partial V$. By [6; Proposition 2.5], this inequality holds for all $x \in E$ and $y \in X \setminus V$, namely, $K_\lambda(x, \cdot) \leq \alpha$ on $X \setminus V$ for all $x \in E$.

There exists a (unique) compactification \bar{X}^λ of X such that every $K_\lambda(x, \cdot)$ has a continuous extension to \bar{X}^λ and $\{K_\lambda(x, \cdot)\}_{x \in X}$ separates points of $\partial^\lambda X = \bar{X}^\lambda \setminus X$. Then $K_\lambda(\cdot, \eta) \in \mathcal{H}_{\lambda, 1}$ for any $\eta \in \partial^\lambda X$. \bar{X}^λ is metrizable; in fact, for a countable dense set $\{x_j\}$ in X , $\{K_\lambda(x_j, \cdot)\}_j$ separates points of $\partial^\lambda X$.

LEMMA 2.2. Let $u \in \mathcal{H}_\lambda$ and U be an open set in X . Then there exists a nonnegative measure μ_U on \bar{X}^λ such that $\text{Supp } \mu_U \subset \partial^\lambda U$ (= the boundary of U in \bar{X}^λ), $\mu_U(\bar{X}^\lambda) = \int_X R_U u \, d\lambda$ and

$$R_U u = \int_{\bar{X}^\lambda} K_\lambda(\cdot, \eta) \, d\mu_U(\eta).$$

In particular, for each $u \in \mathcal{H}_\lambda$ there is a nonnegative measure μ on $\partial^\lambda X$ such that $\mu(\partial^\lambda X) = \int_X u \, d\lambda$ and

$$u = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) \, d\mu(\eta).$$

PROOF. Let $\{X_n\}$ be an exhaustion of X and set $U_n = U \cap X_n$. By Lemma 1.1, for each n , there is a nonnegative measure μ_n such that $\text{Supp } \mu_n \subset \partial U_n$ and $R_{U_n} u = G\mu_n$. Let $\nu_n = (G^*\lambda)\mu_n$. Since

$$(2.1) \quad \nu_n(X) = \int G^*\lambda \, d\mu_n = \int R_{U_n} u \, d\lambda \leq \int u \, d\lambda < \infty,$$

$\{\nu_n\}$ has a vaguely convergent subsequence as measures on \bar{X}^λ . Let μ_U be its limit measure and set $v = \int K_\lambda(\cdot, y) \, d\mu_U(y)$. Then, $\text{Supp } \mu_U \subset \partial^\lambda U$ and (2.1) implies that $\mu_U(\bar{X}^\lambda) = \int R_U u \, d\lambda$, since $R_{U_n} u \uparrow R_U u$. Also, since $R_{U_n} u = \int K_\lambda(\cdot, y) \, d\nu_n(y)$ and $K_\lambda(x, \cdot)$ is continuous on $\bar{X}^\lambda \setminus \{x\}$ and lower semicontinuous on \bar{X}^λ , we see that $v = R_U u = u$ on U and $v \leq R_U u$ on X . Since $v \in \mathcal{S}_+$, it follows that $v = R_U u$.

LEMMA 2.3. Let U be an open set in X .

- (i) For each $x \in X$, $\eta \mapsto [R_U K_\lambda(\cdot, \eta)](x)$ is lower semicontinuous on \bar{X}^λ .
- (ii) If U is relatively compact, then, for any $\eta \in \partial^\lambda X$, $R_U K_\lambda(\cdot, y) \rightarrow R_U K_\lambda(\cdot, \eta)$ uniformly on X as $y \rightarrow \eta$ ($y \in \bar{X}^\lambda$).

PROOF. By Lemma 2.1 and [1; Theorem 11.1.1], if U is relatively

compact, then $K_\lambda(\cdot, y) \rightarrow K_\lambda(\cdot, \eta)$ uniformly on \bar{U} as $y \rightarrow \eta$ ($\eta \in \partial^\lambda X$, $y \in \bar{X}^\lambda$). Hence we have (ii) of the lemma. Then (i) follows from the fact that $R_U K_\lambda(\cdot, \eta) = \lim_{n \rightarrow \infty} R_{U \cap X_n} K_\lambda(\cdot, \eta)$ for an exhaustion $\{X_n\}$ of X .

Let $\eta \in \partial^\lambda X$ and U be a relatively compact open set in X . By Lemma 1.1, there is a unique nonnegative measure ε_η^U such that $\text{Supp } \varepsilon_\eta^U \subset \partial U$ and

$$R_U K_\lambda(\cdot, \eta) = \int K_\lambda(\cdot, y) d\varepsilon_\eta^U(y).$$

Note that $\varepsilon_\eta^U(X) = \int R_U K_\lambda(\cdot, \eta) d\lambda \leq 1$.

LEMMA 2.4. *Let U be a relatively compact open set in X . Then $\eta \mapsto \int_{\partial U} f d\varepsilon_\eta^U$ is continuous on $\partial^\lambda X$ for each $f \in \mathcal{C}(\partial U)$.*

PROOF. If $f = (p/G^*\lambda)|_{\partial U}$ with $p \in \mathcal{P}_C^*$ such that $\text{Supp } \sigma^*(p)$ is compact, then $f = \int_X K_\lambda(x, \cdot) dv(x)$ on ∂U , where $v = \sigma^*(p)$, so that

$$\int_{\partial U} f d\varepsilon_\eta^U = \int_X \left\{ \int_{\partial U} K_\lambda(x, y) d\varepsilon_\eta^U(y) \right\} dv(x) = \int_X R_U K_\lambda(\cdot, \eta) dv.$$

Hence, in view of Lemma 2.3 (ii), the assertion of the lemma holds for such an f . Since $\varepsilon_\eta^U(\partial U) \leq 1$ for all $\eta \in \partial^\lambda X$, [1; Theorem 2.3.1] implies that the assertion of the lemma holds for any $f \in \mathcal{C}(\partial U)$.

By the above lemma, for any nonnegative measure μ on $\partial^\lambda X$ and a relatively compact open set U in X ,

$$(2.2) \quad \mu^U(f) = \int_{\partial^\lambda X} \left(\int_{\partial U} f d\varepsilon_\eta^U \right) d\mu(\eta) \quad \text{for } f \in \mathcal{C}(\partial U)$$

defines a nonnegative measure μ^U such that $\text{Supp } \mu^U \subset \partial U$ and $\mu^U(\partial U) \leq \mu(\partial^\lambda X)$.

LEMMA 2.5. *Let μ be a nonnegative measure on $\partial^\lambda X$ and let*

$$u = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\mu(\eta).$$

Then for any open set U in X ,

$$(2.3) \quad R_U u = \int_{\partial^\lambda X} R_U K_\lambda(\cdot, \eta) d\mu(\eta).$$

Furthermore, if U is relatively compact, then

$$(2.4) \quad R_U u = \int_{\partial U} K_\lambda(\cdot, y) d\mu^U(y),$$

where μ^U is the nonnegative measure defined above.

PROOF. First, let U be relatively compact. Since (2.2) holds for any nonnegative lower semicontinuous function, we have

$$\begin{aligned} \int_{\partial U} K_\lambda(x, y) d\mu^U(y) &= \int_{\partial^\lambda X} \left\{ \int_{\partial U} K_\lambda(x, y) d\varepsilon_\eta^U(y) \right\} d\mu(\eta) \\ &= \int_{\partial^\lambda X} [R_U K_\lambda(\cdot, \eta)](x) d\mu(\eta) \end{aligned}$$

for any $x \in X$. On the other hand, by Lemma 1.2,

$$\begin{aligned} \int_{\partial^\lambda X} [R_U K_\lambda(\cdot, \eta)](x) d\mu(\eta) &= \int_{\partial^\lambda X} \left\{ \int_{\bar{U}} K_\lambda(z, \eta) d\delta_x^U(z) \right\} d\mu(\eta) \\ &= \int_{\bar{U}} u(z) d\delta_x^U(z) = (R_U u)(x). \end{aligned}$$

Hence, (2.3) and (2.4) hold when U is relatively compact. To prove (2.3) for any open set U , it is enough to note that $R_U v = \lim_{n \rightarrow \infty} R_{U \cap X_n} v$ for $v \in \mathcal{S}_+$ and an exhaustion $\{X_n\}$ of X .

§3. Reduced functions for boundary sets

For a closed set $F \subset \partial^\lambda X$ let $\mathfrak{B}(F)$ be the set of open neighborhoods of F in \bar{X}^λ , and for $v \in \mathcal{S}_+$, let

$$R_F v = \inf \{ w \in \mathcal{S}_+ \mid w \geq v \text{ on } V \cap X \text{ for some } V \in \mathfrak{B}(F) \}.$$

By Perron's theorem ([1; Theorem 2.2.1]) we see that $R_F v \in \mathcal{H}(X)$. If $u \in \mathcal{H}_\lambda$, then $R_{\partial^\lambda X} u = u$, since X is an MP-set (cf. [1; Corollary 2.3.3]). Obviously, if $F_1 \subset F_2$ and $v_1 \leq v_2$, then $R_{F_1} v_1 \leq R_{F_2} v_2$. Note that if $V_n \in \mathfrak{B}(F)$, $n = 1, 2, \dots$, satisfy

$$(3.1) \quad \bar{V}_{n+1}^\lambda \subset V_n, \quad n = 1, 2, \dots, \quad \text{and} \quad \bigcap_{n=1}^\infty V_n = F,$$

then $R_F v = \lim_{n \rightarrow \infty} R_{V_n \cap X} v$. Thus, by [1; Theorem 4.2.1], we obtain

LEMMA 3.1. *If F is a closed set in $\partial^\lambda X$ and $v_1, v_2 \in \mathcal{S}_+$, then*

$$R_F(v_1 + v_2) = R_F v_1 + R_F v_2.$$

LEMMA 3.2. *If F is a closed set in $\partial^\lambda X$ and $v \in \mathcal{S}_+$, then*

$$R_F(R_F v) = R_F v.$$

PROOF. Obviously, $R_F(R_F v) \leq R_F v$. For any $V \in \mathfrak{B}(F)$,

$$R_F(R_{V \cap X} v) = R_F v,$$

since $R_{V \cap X} v = v$ on $V \cap X$. Since $R_{V \cap X} v - R_F v \in \mathcal{S}_+$, using Lemma 3.1, we have

$$\begin{aligned} R_F v &= R_F(R_{V \cap X} v) = R_F(R_{V \cap X} v - R_F v) + R_F(R_F v) \\ &\leq R_{V \cap X} v - R_F v + R_F(R_F v). \end{aligned}$$

Taking the infimum on V , we obtain $R_F v \leq R_F(R_F v)$.

COROLLARY 3.1. *If F_1, F_2 are closed sets in $\partial^\lambda X$ such that $F_1 \subset F_2$ and if $v \in \mathcal{S}_+$, then*

$$R_{F_1}(R_{F_2} v) = R_{F_2}(R_{F_1} v) = R_{F_1} v.$$

PROPOSITION 3.1. *For a closed set F in $\partial^\lambda X$ and $u \in \mathcal{H}_\lambda$, there exists a nonnegative measure μ on $\partial^\lambda X$ such that $\text{Supp } \mu \subset F$, $\mu(F) = \int_X R_F u d\lambda$ and*

$$R_F u = \int_F K_\lambda(\cdot, \eta) d\mu(\eta).$$

PROOF. Let $V_n \in \mathfrak{B}(F)$ satisfy (3.1). Applying Lemma 2.2 with $U = V_n \cap X$ and taking a vaguely convergent subsequence of the corresponding measures, we easily obtain the proposition.

COROLLARY 3.2. *Let $u \in \mathcal{H}_\lambda$ and $\eta \in \partial^\lambda X$. If $R_{(\eta)} u = u$, then $u = (\int u d\lambda) K_\lambda(\cdot, \eta)$. If in addition $u \neq 0$, then $\int K_\lambda(\cdot, \eta) d\lambda = 1$.*

PROOF. By the above proposition, $R_{(\eta)} u = c K_\lambda(\cdot, \eta)$ with $c = \int_X R_{(\eta)} u d\lambda$.

PROPOSITION 3.2. *Let μ be a nonnegative measure on $\partial^\lambda X$ and let*

$$u = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\mu(\eta).$$

Then, for any closed set F in $\partial^\lambda X$,

$$R_F u = \int_{\partial^\lambda X} R_F K_\lambda(\cdot, \eta) d\mu(\eta).$$

PROOF. Taking $V_n \in \mathfrak{B}(F)$ satisfying (3.1), applying (2.3) in Lemma 2.5 with $U = V_n \cap X$ and letting $n \rightarrow \infty$, we obtain the required result by Lebesgue's convergence theorem.

§4. Minimal boundary points

We say that $u \in \mathcal{H}(X)$, $u \geq 0$, is minimal if $u \neq 0$ and $0 \leq v \leq u$ with $v \in \mathcal{H}(X)$ implies $v = cu$ (c : constant).

LEMMA 4.1. *If $u \in \mathcal{H}_\lambda$ is minimal and F is a closed set in $\partial^\lambda X$, then either $R_F u = 0$ or $R_F u = u$.*

PROOF. Since $R_F u \in \mathcal{H}_\lambda$ and $0 \leq R_F u \leq u$, $R_F u = cu$. Using Lemma 3.2, we see that $c = 0$ or 1 .

LEMMA 4.2. *Let F be a closed set in $\partial^\lambda X$. If $u \in \mathcal{H}_\lambda$ is minimal and $R_{(\eta)} u = 0$ for all $\eta \in F$, then $R_F u = 0$.*

PROOF. For each $\eta \in F$, there is $V_\eta \in \mathfrak{B}(\{\eta\})$ such that $R_{V_\eta \cap X} u \neq u$. We can cover F by a finite number of closed sets F_j , $j = 1, \dots, k$ such that $F_j \subset V_\eta$ for some $\eta \in F$ for each j . Then $R_{F_j} u = 0$ by the above lemma, and hence $R_F u \leq \sum_j R_{F_j} u = 0$.

PROPOSITION 4.1. *If $u \in \mathcal{H}_\lambda$ is minimal, then there exists a unique $\eta \in \partial^\lambda X$ such that $R_{(\eta)} u = u$, and hence $u = (\int u d\lambda) K_\lambda(\cdot, \eta)$ (by Corollary 3.2).*

PROOF. The uniqueness follows from Corollary 3.2. By Lemma 4.1, $R_{(\eta)} u = u$ or 0 for each $\eta \in \partial^\lambda X$. If $R_{(\eta)} u = 0$ for all $\eta \in \partial^\lambda X$, then $R_{\partial^\lambda X} u = 0$ by the above lemma, which implies $u = 0$. Thus $R_{(\eta)} u = u$ for some $\eta \in \partial^\lambda X$.

We shall say that $\eta \in \partial^\lambda X$ is a λ -minimal point if $K_\lambda(\cdot, \eta)$ is minimal and $\int_X K_\lambda(\cdot, \eta) d\lambda = 1$. Let $\partial_1^\lambda X$ be the set of all λ -minimal points and let $\partial_0^\lambda X = \partial^\lambda X \setminus \partial_1^\lambda X$.

PROPOSITION 4.2.

$$\partial_1^\lambda X = \{\eta \in \partial^\lambda X \mid K_\lambda(\cdot, \eta) \neq 0 \text{ and } R_{(\eta)} K_\lambda(\cdot, \eta) = K_\lambda(\cdot, \eta)\}.$$

PROOF. Let A be the set in the right hand side. By Proposition 4.1, $\partial_1^\lambda X \subset A$. Let $\eta \in A$ and suppose $0 \leq u \leq K_\lambda(\cdot, \eta)$ with $u \in \mathcal{H}(X)$. Put $v = K_\lambda(\cdot, \eta) - u$. Then $v \in \mathcal{H}_\lambda$ and by Lemma 3.1

$$R_{(\eta)} u + R_{(\eta)} v = R_{(\eta)} K_\lambda(\cdot, \eta) = K_\lambda(\cdot, \eta) = u + v.$$

It follows that $R_{(\eta)} u = u$, and thus $\eta \in \partial_1^\lambda X$ by virtue of Corollary 3.2.

PROPOSITION 4.3. $\partial_0^\lambda X = \{\eta \in \partial^\lambda X \mid R_{(\eta)} K_\lambda(\cdot, \eta) = 0\}$.

PROOF. If $R_{(\eta)} K_\lambda(\cdot, \eta) = 0$, then $\eta \notin \partial_1^\lambda X$ by the above proposition. Conversely, suppose $\eta \in \partial_0^\lambda X$. If $K_\lambda(\cdot, \eta) = 0$, then obviously $R_{(\eta)} K_\lambda(\cdot, \eta) = 0$. If $K_\lambda(\cdot, \eta) \neq 0$, then by Proposition 3.1 $R_{(\eta)} K_\lambda(\cdot, \eta) = cK_\lambda(\cdot, \eta)$ with

$0 \leq c \leq 1$. Since $\eta \notin \partial_1^\lambda X$, $c \neq 1$. Using Lemma 3.2, we see that $c^2 = c$, so that $c = 0$.

PROPOSITION 4.4. *The set $\partial_0^\lambda X$ is an F_σ -set. For any $u \in \mathcal{H}_\lambda$ and for any closed set F contained in $\partial_0^\lambda X$, $R_F u = 0$.*

PROOF. First, note that if there is $\zeta_0 \in \partial_0^\lambda X$ such that $K_\lambda(\cdot, \zeta_0) = 0$, then $R_{\{\zeta_0\}} u = 0$ for any $u \in \mathcal{H}_\lambda$ by Proposition 3.1.

For an open set U in \bar{X}^λ , $0 < t < 1$ and $x \in X$, let

$$F_{U,x,t} = \{\eta \in \partial^\lambda X \mid [R_{U \cap X} K_\lambda(\cdot, \eta)](x) \leq t K_\lambda(x, \eta)\}.$$

By Lemma 2.3, $F_{U,x,t}$ is a closed set. Let $P_x = \{\eta \in \partial^\lambda X \mid K_\lambda(x, \eta) > 0\}$ and set $A_{U,x,t} = F_{U,x,t} \cap U \cap P_x$. Then $A_{U,x,t}$ is an F_σ -set. We show that if F is a closed set contained in $A_{U,x,t}$ and if $u \in \mathcal{H}_\lambda$, then $R_F u = 0$. In fact, by Proposition 3.1, there is a nonnegative measure μ with $\text{Supp } \mu \subset F$ such that $R_F u = \int K_\lambda(\cdot, \eta) d\mu(\eta)$. Since $[R_F K_\lambda(\cdot, \eta)](x) \leq t K_\lambda(x, \eta)$, Lemma 3.2 and Proposition 3.2 imply that $(R_F u)(x) \leq t(R_F u)(x)$, and hence $(R_F u)(x) = 0$, or $\int_F K_\lambda(x, \eta) d\mu(\eta) = 0$. Since $F \subset P_x$, it follows that $\mu = 0$, i.e., $R_F u = 0$.

Choose a countable base $\{U_n\}$ of open sets in X , a countable dense set $\{x_k\}$ in X and a sequence $\{t_m\}$ of positive numbers such that $t_m \uparrow 1$. Then, using Propositions 4.2 and 4.3, we see that $\partial_0^\lambda X \setminus \{\zeta_0\} = \bigcup_{n,k,m} A_{U_n, x_k, t_m}$. This, together with the above observation, implies the required results.

REMARK 4.1. If we define a mapping $\varphi: \partial^\lambda X \rightarrow \mathcal{H}_{\lambda,1}$ by $\varphi(\eta) = K_\lambda(\cdot, \eta)$, then φ is injective and continuous. Let $A_e^\lambda = \{u \in \mathcal{H}_{\lambda,1} \mid u \text{ is minimal and } \int u d\lambda = 1\}$. By Proposition 4.1, $\varphi(\partial_1^\lambda X) = A_e^\lambda$. Thus, we can see that $\partial_1^\lambda X$ is a G_δ -set by a general theory (cf. [3]).

LEMMA 4.4. *If $\eta \in \partial_1^\lambda X$ and F is a closed set in $\partial^\lambda X$, then*

$$R_F K_\lambda(\cdot, \eta) = \begin{cases} K_\lambda(\cdot, \eta) & \text{if } \eta \in F \\ 0 & \text{if } \eta \notin F \end{cases}.$$

PROOF. If $\eta \in F$, then $R_F K_\lambda(\cdot, \eta) = K_\lambda(\cdot, \eta)$ by Proposition 4.2. If $\eta \notin F$, then $R_{\{\zeta\}} K_\lambda(\cdot, \eta) = 0$ for any $\zeta \in F$ by Lemma 4.1 and the uniqueness in Proposition 4.1. Hence $R_F K_\lambda(\cdot, \eta) = 0$ by Lemma 4.2.

LEMMA 4.5. *Let $\eta \in \partial_1^\lambda X$ and let $\{X_n\}$ be an exhaustion of X . Then $\varepsilon_\eta^{X_n} \rightarrow \varepsilon_\eta$ (the unit mass at η) vaguely.*

PROOF. Since $\varepsilon_\eta^{X_n}(\bar{X}^\lambda) \leq 1$, $\{\varepsilon_\eta^{X_n}\}$ is vaguely relatively compact. Let μ be any limit measure. Then $\text{Supp } \mu \subset \partial^\lambda X$. Letting $n \rightarrow \infty$ in the equality $\int_{\partial X_n} K_\lambda(\cdot, y) d\varepsilon_\eta^{X_n}(y) = R_{X_n} K_\lambda(\cdot, \eta)$, we have

$$\int_{\partial^\lambda X} K_\lambda(\cdot, \zeta) d\mu(\zeta) = K_\lambda(\cdot, \eta).$$

Since $K_\lambda(\cdot, \eta)$ is minimal and $\mu(\partial^\lambda X) \leq 1$, it follows that $\mu = \varepsilon_\eta$.

§5. Canonical representations

A signed measure ν on $\partial^\lambda X$ is called a *canonical measure* (with respect to λ) if $|\nu|(\partial_0^\lambda X) = 0$.

LEMMA 5.1. *If μ is a canonical nonnegative measure on $\partial^\lambda X$ and*

$$u = \int K_\lambda(\cdot, \eta) d\mu(\eta),$$

then, for any closed set F in $\partial^\lambda X$

$$R_F u = \int_F K_\lambda(\cdot, \eta) d\mu(\eta) \quad \text{and} \quad \mu(F) = \int R_F u d\lambda.$$

PROOF. By Proposition 3.2 and Lemma 4.4, we have the first equality. Integrating both sides by λ , we obtain the second.

COROLLARY 5.1 (Uniqueness of the canonical representation). *If*

$$\int K_\lambda(\cdot, \eta) d\mu_1(\eta) = \int K_\lambda(\cdot, \eta) d\mu_2(\eta)$$

for canonical nonnegative measures μ_1 and μ_2 , then $\mu_1 = \mu_2$.

THEOREM 5.1. *If $u \in \mathcal{H}_\lambda$, then there exists a unique canonical nonnegative measure μ_u on $\partial^\lambda X$ such that*

$$u = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\mu_u(\eta).$$

PROOF. The uniqueness is given in the above corollary.

Let $\partial_0^\lambda X = \bigcup_{n=1}^\infty A_n$ with closed sets A_n such that $A_n \subset A_{n+1}$. By Proposition 4.4, $R_{A_n} u = 0$. Hence, given an exhaustion $\{X_n\}$ of X , we can find $v_n \in \mathcal{S}_+$ such that $0 \leq v_n \leq u$, $v_n = u$ on $V_n \cap X$ for some $V_n \in \mathfrak{B}(A_n)$ and $v_n \leq 2^{-n}$ on X_n (cf. the proof of [1; Proposition 5.3.2]). Set $v = \sum_{n=1}^\infty v_n$. Then $v \in \mathcal{S}_+$ and v is locally bounded on X . For $m > 0$, set $U_m = \bigcup_{n=1}^\infty (V_n \cap \dots \cap V_{n+m})$ and $F_m = \partial^\lambda X \setminus U_m$. Then, F_m is a closed set contained in $\partial_1^\lambda X$ and $F_m \subset F_{m+1}$ for each m . By Proposition 3.1, there is a nonnegative measure μ_m such that $\text{Supp } \mu_m \subset F_m$ (so that μ_m is canonical), $\mu_m(F_m) = \int R_{F_m} u d\lambda$ and $R_{F_m} u = \int K_\lambda(\cdot, \eta) d\mu_m(\eta)$ for each m . By Corollary 3.1 and Lemma 5.1, we have

$$R_{F_m} u = R_{F_m}(R_{F_{m+1}} u) = R_{F_m} \left(\int K_\lambda(\cdot, \eta) d\mu_{m+1}(\eta) \right) = \int_{F_m} K_\lambda(\cdot, \eta) d\mu_{m+1}(\eta).$$

By Corollary 5.1, it follows that $\mu_{m+1}|_{F_m} = \mu_m$. Since $\mu_m(F_m) \leq \int u d\lambda < \infty$, there is a nonnegative measure μ on $\partial^\lambda X$ such that $\mu|_{F_m} = \mu_m$ for all m and $\mu(\partial^\lambda X \setminus \bigcup_{m=1}^\infty F_m) = 0$. Then, μ is a canonical measure and

$$(5.1) \quad \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\mu(\eta) = \lim_{m \rightarrow \infty} \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\mu_m(\eta) = \lim_{m \rightarrow \infty} R_{F_m} u \leq u.$$

For fixed m , take $w \in \mathcal{S}_+$ such that $0 \leq w \leq u$ on X , $w = u$ on $W \cap X$ for some $W \in \mathfrak{B}(F_m)$. Since $v \geq mu$ on $U_m \cap X$ and $U_m \cup W \supset \partial^\lambda X$, we see that $v/m + w \geq u$ on X . Hence, $v/m + R_{F_m} u \geq u$ for any m . This, together with (5.1), implies that $u = \int K_\lambda(\cdot, \eta) d\mu(\eta)$.

REMARK 5.1. In view of Remark 4.1, the above theorem can also be obtained through a general theory (cf. [3; Theorem 2.5]).

PROPOSITION 5.1. *If μ is a canonical nonnegative measure on $\partial^\lambda X$ and $\{X_n\}$ is an exhaustion of X , then $\mu^{X_n} \rightarrow \mu$ vaguely, where μ^U denotes the measure defined by (2.2).*

PROOF. Since $\varepsilon_\eta^{X_n}(\bar{X}^\lambda) \leq 1$, the functions $\eta \mapsto \int_{\partial X_n} f d\varepsilon_\eta^{X_n}$ are uniformly bounded on $\partial^\lambda X$ for each $f \in \mathcal{C}(\bar{X}^\lambda)$. Hence, we obtain the required result by Lemma 4.5 and Lebesgue's convergence theorem.

COROLLARY 5.2. *Let $u \in \mathcal{H}_\lambda$ and let $\{X_n\}$ be an exhaustion of X . If $R_{X_n} u = \int_{\partial X_n} K_\lambda(\cdot, y) d\mu_n(y)$, then $\{\mu_n\}$ vaguely converges to μ_u (the canonical measure representing u).*

PROOF. By Lemma 2.5 and the uniqueness in Lemma 1.1, $\mu_n = \mu_u^{X_n}$.

§6. Minimal fine limits

Given $\eta \in \partial_1^\lambda X$, a set $A \subset X$ is said to be \mathcal{H}_λ -minimal thin or, simply \mathcal{H}_λ -thin at η , if there is an open set $U \subset X$ such that $A \subset U$ and $R_U K_\lambda(\cdot, \eta) \neq K_\lambda(\cdot, \eta)$.

LEMMA 6.1. *For an open set U in X , it is \mathcal{H}_λ -thin at $\eta \in \partial_1^\lambda X$ if and only if $R_U K_\lambda(\cdot, \eta) \in \mathcal{P}$.*

PROOF. The "if" part is obvious, since $K_\lambda(\cdot, \eta) \in \mathcal{H}(X)$, $\neq 0$. To prove the "only if" part, let $v = R_U K_\lambda(\cdot, \eta)$. Since $v \in \mathcal{S}_+$, $v = h + p$ with $h \in \mathcal{H}(X)$ and $p \in \mathcal{P}$. Since $R_U K_\lambda(\cdot, \eta) \neq K_\lambda(\cdot, \eta)$ and $K_\lambda(\cdot, \eta)$ is minimal, $h = cK_\lambda(\cdot, \eta)$

with $0 \leq c < 1$. Since $R_U v = v$, we have

$$cK_\lambda(\cdot, \eta) + p = R_U(cK_\lambda(\cdot, \eta) + p) = cv + R_U p = c^2 K_\lambda(\cdot, \eta) + cp + R_U p,$$

which implies that $c = c^2$. Hence, $c = 0$, i.e., $v = p \in \mathcal{P}$.

LEMMA 6.2. For $\eta \in \partial_1^\lambda X$, $\mathfrak{B}_\eta^F = \{V \subset X \mid X \setminus V \text{ is } \mathcal{H}_\lambda\text{-thin at } \eta\}$ is a filter.

PROOF. It is enough to verify that if U_1 and U_2 are open sets which are \mathcal{H}_λ -thin at η , then so is $U_1 \cup U_2$; and this is easily seen by the inequality $R_{U_1 \cup U_2} K_\lambda(\cdot, \eta) \leq R_{U_1} K_\lambda(\cdot, \eta) + R_{U_2} K_\lambda(\cdot, \eta)$ and the above lemma.

The limit (resp. upper limit, lower limit) of a function f on X with respect to the filter \mathfrak{B}_η^F will be denoted by

$$\text{F-lim}_{x \rightarrow \eta} f(x) \quad (\text{resp. } \text{F-limsup}_{x \rightarrow \eta} f(x), \text{F-liminf}_{x \rightarrow \eta} f(x)).$$

LEMMA 6.3. Let $\eta \in \partial_1^\lambda X$. Then, for any neighborhood V of η in \bar{X}^λ , $X \setminus V$ is \mathcal{H}_λ -thin at η .

PROOF. Choose $W \in \mathfrak{B}(\{\eta\})$ such that $\bar{W}^\lambda \subset V$ and let $U = X \setminus \bar{W}$. By Lemma 2.2, there is a nonnegative measure μ on \bar{X}^λ such that $\text{Supp } \mu \subset \partial^\lambda U \subset \bar{X}^\lambda \setminus W$,

$$R_U K_\lambda(\cdot, \eta) = \int_{\bar{X}^\lambda} K_\lambda(\cdot, \zeta) d\mu(\zeta) \quad \text{and} \quad \mu(\bar{X}^\lambda) = \int R_U K_\lambda(\cdot, \eta) d\lambda.$$

Put $u = \int_{\partial^\lambda X} K_\lambda(\cdot, \zeta) d\mu(\zeta)$ and $p = \int_X K_\lambda(\cdot, \zeta) d\mu(\zeta)$. Then, $u \in \mathcal{H}_\lambda$ and $0 \leq u \leq K_\lambda(\cdot, \eta)$, so that $u = cK_\lambda(\cdot, \eta)$ with $0 \leq c \leq 1$ by the minimality of $K_\lambda(\cdot, \eta)$. On the other hand,

$$\mu(\partial^\lambda X) = \mu(\bar{X}^\lambda) - \mu(X) = \int R_U K_\lambda(\cdot, \eta) d\lambda - \int p d\lambda = \int u d\lambda = c.$$

Hence, if $c \neq 0$, then the minimality of $K_\lambda(\cdot, \eta)$ implies that $\mu = c\varepsilon_\eta$, which contradicts the choice of μ . Hence, $c = 0$, and so $R_U K_\lambda(\cdot, \eta) = p \in \mathcal{P}$, i.e., U is \mathcal{H}_λ -thin at η by Lemma 6.1. Hence $X \setminus V$ is \mathcal{H}_λ -thin at η .

COROLLARY 6.1. For any extended real valued function f on X and for any $\eta \in \partial_1^\lambda X$,

$$\liminf_{x \rightarrow \eta, x \in X} f(x) \leq \text{F-liminf}_{x \rightarrow \eta} f(x).$$

Let $1 = h_1 + G(\sigma(1))$ with $h_1 \in \mathcal{H}(X)$. Then $h_1 \in \mathcal{H}_\lambda$, so that there is a unique canonical nonnegative measure ω_1^λ on $\partial^\lambda X$ such that

$$h_1 = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) d\omega_1^\lambda(\eta).$$

REMARK 6.1. If $h_1 = 0$, then $\omega_1^\lambda = 0$ so that all the results in the rest of this section become trivial.

LEMMA 6.4. For an ω_1^λ -measurable set E in $\partial_1^\lambda X$, let

$$u_E = \int_E K_\lambda(\cdot, \eta) d\omega_1^\lambda(\eta).$$

Then, an open set U in X is \mathcal{H}_λ -thin at ω_1^λ -a.e. $\eta \in E$ if and only if $R_U u_E \in \mathcal{P}$.

PROOF. By Lemma 2.5,

$$R_U u_E = \int_E R_U K_\lambda(\cdot, \eta) d\omega_1^\lambda(\eta).$$

Let $A = \{\eta \in E \mid U \text{ is not } \mathcal{H}_\lambda\text{-thin at } \eta\}$. Then, A is ω_1^λ -measurable and

$$R_U u_E = \int_{A \cap E} K_\lambda(\cdot, \eta) d\omega_1^\lambda(\eta) + \int_{E \setminus A} R_U K_\lambda(\cdot, \eta) d\omega_1^\lambda(\eta).$$

The first integral in the above belongs to \mathcal{H}_λ and, by Lemma 6.1, the second integral belongs to \mathcal{P} . Hence, $R_U u_E \in \mathcal{P}$ if and if $\omega_1^\lambda(A \cap E) = 0$.

LEMMA 6.5. For any $p \in \mathcal{P}$ and $\varepsilon > 0$, the set

$$V_{p,\varepsilon} = \{x \in X \mid p(x) > \varepsilon h_1(x)\}$$

is \mathcal{H}_λ -thin at ω_1^λ -a.e. $\eta \in \partial_1^\lambda X$.

PROOF. $V_{p,\varepsilon}$ is an open set in X and $p \geq \varepsilon R_{V_{p,\varepsilon}} h_1$. Hence $R_{V_{p,\varepsilon}} h_1 \in \mathcal{P}$, so that the lemma follows from the previous one.

PROPOSITION 6.1. If $p \in \mathcal{P}$, then $F\text{-}\lim_{x \rightarrow \eta} p(x) = 0$ for ω_1^λ -a.e. $\eta \in \partial_1^\lambda X$.

PROOF. By the above lemma, for any $\varepsilon > 0$, $F\text{-}\limsup_{x \rightarrow \eta} p(x) < \varepsilon$ for a.e. $\eta \in \partial_1^\lambda X$, since $h_1 \leq 1$.

COROLLARY 6.2. $F\text{-}\lim_{x \rightarrow \eta} h_1(x) = 1$ for ω_1^λ -a.e. $\eta \in \partial_1^\lambda X$.

PROOF. It is enough to note that $1 - h_1 \in \mathcal{P}$.

PROPOSITION 6.2 (cf. [3; Proposition 2.20] and [8; §2, Satz 3]). For a bounded ω_1^λ -measurable function φ on $\partial^\lambda X$, let

$$h_\varphi = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) \varphi(\eta) d\omega_1^\lambda(\eta).$$

Then

$$\text{F-lim}_{x \rightarrow \eta} h_\varphi(x) = \varphi(\eta) \quad \text{for } \omega_1^\lambda\text{-a.e. } \eta \in \partial_1^\lambda X.$$

PROOF. It is enough to consider the case $\varphi = \chi_E$, the characteristic function of an ω_1^λ -measurable set E . Let $u = h_{\chi_E}$ and $v = h_1 - u = h_{\chi_{E^c}}$, where $E' = \partial_1^\lambda X \setminus E$. For $n > 0$, $V_n = \{x \in X \mid nu(x) > v(x)\}$ is an open set in X and $v_n \equiv \min\{nu, v\} \in \mathcal{S}_+$. Let $v_n = u_n + p_n$ with $u_n \in \mathcal{H}_\lambda$ and $p_n \in \mathcal{P}$. Let μ_n be the canonical measure representing u_n . Then, for any closed set F contained in E' , Lemma 5.1 implies

$$\mu_n(F) = \int R_F u_n d\lambda \leq \int R_F v_n d\lambda \leq n \int R_F u d\lambda = n\omega_1^\lambda(F \cap E) = 0.$$

Hence $\mu_n(E') = 0$. Similarly, using the inequality $v_n \leq v$, we have $\mu_n(E) = 0$. Thus $\mu_n = 0$, so that $v_n \in \mathcal{P}$. Since $R_{V_n} v \leq v_n$, it follows that $R_{V_n} v \in \mathcal{P}$, and hence V_n is \mathcal{H}_λ -thin at ω_1^λ -a.e. $\eta \in \partial_1^\lambda X \setminus E$ by Lemma 6.4. Since $u \leq v/n \leq 1/n$ on $X \setminus V_n$, letting $n \rightarrow \infty$, we see that

$$\text{F-lim}_{x \rightarrow \eta} u(x) = 0 \quad \text{for } \omega_1^\lambda\text{-a.e. } \eta \in E'.$$

Similar arguments show that $\text{F-lim}_{x \rightarrow \eta} v(x) = 0$ for ω_1^λ -a.e. $\eta \in E$. Since $u = h_1 - v$, Corollary 6.2 implies that $\text{F-lim}_{x \rightarrow \eta} u(x) = 1$ for ω_1^λ -a.e. $\eta \in E$. Hence the required assertion holds for $\varphi = \chi_E$.

COROLLARY 6.3. *If $u \in \mathcal{H}(X)$ is bounded, then*

$$u = \int_{\partial^\lambda X} K_\lambda(\cdot, \eta) \varphi_u(\eta) d\omega_1^\lambda(\eta)$$

with $\varphi_u(\eta) = \text{F-lim}_{x \rightarrow \eta} u(x)$, which exists for ω_1^λ -a.e. $\eta \in \partial_1^\lambda X$.

PROOF. We may assume that $u \geq 0$. If μ_u is the canonical measure representing u and if $u \leq M$, then $\mu_u(F) \leq M\omega_1^\lambda(F)$ for any closed set F in $\partial^\lambda X$ by Lemma 5.1. Hence, μ_u is absolutely continuous with respect to ω_1^λ and $\mu_u = \varphi_u \omega_1^\lambda$ for some ω_1^λ -measurable function φ_u with $0 \leq \varphi_u \leq M$. Thus, $u = h_{\varphi_u}$ and $\text{F-lim}_{x \rightarrow \eta} u(x) = \varphi_u(\eta)$ for ω_1^λ -a.e. $\eta \in \partial_1^\lambda X$.

§7. Function classes \mathcal{F}_λ and \mathcal{F}

Let \mathcal{A}^* be the set of all standard \mathcal{H}^* -reference measures, namely the set of all nonnegative measures λ on X such that $\lambda(X) < \infty$ and $G\lambda$ is positive bounded continuous on X . Let $\mathcal{P}_{C,0} = \{p \in \mathcal{P}_C \mid \text{Supp } \sigma(p) \text{ is compact}\}$.

LEMMA 7.1. *Let ν be a nonnegative measure on X such that*

$$\int p \, d\nu < \infty \quad \text{for any } p \in \mathcal{P}_{C,0}.$$

Then there exists $\lambda \in A^$ such that $\int G\lambda \, d\nu < \infty$.*

PROOF. Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable base of the open sets in X and let $J = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid \bar{U}_m \text{ is compact and } \bar{U}_m \subset U_n\}$. For each $(m, n) \in J$, choose $\varphi_{(m,n)} \in \mathcal{C}_0(X)$ such that $0 \leq \varphi_{(m,n)} \leq 1$ on X , $\varphi_{(m,n)} = 1$ on \bar{U}_m and $\text{Supp } \varphi_{(m,n)} \subset U_n$. Let $p_{(m,n)} = R\varphi_{(m,n)}$. Then $p_{(m,n)} \in \mathcal{P}_{C,0}$, $0 \leq p_{(m,n)} \leq 1$ on X and $p_{(m,n)} = 1$ on \bar{U}_m . Put $\mu_{(m,n)} = \sigma(p_{(m,n)})$. By assumption, $a_{(m,n)} \equiv \int p_{(m,n)} \, d\nu < \infty$. Choose $\varepsilon_{(m,n)} > 0$, $(m, n) \in J$, such that

$$\sum_{(m,n) \in J} \varepsilon_{(m,n)} \max\{a_{(m,n)}, \mu_{(m,n)}(X), 1\} < \infty$$

and set $\lambda = \sum_{(m,n) \in J} \varepsilon_{(m,n)} \mu_{(m,n)}$. Then, we see easily that this λ has the required properties.

The mutual gradient measure $\delta_{[f,g]}$ and the gradient measure δ_f for $f, g \in \mathcal{R}(X)$ are defined by (see [6])

$$\delta_{[f,g]} = \frac{1}{2} \{f\sigma(g) + g\sigma(f) - \sigma(fg) - fg\sigma(1)\} \quad \text{and} \quad \delta_f = \delta_{[f,f]}.$$

We know that $\delta_f \geq 0$ ([6; Theorem 3.1]). If we define $\delta_{[f,g]}^*$ for $f, g \in \mathcal{R}^*(X)$ similarly in terms of σ^* , then $\delta_{[f,g]} = \delta_{[f,g]}^*$ whenever $f, g \in \mathcal{R}(X) \cap \mathcal{R}^*(X)$ ([7; Theorem 2.1]). Therefore, we can define $\delta_{[f,g]}$ for $f \in \mathcal{R}(X) \cap \mathcal{R}^*(X)$ and $g \in \mathcal{R}(X) + \mathcal{R}^*(X)$ in such a way that the mapping $g \mapsto \delta_{[f,g]}$ is linear on $\mathcal{R}(X) + \mathcal{R}^*(X)$.

Let $\mathcal{P}_F = \{p \in \mathcal{P}_C \mid \sigma(p)(X) < \infty\}$ and $\mathcal{Q}_F = \mathcal{P}_F - \mathcal{P}_F$. We consider the function classes

$$\mathcal{F}_\lambda = \{q/G\lambda \mid q \in \mathcal{Q}_F\} \quad \text{for } \lambda \in A^*, \quad \text{and} \quad \mathcal{F} = \bigcup_{\lambda \in A^*} \mathcal{F}_\lambda.$$

LEMMA 7.2. *Suppose $f \in \mathcal{R}(X)$ and $g \in \mathcal{Q}_F$ satisfy the following conditions (i), (ii) and (iii):*

- (i) f is bounded;
- (ii) $\int_X |g| \, d|\sigma(f) - f\sigma(1)| < \infty$;
- (iii) $\delta_f(X) < \infty$ and $\delta_g(X) < \infty$.

Then $fg \in \mathcal{Q}_F$.

PROOF. Let $\nu = \sigma(fg)$. Since $\nu = f\sigma(g) + g\sigma(f) - fg\sigma(1) - 2\delta_{[f,g]}$,

$$|v|(X) \leq \int |f| d|\sigma(g)| + \int |g| d|\sigma(f) - f\sigma(1)| + 2|\delta_{[f,g]}|(X).$$

Since f is bounded and $|\sigma(g)|(X) < \infty$, $\int |f| d|\sigma(g)| < \infty$. By conditions (ii) and (iii), the last two integrals in the above are finite. Hence, $|v|(X) < \infty$. Thus, $Gv \in \mathcal{Q}_F$ and $u \equiv fg - Gv \in \mathcal{H}(X)$. If $|f| \leq M$ and $g = p_1 - p_2$ with $p_1, p_2 \in \mathcal{P}_F$, then $|u| \leq M(p_1 + p_2) + G|v| \in \mathcal{P}_C$. It follows that $u = 0$, and thus $fg = Gv \in \mathcal{Q}_F$.

PROPOSITION 7.1. *If $f \in \mathcal{R}(X)$ satisfies the following three conditions (i), (ii) and (iii), then $f \in \mathcal{F}$:*

- (i) f is bounded;
- (ii) $\delta_f(X) < \infty$;
- (iii) $\int p d|\sigma(f) - f\sigma(1)| < \infty$ for any $p \in \mathcal{P}_{C,0}$.

PROOF. In view of condition (iii), by Lemma 7.1, we find $\lambda \in A^*$ such that $\int G\lambda d|\sigma(f) - f\sigma(1)| < \infty$. Note that $\delta_{G\lambda}(X) \leq \int G\lambda d\lambda < \infty$ by [7; Theorem 3.1]. Hence, by Lemma 7.2, $fG\lambda \in \mathcal{Q}_F$, i.e., $f \in \mathcal{F}_\lambda$.

REMARK 7.1. Condition (iii) in Proposition 7.1 is valid if one of the following is satisfied:

- (a) $G^*|\sigma(f) - f\sigma(1)|$ is locally bounded;
- (b) $|\sigma(f) - f\sigma(1)|(X) < \infty$;
- (c) $|\sigma(f)|(X) < \infty$ and $\int f^2 d\sigma(1) < \infty$;
- (d) $\int G|\sigma(f)| d|\sigma(f)| < \infty$ and $\int f^2 d\sigma(1) < \infty$.

PROOF. If (a) is satisfied, then $\int p d|\sigma(f) - f\sigma(1)| = \int G^*|\sigma(f) - f\sigma(1)| d\sigma(p) < \infty$ for any $p \in \mathcal{P}_{C,0}$. Since any $p \in \mathcal{P}_{C,0}$ is bounded, (b) implies (iii) of Proposition 7.1. Also, by [7; Theorem 3.1], $\int p^2 d\sigma(1) \leq \int p d\sigma(p) < \infty$ for any $p \in \mathcal{P}_{C,0}$. Hence, $\int f^2 d\sigma(1) < \infty$ implies $\int p|f| d\sigma(1) < \infty$ for any $p \in \mathcal{P}_{C,0}$ by Schwarz's inequality, and, in view of [7; Proposition 2.2], $\int G|\sigma(f)| d|\sigma(f)| < \infty$ implies $\int p d|\sigma(f)| < \infty$ for any $p \in \mathcal{P}_{C,0}$. Thus, each of (c) and (d) implies (iii) of Proposition 7.1.

Let

$$\mathcal{H}_E = \left\{ u \in \mathcal{H}(X) \mid \delta_u(X) + \int u^2 d\sigma(1) < \infty \right\},$$

$$\mathcal{P}_I = \left\{ p \in \mathcal{P}_C \mid \int p d\sigma(p) < \infty \right\},$$

$$\mathcal{Q}_I = \mathcal{P}_I - \mathcal{P}_I, \text{ and}$$

$$\mathcal{R}_{EB} = \{ f \in \mathcal{H}_E + \mathcal{Q}_I \mid f: \text{bounded} \}.$$

COROLLARY 7.1. Any $f \in \mathcal{R}_{EB}$ satisfies conditions (i), (ii) and (iii) of Proposition 7.1; so that $\mathcal{R}_{EB} \subset \mathcal{F}$.

PROOF. Let $f \in \mathcal{R}_{EB}$. By definition, (i) of Proposition 7.1 is satisfied. By [7; Theorem 3.1], we see that $\delta_f(X) + \int f^2 d\sigma(1) < \infty$. Therefore, (ii) and (iii) of Proposition 7.1 are valid in view of (d) in the above Remark.

§8. Green's formulae

Given $\lambda \in \Lambda^*$, we denote by $K_\lambda^*(x, y)$ the adjoint λ -Martin kernel, namely,

$$K_\lambda^*(x, y) = \frac{G(x, y)}{G\lambda(x)}, \quad x, y \in X.$$

The adjoint λ -Martin compactification $\bar{X}^{*\lambda}$ is defined by $\{K_\lambda^*(\cdot, y)\}_{y \in X}$ and, for $\xi \in \partial^{*\lambda}X = \bar{X}^{*\lambda} \setminus X$, $K_\lambda^*(\xi, \cdot) \in \mathcal{H}_{\lambda, 1}^*$.

Now, let $1 = h_1^* + G^*(\sigma^*(1))$ with $h_1^* \in \mathcal{H}^*(X)$. For any $\lambda \in \Lambda^*$, $h_1^* \in \mathcal{H}_\lambda^*$. By Theorem 5.1 (applied to \mathcal{H}^*), there is a unique canonical nonnegative measure $\omega_1^{*\lambda}$ on $\partial^{*\lambda}X$ such that

$$h_1^* = \int_{\partial^{*\lambda}X} K_\lambda^*(\xi, \cdot) d\omega_1^{*\lambda}(\xi).$$

LEMMA 8.1. Let $\lambda \in \Lambda^*$. If $f = p/G\lambda$ with $p \in \mathcal{P}_F$, then

$$(8.1) \quad \tilde{f}(\xi) = \int_X K_\lambda^*(\xi, y) d\sigma(fG\lambda)(y)$$

belongs to $L^1(\omega_1^{*\lambda})$; and hence (8.1) is defined $\omega_1^{*\lambda}$ -a.e. on $\partial^{*\lambda}X$ and $\tilde{f} \in L^1(\omega_1^{*\lambda})$ for any $f \in \mathcal{F}_\lambda$.

PROOF. If $f = p/G\lambda$ with $p \in \mathcal{P}_F$, then

$$\begin{aligned} \int_{\partial^{*\lambda}X} \tilde{f} d\omega_1^{*\lambda} &= \int_X \left\{ \int_{\partial^{*\lambda}X} K_\lambda^*(\xi, y) d\omega_1^{*\lambda}(\xi) \right\} d\sigma(p)(y) \\ &= \int_X h_1^* d\sigma(p) \leq \sigma(p)(X) < \infty. \end{aligned}$$

We define

$$H^*f = \int_{\partial^{*\lambda}X} K_\lambda^*(\xi, \cdot) \tilde{f}(\xi) d\omega_1^{*\lambda}(\xi)$$

for $f \in \mathcal{F}_\lambda$. Then $H^*f \in \mathcal{H}_\lambda^* - \mathcal{H}_\lambda^*$. Obviously, $H^*1 = h_1^*$.

PROPOSITION 8.1. Let $\{X_n\}$ be an exhaustion of X and let τ_n^* be the

nonnegative measure on ∂X_n such that $R_{X_n}^* h_1^* = G^* \tau_n^*$ (cf. Lemma 1.1). Then, for any $f \in \mathcal{F}$,

$$H^* f(y) = \lim_{n \rightarrow \infty} G^*(f \tau_n^*)(y) \quad \text{for all } y \in X.$$

PROOF. We may assume that $f = p/G\lambda$ with $p \in \mathcal{P}_F$ and $\lambda \in \Lambda^*$. Let $\mu = \sigma(p) = \sigma(fG\lambda)$. Then for any $y \in X$,

$$\begin{aligned} H^* f(y) &= \int_{\partial^{*\lambda} X} K_\lambda^*(\xi, y) \tilde{f}(\xi) d\omega_1^{*\lambda}(\xi) \\ &= \int_{\partial^{*\lambda} X} K_\lambda^*(\xi, y) \left\{ \int_X K_\lambda^*(\xi, z) d\mu(z) \right\} d\omega_1^{*\lambda}(\xi) \\ &= \int_X \left\{ \int_{\partial^{*\lambda} X} K_\lambda^*(\xi, y) K_\lambda^*(\xi, z) d\omega_1^{*\lambda}(\xi) \right\} d\mu(z). \end{aligned}$$

By Corollary 5.2, $\{(G\lambda)\tau_n^*\}$ vaguely converges to $\omega_1^{*\lambda}$. Since $\xi \mapsto K_\lambda^*(\xi, y) K_\lambda^*(\xi, z)$ is continuous near $\partial^{*\lambda} X$ for fixed $y, z \in X$, we have

$$\begin{aligned} &\int_{\partial^{*\lambda} X} K_\lambda^*(\xi, y) K_\lambda^*(\xi, z) d\omega_1^{*\lambda}(\xi) \\ &= \lim_{n \rightarrow \infty} \int_{\partial X_n} K_\lambda^*(x, y) K_\lambda^*(x, z) G\lambda(x) d\tau_n^*(x) \\ &= \lim_{n \rightarrow \infty} \int_{\partial X_n} K_\lambda^*(x, y) G(x, z) d\tau_n^*(x). \end{aligned}$$

For any relatively compact neighborhood V of y , there is $c > 0$ such that $K_\lambda^*(x, y) \leq c$ for $x \in X \setminus V$ (cf. Lemma 2.1). Then, for $X_n \supset \bar{V}$

$$0 \leq \int_{\partial X_n} K_\lambda^*(x, y) G(x, z) d\tau_n^*(x) \leq c \int_{\partial X_n} G(x, z) d\tau_n^*(x) = c(R_{X_n}^* h_1^*)(z) \leq c$$

for all $z \in X$. Since $\mu(X) < \infty$, Lebesgue's convergence theorem implies

$$\begin{aligned} H^* f(y) &= \lim_{n \rightarrow \infty} \int_X \left\{ \int_{\partial X_n} K_\lambda^*(x, y) G(x, z) d\tau_n^*(x) \right\} d\mu(z) \\ &= \lim_{n \rightarrow \infty} \int_{\partial X_n} K_\lambda^*(x, y) G\mu(x) d\tau_n^*(x) = \lim_{n \rightarrow \infty} \int_{\partial X_n} G(x, y) f(x) d\tau_n^*(x). \end{aligned}$$

COROLLARY 8.1. For $f \in \mathcal{F}$, $H^* f$ is independent of the choice of $\lambda \in \Lambda^*$; $H^* f \geq 0$ if $f \geq 0$; $|H^* f| \leq \|f\|_\infty h_1^*$.

PROPOSITION 8.2. Suppose $f \in \mathcal{R}(X)$ and $g \in \mathcal{L}_F$ satisfy conditions (i), (ii) and

(iii) in Lemma 7.2 and f also satisfies condition (iii) in Proposition 7.1. Then

$$(8.2) \quad \sigma(fg)(X) = \int_X H^* f d\sigma(g) + \int_X fg d\sigma^*(1).$$

PROOF. By Proposition 7.1 and Lemma 7.2, we see that $f \in \mathcal{F}$ and $fg \in \mathcal{Q}_F$. Thus, $|\sigma(fg)|(X) < \infty$. Let $g = p_1 - p_2$ with $p_j \in \mathcal{P}_F$ ($j = 1, 2$); let $|f| \leq M$ on X . Then, using Corollary 8.1, we have

$$\begin{aligned} \int_X |H^* f| d|\sigma(g)| &\leq M\{\sigma(p_1)(X) + \sigma(p_2)(X)\} < \infty, \\ \int_X |fg| d\sigma^*(1) &\leq M \int_X (p_1 + p_2) d\sigma^*(1) = M \int_X G^*(\sigma^*(1)) d\sigma(p_1 + p_2) \\ &\leq M\{\sigma(p_1)(X) + \sigma(p_2)(X)\} < \infty. \end{aligned}$$

Thus, every term in (8.2) is well-defined and finite valued. Since $R_{X_n}^* h_1^* + G^*(\sigma^*(1)) \uparrow 1$, we have

$$\begin{aligned} \sigma(fg)(X) &= \lim_{n \rightarrow \infty} \int_X \{R_{X_n}^* h_1^* + G^*(\sigma^*(1))\} d\sigma(fg) \\ &= \lim_{n \rightarrow \infty} \left\{ \int_X G^* \tau_n^* d\sigma(fg) + \int_X G^*(\sigma^*(1)) d\sigma(fg) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \int_{\partial X_n} fg d\tau_n^* + \int_X fg d\sigma^*(1) \right\} \\ &= \lim_{n \rightarrow \infty} \int_X G^*(f\tau_n^*) d\sigma(g) + \int_X fg d\sigma^*(1). \end{aligned}$$

By Proposition 8.1, $G^*(f\tau_n^*) \rightarrow H^*f$. Since $\{G^*(f\tau_n^*)\}$ is uniformly bounded and $|\sigma(g)|(X) < \infty$, Lebesgue's convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_X G^*(f\tau_n^*) d\sigma(g) = \int_X H^* f d\sigma(g).$$

THEOREM 8.1. If $f \in \mathcal{R}_{EB}$ and $g \in \mathcal{Q}_{IF} \equiv \mathcal{Q}_I \cap \mathcal{Q}_F$, then

$$\begin{aligned} 2\delta_{[f,g]}(X) + \int_X fg d\sigma(1) + \int_X fg d\sigma^*(1) \\ = \int_X (f - H^*f) d\sigma(g) + \int_X g d\sigma(f). \end{aligned}$$

PROOF. By Corollary 7.1, f satisfies (i) and the first condition in (iii) of

Lemma 7.2 and (iii) of Proposition 7.1. Since $g \in \mathcal{Q}_I$, the second condition in (iii) of Lemma 7.2 is also satisfied. If $f = u + q$ with $u \in \mathcal{H}_E$ and $q \in \mathcal{Q}_I$, then

$$\begin{aligned} \int_X |g| d|\sigma(f)| &\leq \int_X G(|\sigma(g)|) d|\sigma(q)| \\ &\leq \int_X G(|\sigma(g)|) d|\sigma(g)| + \int_X G(|\sigma(q)|) d|\sigma(q)| < \infty \end{aligned}$$

by [7; Proposition 2.2], and

$$\int_X |fg| d\sigma(1) \leq \frac{1}{2} \left\{ \int_X u^2 d\sigma(1) + \int_X q^2 d\sigma(1) \right\} + \int_X g^2 d\sigma(1) < \infty.$$

Thus (ii) of Lemma 7.2 is satisfied. Hence, (8.2) in the previous proposition and the definition of $\delta_{[f,g]}$ yield the required formula.

LEMMA 8.2. *Let $\lambda \in A^*$ and $f = p/G\lambda$ with $p \in \mathcal{P}_F$. Then*

$$\tilde{f}(\xi) = \liminf_{x \rightarrow \xi} f(x) \quad \text{for any } \xi \in \partial_1^{*\lambda} X.$$

PROOF. Let $\mu = \sigma(p)$. Then, $f(x) = \int K_\lambda^*(x, y) d\mu(y)$. Hence, by the lower semicontinuity of $K_\lambda^*(\cdot, y)$ on \bar{X}^λ , we have $\tilde{f}(\xi) \leq \liminf_{x \rightarrow \xi} f(x)$.

To prove the converse inequality, let $\alpha < \liminf_{x \rightarrow \xi} f(x)$. Then there exists a neighborhood V of ξ in $\bar{X}^{*\lambda}$ such that $f(x) > \alpha$ on $V \cap X$. Let $\{X_n\}$ be an exhaustion of X and let ν_n be the nonnegative measure such that $\text{Supp } \nu_n \subset \partial X_n$ and $R_{X_n}^* K_\lambda^*(\xi, \cdot) = \int K_\lambda^*(x, \cdot) d\nu_n(x)$. Note that by Lemma 4.5 applied to \mathcal{H}^* , $\nu_n \rightarrow \varepsilon_\xi$ vaguely as $n \rightarrow \infty$. Hence, $\nu_n(X \cap V) \rightarrow 1$ ($n \rightarrow \infty$). Thus, we have

$$\begin{aligned} \tilde{f}(\xi) &= \int_X K_\lambda^*(\xi, y) d\mu(y) = \lim_{n \rightarrow \infty} \int_X [R_{X_n}^* K_\lambda^*(\xi, \cdot)](y) d\mu(y) \\ &= \lim_{n \rightarrow \infty} \int_X \left\{ \int_{\partial X_n} K_\lambda^*(x, y) d\nu_n \right\} d\mu(y) = \lim_{n \rightarrow \infty} \int_X f(x) d\nu_n(x) \\ &\geq \alpha \limsup_{n \rightarrow \infty} \nu_n(X \cap V) = \alpha. \end{aligned}$$

Hence, $\tilde{f}(\xi) \geq \liminf_{x \rightarrow \xi} f(x)$.

Given $\lambda \in A^*$ and $\xi \in \partial_1^{*\lambda} X$, F^* -lim, F^* -limsup and F^* -liminf are defined with respect to the filter $\mathfrak{B}_\xi^{*F} = \{V \subset X \mid X \setminus V \text{ is } \mathcal{H}_\lambda^* \text{-thin at } \xi\}$.

PROPOSITION 8.3. *Let $\lambda \in A^*$ and $f \in \mathcal{F}_\lambda$. Then*

$$F^* \text{-lim}_{x \rightarrow \xi} f(x) = \tilde{f}(\xi) \quad \text{for } \omega_1^{*\lambda} \text{-a.e. } \xi \in \partial_1^{*\lambda} X.$$

PROOF. Since $\tilde{f}(\xi)$ is finite for $\omega_1^{*\lambda}$ -a.e. $\xi \in \partial_1^{*\lambda} X$, it is enough to show that the equality $F^*\text{-}\lim_{x \rightarrow \xi} f(x) = \tilde{f}(\xi)$ holds for all $\xi \in \partial_1^{*\lambda} X$ in case $f = p/G\lambda$ with $p \in \mathcal{P}_F$. By Corollary 6.1 (applied to \mathcal{H}^*),

$$\liminf_{x \rightarrow \xi} f(x) \leq F^*\text{-}\liminf_{x \rightarrow \xi} f(x) \quad \text{for all } \xi \in \partial_1^{*\lambda} X.$$

Hence, together with Lemma 8.2, we have

$$\tilde{f}(\xi) \leq F^*\text{-}\liminf_{x \rightarrow \xi} f(x) \quad \text{for all } \xi \in \partial_1^{*\lambda} X.$$

To prove $F^*\text{-}\limsup_{x \rightarrow \xi} f(x) \leq \tilde{f}(\xi)$, we may assume that $\tilde{f}(\xi) < \infty$. Let $\alpha > \tilde{f}(\xi)$ and set $U_\alpha = \{x \in X \mid f(x) > \alpha\}$. Then U_α is an open set in X . Using Lemma 1.3, we have

$$\begin{aligned} (8.3) \quad [R_{\bar{U}_\alpha}^* K_\lambda^*(x, \cdot)](y) &= \frac{1}{(G\lambda)(x)} [R_{\bar{U}_\alpha}^* G(x, \cdot)](y) \\ &= \frac{1}{(G\lambda)(x)} [R_{U_\alpha} G(\cdot, y)](x). \end{aligned}$$

Let $\{X_n\}$ be an exhaustion of X and consider the measures $\mu_{x,n} \equiv \delta_x^{U_\alpha \cap X_n}$ defined in Lemma 1.2. Then, since $\text{Supp } \mu_{x,n} \subset \bar{U}_\alpha$ and $G\lambda \leq p/\alpha$ on \bar{U}_α , we have

$$\begin{aligned} \int_X [R_{U_\alpha} G(\cdot, y)](x) d\lambda(y) &= \lim_{n \rightarrow \infty} \int_X [R_{U_\alpha \cap X_n} G(\cdot, y)](x) d\lambda(y) \\ &= \lim_{n \rightarrow \infty} \int_X \left\{ \int_{\bar{U}_\alpha} G(z, y) d\mu_{x,n}(z) \right\} d\lambda(y) \\ &= \lim_{n \rightarrow \infty} \int_{\bar{U}_\alpha} G\lambda d\mu_{x,n} \leq \frac{1}{\alpha} \lim_{n \rightarrow \infty} \int_{\bar{U}_\alpha} p d\mu_{x,n} \\ &= \frac{1}{\alpha} \lim_{n \rightarrow \infty} [R_{U_\alpha \cap X_n} p](x) \leq \frac{1}{\alpha} p(x). \end{aligned}$$

Hence, by (8.3),

$$\int_X R_{\bar{U}_\alpha}^* K_\lambda^*(x, \cdot) d\lambda \leq \frac{1}{\alpha} f(x).$$

Thus, using Lemma 2.3 (applied to \mathcal{H}^*) and Lemma 8.2, we have

$$\int_X R_{\bar{U}_\alpha}^* K_\lambda^*(\xi, \cdot) d\lambda \leq \frac{1}{\alpha} \liminf_{x \rightarrow \xi} f(x) = \frac{1}{\alpha} \tilde{f}(\xi) < 1 = \int_X K_\lambda^*(\xi, \cdot) d\lambda,$$

which implies that $R_{\bar{U}_\alpha}^* K_\lambda^*(\xi, \cdot) \neq K_\lambda^*(\xi, \cdot)$, i.e., U_α is \mathcal{H}_λ^* -thin at ξ . Hence, $F^*\text{-}\limsup_{x \rightarrow \xi} f(x) \leq \alpha$, and thus $F^*\text{-}\limsup_{x \rightarrow \xi} f(x) \leq \tilde{f}(\xi)$. This completes the proof.

LEMMA 8.3. *If $f \in \mathcal{F} \cap (\mathcal{H}^*(X) + \mathcal{Q}_C^*)$, where $\mathcal{Q}_C^* = \mathcal{P}_C^* - \mathcal{D}_C^*$, and if f is bounded, then $f = H^*f + G^*(\sigma^*(f))$.*

PROOF. Let $f \in \mathcal{F}_\lambda$, $\lambda \in \Lambda^*$. Since $f \in \mathcal{H}^*(X) + \mathcal{Q}_C^*$, $f = u^* + G^*(\sigma^*(f))$ with $u^* \in \mathcal{H}^*(X)$. Then u^* is bounded on X . Hence, by Corollary 6.3 applied to \mathcal{H}^* ,

$$u^* = \int_{\partial^{*\lambda}X} K_\lambda^*(\xi, \cdot) \varphi(\xi) d\omega_1^{*\lambda}(\xi)$$

with $\varphi(\xi) = F^*\text{-}\lim_{y \rightarrow \xi} u^*(y)$, which exists $\omega_1^{*\lambda}$ -a.e. $\xi \in \partial_1^{*\lambda}X$. By Proposition 6.1 (applied to \mathcal{H}^*),

$$F^*\text{-}\lim_{y \rightarrow \xi} G^*(\sigma^*(f))(y) = 0 \quad \text{for } \omega_1^{*\lambda}\text{-a.e. } \xi \in \partial_1^{*\lambda}X.$$

Hence

$$F^*\text{-}\lim_{y \rightarrow \xi} f(y) = \varphi(\xi) \quad \text{for } \omega_1^{*\lambda}\text{-a.e. } \xi \in \partial_1^{*\lambda}X.$$

Thus, by Proposition 8.3, $\tilde{f}(\xi) = \varphi(\xi)$ $\omega_1^{*\lambda}$ -a.e. on $\partial^{*\lambda}X$, and hence $H^*f = u^*$.

We write $\hat{\sigma}(f) = \{\sigma(f) + \sigma^*(f)\}/2$ for $f \in \mathcal{R}(X) \cap \mathcal{R}^*(X)$.

THEOREM 8.2. *If $f \in \mathcal{R}_{EB} \cap (\mathcal{H}^*(X) + \mathcal{Q}_C^*)$ and $g \in \mathcal{Q}_{IF}$, then*

$$(8.4) \quad \delta_{[f,g]}(X) + \int_X fg \, d\hat{\sigma}(1) = \int_X g \, d\hat{\sigma}(f).$$

PROOF. By Corollary 7.1, $f \in \mathcal{F}$. Thus, by Lemma 8.3, $f - H^*f = G^*(\sigma^*(f))$. Therefore,

$$\int_X (f - H^*f) \, d\sigma(g) = \int_X G^*(\sigma^*(f)) \, d\sigma(g) = \int_X g \, d\sigma^*(f).$$

Hence, (8.4) follows from Theorem 8.1.

COROLLARY 8.2. *If $f \in \mathcal{R}_{EB} \cap \mathcal{R}_{EB}^*$ and $g \in \mathcal{Q}_{IF} + \mathcal{Q}_{IF}^*$, then (8.4) holds.*

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