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Notes on elements of $U(1, n; C)$

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Introduction

Let C be the field of complex numbers. Let $V = V^{1,n}(C)$ $(n \geqq 1)$ denote the vector space C^{n+1}, together with the unitary structure defined by the Hermitian form

$$\Phi(z^*, w^*) = -\overline{z_0^*}w_0^* + \overline{z_1^*}w_1^* + \cdots + \overline{z_n^*}w_n^*$$

for $z^* = (z_0^*, z_1^*, \ldots, z_n^*)$ and $w^* = (w_0^*, w_1^*, \ldots, w_n^*)$ in V. An automorphism g of V, that is, a linear bijection such that $\Phi(g(z^*), g(w^*)) = \Phi(z^*, w^*)$ for z^*, $w^* \in V$, will be called a *unitary transformation.* We denote the group of all unitary transformations by $U(1, n; C)$. Let $V_0 = \{z^* \in V | \Phi(z^*, z^*) = 0\}$ and $V_- = \{z^* \in V | \Phi(z^*, z^*) < 0\}$. It is clear that V_0 and V_- are invariant under $U(1, n; C)$. Set $V^* = V_- \cup V_0 - \{0\}$. Let $\pi: V^* \to \pi(V^*)$ be the projection map defined by $\pi(z_0^*, z_1^*, \ldots, z_n^*) = (z_1^* z_0^{*-1}, z_2^* z_0^{*-1}, \ldots, z_n^* z_0^{*-1})$. Set $H^n(C) = \pi(V_-)$. Let $\overline{H^n(C)}$ denote the closure of $H^n(C)$ in the projective space $\pi(V^*)$. An element g of $U(1, n; C)$ operates in $\pi(V^*)$, leaving $\overline{H^n(C)}$ invariant. Since $H^n(C)$ is identified with the complex unit ball $B^n = B^n(C) = \{z = (z_1, z_2, \ldots, z_n) \in C^n | \|z\|^2 = \Sigma_{k=1}^n |z_k|^2 < 1\}$, we regard a unitary transformation as a transformation operating on B^n. We introduce the Bergman metric

$$g_{ij}(z) = \delta_{ij}(1 - \|z\|^2)^{-1} + \bar{z}_i z_j (1 - \|z\|^2)^{-2}$$

for $z = (z_1, z_2, \ldots, z_n) \in B^n$. Using this metric, we see that the holomorphic sectional curvature is -4. The distance $d(z, w)$ for z, $w \in B^n$ is defined by the use of the Hermitian form Φ as follows:

$$d(z, w) = \cosh^{-1} [|\Phi(z^*, w^*)| \{\Phi(z^*, z^*)\Phi(w^*, w^*)\}^{-1/2}],$$

where $z^* \in \pi^{-1}(z)$ and $w^* \in \pi^{-1}(w)$ (see [3; Proposition 2.4.4]).

Many results on Möbius transformations and discrete groups are shown in [1] and [6]. Our purpose of this paper is to find analogous results for elements of $U(1, n; C)$ and discrete subgroups of $U(1, n; C)$. In Section 1 we shall prove that an element of $U(1, n; C)$ can be decomposed into two special

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elements. Using this fact, we obtain a distortion theorem for unitary transformations in Theorem 1.3. Elements of $U(1, n; \boldsymbol{C})$ and $\tilde{U}(1, n; \boldsymbol{C})$ are classified into three types (see Section 1 for the definition of $\tilde{U}(1, n; \boldsymbol{C})$). We discuss the properties of elements of each type. In Section 2 we shall study the properties of elements of discrete subgroups of $U(1, n; \boldsymbol{C})$ and $\tilde{U}(1, n; \boldsymbol{C})$ and show in Theorem 2.2 the existence of a domain where the action of a discrete subgroup is equal to the action of the cyclic group generated by a translation. Using the G-duality, we shall state in Theorem 2.16 that the fixed points of loxodromic elements of a discrete subgroup of $U(1, n; \boldsymbol{C})$ are dense in $L(G) \times L(G)$.

1. Elements of $U(1, n; \boldsymbol{C})$

We define the *norm* of an element $g = (a_{ij})_{i,j=1,2,\dots,n+1}$ in $U(1, n; \boldsymbol{C})$ by $\|g\| = (\Sigma_{i,j=1}^{n+1} |a_{ij}|^2)^{1/2}$. Noting that $\Phi(z^*, w^*)$ is invariant under $U(1, n; \boldsymbol{C})$, we see that

(1) $$-|a_{11}|^2 + \sum_{k=2}^{n+1} |a_{k1}|^2 = -1\,,$$

(2) $$-|a_{1j}|^2 + \sum_{k=2}^{n+1} |a_{kj}|^2 = 1 \qquad \text{for } j = 2, 3, \dots, n+1\,,$$

(3) $$-\overline{a_{1i}}a_{1j} + \sum_{k=2}^{n+1} \overline{a_{ki}}a_{kj} = 0 \qquad \text{for } i \neq j,\ i, j = 1, 2, \dots, n+1\,,$$

(4) $$-|a_{11}|^2 + \sum_{k=2}^{n+1} |a_{1k}|^2 = -1\,,$$

(5) $$-|a_{21}|^2 + \sum_{k=2}^{n+1} |a_{2k}|^2 = 1\,.$$

PROPOSITION 1.1 (cf. [2; Theorem 2]). *For $g \in U(1, n; \boldsymbol{C})$,*

$$\|g\|^2 = \|I\|^2 + 4\sinh^2 d(0, g(0))\,,$$

where I is the unit matrix.

PROOF. First we note that $d(0, g(0)) = \cosh^{-1} |a_{11}|$ by (1) and $\|I\|^2 = n + 1$. Therefore we have

$$\|I\|^2 + 4\sinh^2 d(0, g(0)) = 4|a_{11}|^2 + (n-3)\,.$$

It follows from the equalities (1), (2) and (4) that

$$\begin{aligned}\|g\|^2 &= \sum_{k=1}^{n+1} |a_{k1}|^2 + \sum_{k=1}^{n+1} |a_{k2}|^2 + \cdots + \sum_{k=1}^{n+1} |a_{k,n+1}|^2 \\ &= (-1 + 2|a_{11}|^2) + (1 + 2|a_{12}|^2) + \cdots + (1 + 2|a_{1,n+1}|^2) \\ &= (n-1) + 2\sum_{k=1}^{n+1} |a_{1k}|^2 \\ &= (n-1) + 2(2|a_{11}|^2 - 1) \\ &= 4|a_{11}|^2 + (n-3)\,.\end{aligned}$$

Thus we complete the proof.

Next we shall show that any element of $U(1, n; \boldsymbol{C})$ can be expressed as the product of two special elements. Before stating our theorem, we shall give notation. We denote by $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$ the subgroup $\left\{\begin{bmatrix} \alpha & 0 \\ 0 & A \end{bmatrix} \middle| |\alpha| = 1, A \in U(n; \boldsymbol{C})\right\}$ of $U(1, n; \boldsymbol{C})$.

THEOREM 1.2. *Let $g = (a_{ij})_{i,j=1,2,\ldots,n+1}$ be an element of $U(1, n; \boldsymbol{C})$ and let p be a point of B^n such that $g(p) = 0$. Then the element g is decomposed into two elements f_p and u in $U(1, n; \boldsymbol{C})$ such that*

1) $g = uf_p$;
2) $f_p(p) = 0$, $f_p(0) = p$ *and* $f_p^2 =$ *identity*;
3) $u \in U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$.

PROOF. Without loss of generality, we may assume that $p = (a, 0, \ldots, 0) \in B^n$. Set

$$f_p = \begin{bmatrix} \alpha_{11} & -\bar{a}\alpha_{11} & 0 \\ a\alpha_{11} & -\alpha_{11} & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},$$

where $\alpha_{11}^2 = (1 - |a|^2)^{-1}$. It is easy to show that f_p is an element of $U(1, n; \boldsymbol{C})$ such that $f_p(p) = 0$, $f_p(0) = p$ and $f_p^2 =$ identity. Next we shall show that there exists an element $u \in U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$ such that $g = uf_p$. To prove this, we have only to show that gf_p^{-1} belongs to $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$. We denote $(a_{ij})_{i=1,2,j=3,\ldots,n+1}$ and $(a_{ij})_{i=3,\ldots,n+1,j=3,\ldots,n+1}$ by A_1 and A_2, respectively. It is seen that

$$u = gf_p^{-1} = \begin{bmatrix} a_{11}\alpha_{11} + a_{12}a\alpha_{11} & -a_{11}\bar{a}\alpha_{11} - a_{12}\alpha_{11} & \\ a_{21}\alpha_{11} + a_{22}a\alpha_{11} & -a_{21}\bar{a}\alpha_{11} - a_{22}\alpha_{11} & A_1 \\ \cdots & \cdots & \\ a_{n+1,1}\alpha_{11} + a_{n+1,2}a\alpha_{11} & -a_{n+1,1}\bar{a}\alpha_{11} - a_{n+1,2}\alpha_{11} & A_2 \end{bmatrix}.$$

Then $g(p) = 0$ implies $a_{i1} + a_{i2}a = 0$ for $i \geqq 2$. Therefore the $(i, 1)$-component of u is equal to 0 for $i \geqq 2$. By (1), $|a_{11}\alpha_{11} + a_{12}a\alpha_{11}| = 1$. It follows from (4) that the $(1, j)$-component of u equals 0 for $j \geqq 2$. Using (2) and (3), we see that gf_p^{-1} has the form $\begin{bmatrix} b & 0 \\ 0 & B \end{bmatrix}$, where $|b| = 1$ and $B \in U(n; \boldsymbol{C})$. Thus u belongs to $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$.

Given any points $z, w \in \overline{B^n}$, define $d^*(z, w)$ by

$$d^*(z, w) = \{|z_0^*|^{-1}|w_0^*|^{-1}|\Phi(z^*, w^*)|\}^{1/2},$$

where $z^* = (z_0^*, z_1^*, \ldots, z_n^*) \in \pi^{-1}(z)$ and $w^* = (w_0^*, w_1^*, \ldots, w_n^*) \in \pi^{-1}(w)$. Note that $d^*(z, w)$ does not depend on the choice of z^* and w^*.

THEOREM 1.3 (cf. [1; Theorem 3.6.1]). *If* $g \in U(1, n; \boldsymbol{C})$, *then*

$$\sup_{z, w \in B^n, z \neq w} \frac{d^*(g(z), g(w))}{d^*(z, w)} = \exp(d(0, g(0))) .$$

After showing a lemma, we shall prove Theorem 1.3.

LEMMA 1.4. *Let* f_p *be defined as in the proof of Theorem* 1.2. *Then*

$$\sup_{z, w \in B^n, z \neq w} \frac{d^*(f_p(z), f_p(w))}{d^*(z, w)} = \exp(d(0, f_p(0))) .$$

PROOF. It is seen that

$$\begin{aligned}
&d^*(f_p(z), f_p(w))^2 \\
&= |\alpha_{11} z_0^* - \bar{a}\alpha_{11} z_1^*|^{-1} |\alpha_{11} w_0^* - \bar{a}\alpha_{11} w_1^*|^{-1} | - (\overline{\alpha_{11} z_0^* - \bar{a}\alpha_{11} z_1^*})(\alpha_{11} w_0^* - \bar{a}\alpha_{11} w_1^*) \\
&\quad + (\overline{a\alpha_{11} z_0^* - \alpha_{11} z_1^*})(a\alpha_{11} w_0^* - \alpha_{11} w_1^*) + \textstyle\sum_{k=2}^n \overline{z_k^*} w_k^* | \\
&= |\alpha_{11}|^{-2} |z_0^*(1 - \bar{a} z_1^* z_0^{*-1})|^{-1} |w_0^*(1 - \bar{a} w_1^* w_0^{*-1})|^{-1} | - \overline{z_0^*} w_0^* + \textstyle\sum_{k=1}^n \overline{z_k^*} w_k^* | \\
&= (1 - |a|^2) |1 - \bar{a} z_1|^{-1} |1 - \bar{a} w_1|^{-1} d^*(z, w)^2 .
\end{aligned}$$

Since $|1 - \bar{a} z_1| \geqq 1 - |a|$ and $|1 - \bar{a} w_1| \geqq 1 - |a|$, $\{(1 + |a|)(1 - |a|)^{-1}\}^{1/2}$ is the supremum of $d^*(f_p(z), f_p(w))/d^*(z, w)$ over $z, w \in B^n$. We observe that

$$\begin{aligned}
\exp(d(0, f_p(0))) &= \exp(\log(|\alpha_{11}| + (|\alpha_{11}|^2 - 1)^{1/2}) \\
&= \{(1 + |a|)(1 - |a|)^{-1}\}^{1/2} ,
\end{aligned}$$

and conclude our lemma.

PROOF OF THEOREM 1.3. Let g be an element of $U(1, n; \boldsymbol{C})$ and let p be a point of B^n such that $g(p) = 0$. As in Theorem 1.2 we decompose g into uf_p. It is easy to check $d^*(u(\zeta), u(\omega)) = d^*(\zeta, \omega)$ for any $\zeta, \omega \in B^n$. Hence $d^*(g(z), g(w)) = d^*(uf_p(z), uf_p(w)) = d^*(f_p(z), f_p(w))$. Therefore it follows from Lemma 1.4 that

$$\begin{aligned}
\sup_{z, w \in B^n, z \neq w} \frac{d^*(g(z), g(w))}{d^*(z, w)} &= \sup_{z, w \in B^n, z \neq w} \frac{d^*(f_p(z), f_p(w)))}{d^*(z, w)} \\
&= \exp(d(0, f_p(0))) = \exp(d(u(0), uf_p(0))) \\
&= \exp(d(0, g(0))) .
\end{aligned}$$

Thus our theorem is proved.

Now we set $K = \exp(d(0, g(0)))$. It follows from Proposition 1.1 that $\|g\|^2 = \|I\|^2 + (K - 1/K)^2$. If $\|g\|^2 = \|I\|^2$, then $K = 1$. This equality together with (1) implies that the absolute value of the (1, 1)-component of g is 1.

Hence it follows from (1), (2), (3) and (4) that $g \in U(1; C) \times U(n; C)$. Assume that g is an element of $U(1; C) \times U(n; C)$. Then we see that $\|g\|^2 = \|I\|^2$. Thus we have

PROPOSITION 1.5 *The following statements are equivalent to one another*:
1) $\|g\|^2 = \|I\|^2$;
2) $g \in U(1; C) \times U(n; C)$;
3) $g(0) = 0$.

Elements of $U(1, n; C)$ are classified into three types by S. S. Chen and L. Greenberg [3]. We shall discuss the properties of these types.

DEFINITION 1.6. Let g be an element of $U(1, n; C)$ which is not the identity. We shall call g *elliptic* if it has a fixed point in B^n and g *parabolic* if it has exactly one fixed point and this lies on ∂B^n. An element g will be called *loxodromic* if it has exactly two fixed points and they lie on ∂B^n. If g is conjugate to an element having the form

$$\begin{bmatrix} \lambda \cosh t & \lambda \sinh t & 0 \\ \lambda \text{ sinft } t & \lambda \cosh t & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \qquad (\lambda = \pm 1, t \in \mathbf{R} - \{0\}),$$

then g is called *hyperbolic*. Hyperbolic elements are special kinds of loxodromic elements.

Now we state properties of each kind of element.

PROPOSITION 1.7 ([3; Proposition 3.2.1]). *Let g be an elliptic element in $U(1, n; C)$. Then*:

(a) *g is conjugate to an element in $U(1; C) \times U(n; C)$.*

(b) *g is semisimple with eigenvalues of absolute value* 1.

PROPOSITION 1.8 (cf. [3; Proposition 3.2.3]). *Let g be a loxodromic element of $U(1, n; C)$. Then*:

(a) *There exist a unique hyperbolic element h and a unique elliptic element e such that $g = he = eh$.*

(b) *Any element in $U(1, n; C)$ which commutes with g also commutes with h and e.*

(c) *g is semisimple with exactly $n - 1$ eigenvalues of absolute value* 1.

(d) *g leaves the geodesic connecting the two fixed points, invariant. We call this geodesic the axis of g and denote it by A_g.*

(e) *g moves every point z in A_g the same distance $T(g) = d(z, g(z))$. This $T(g)$ is called the translation length of g.*

(f) $T(g) = \min_{z \in B^n} d(z, g(z))$.

PROOF. Since (a), (b) and (c) are proved in [3; Proposition 3.2.3], we have only to prove (d), (e) and (f).

(d) Using [3; Proposition 2.1.2], we may assume that the fixed points of g are $(1, 0, \dots, 0)$ and $(-1, 0, \dots, 0)$. By [3; Lemma 3.2.2], g has the form

$$\begin{bmatrix} c\lambda & s\lambda & 0 \\ s\lambda & c\lambda & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $c = \cosh t$, $s = \sinh t$ for some $t \in \boldsymbol{R} - \{0\}$, $|\lambda| = 1$ and $A \in U(n-1; \boldsymbol{C})$. Let $\{e_0, e_1, \dots, e_n\}$ be the standard basis in V. Let $X = e_0\boldsymbol{R} + e_1\boldsymbol{R}$. Since $g(z^*) = ((cz_0^* + sz_1^*)\lambda, (sz_0^* + cz_1^*)\lambda, 0, \dots, 0)$ for $z^* = (z_0^*, z_1^*, 0, \dots, 0)$ in $X \cap V_-$, $\pi(g(z^*))$ is contained in the geodesic $\pi(X \cap V_-)$ (see [3; Proposition 2.4.3]).

(e) A direct computation shows that

$$\begin{aligned} d(z, g(z)) &= \cosh^{-1} [|(-z_0^{*2} + z_1^{*2})c\lambda|\{(-z_0^{*2} + z_1^{*2})^2\}^{-1/2}] \\ &= \cosh^{-1} c \end{aligned}$$

for $z \in A_g$.

(f) Let $z^* = (z_0^*, z_1^*, \dots, z_n^*)$ and let $w^* = g(z^*)$. We shall show that $\min_{z^* \in V_-} d(\pi(z^*), \pi(w^*)) = \cosh^{-1} c$. As Φ is invariant under $U(1, n; \boldsymbol{C})$, $\Phi(z^*, z^*) = \Phi(w^*, w^*)$. Therefore it suffices to prove that $|\Phi(z^*, w^*)| \geqq c|\Phi(z^*, z^*)|$.

Let $A = (a_{ij})_{i,j=2,3,\dots,n}$. Noting that $A \in U(n-1; \boldsymbol{C})$, we obtain

$$\tag{6} \begin{aligned} |\textstyle\sum_{k=2}^n \overline{z_k^*}(\sum_{j=2}^n a_{kj}z_j^*)| &\leqq (\textstyle\sum_{k=2}^n |z_k^*|^2)^{1/2}(\sum_{k=2}^n |\sum_{j=2}^n a_{kj}z_j^*|^2)^{1/2} \\ &= \textstyle\sum_{k=2}^n |z_k^*|^2 . \end{aligned}$$

It is seen that

$$\tag{7} \begin{aligned} &|c(\overline{z_0^*}\lambda z_0^* - \overline{z_1^*}\lambda z_1^*) + s(\overline{z_0^*}\lambda z_1^* - \overline{z_1^*}\lambda z_0^*)|^2 - \{c(|z_0^*|^2 - |z_1^*|^2)\}^2 \\ &= |c(|z_0^*|^2 - |z_1^*|^2) + 2si \operatorname{Im}(\overline{z_0^*}z_1^*)|^2 - \{c(|z_0^*|^2 - |z_1^*|^2)\}^2 \\ &= 4s^2\{\operatorname{Im}(\overline{z_0^*}z_1^*)\}^2 \geqq 0 . \end{aligned}$$

Using (6) and (7), we have

$$\begin{aligned} &|\Phi(z^*, w^*)| \\ &= |-\overline{z_0^*}(c\lambda z_0^* + s\lambda z_1^*) + \overline{z_1^*}(s\lambda z_0^* + c\lambda z_1^*) + \textstyle\sum_{k=2}^n \overline{z_k^*}(\sum_{j=2}^n a_{kj}z_j^*)| \\ &\geqq |c(\overline{z_0^*}\lambda z_0^* - \overline{z_1^*}\lambda z_1^*) + s(\overline{z_0^*}\lambda z_1^* - \overline{z_1^*}\lambda z_0^*)| - |\textstyle\sum_{k=2}^n \overline{z_k^*}(\sum_{j=2}^n a_{kj}z_j^*)| \\ &\geqq c(|z_0^*|^2 - |z_1^*|^2) - \textstyle\sum_{k=2}^n |z_k^*|^2 . \end{aligned}$$

This implies that

$$\begin{aligned}
&|\Phi(z^*, w^*)| - c|\Phi(z^*, z^*)| \\
&\geqq c(|z_0^*|^2 - |z_1^*|^2) - \textstyle\sum_{k=2}^n |z_k^*|^2 - c(|z_0^*|^2 - |z_1^*|^2 - \sum_{k=2}^n |z_k^*|^2) \\
&= (c-1)\textstyle\sum_{k=2}^n |z_k^*|^2 \geqq 0 .
\end{aligned}$$

Thus $\min_{z^* \in V_-} d(\pi(z^*), \pi(g(z^*))) = \cosh^{-1} c$.

To discuss some properties of unitary transformations, it may be more convenient to use another matrix representation for $U(1, n; C)$. By changing the basis of V, we introduce the group $\tilde{U}(1, n; C)$ as follows.

Let

$$D = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix}$$

and define $\tilde{U}(1, n; C)$ by $D^{-1}U(1, n; C)D$. We see that $\tilde{U}(1, n; C)$ is the group of linear transformations which leave $D^{-1}(V_-)$ invariant and that $\tilde{U}(1, n; C)$ is the automorphism group of the Hermitian form

$$\tilde{\Phi}(z^*, w^*) = -(\overline{z_0^*}w_1^* + \overline{z_1^*}w_0^*) + \overline{z_2^*}w_2^* + \cdots + \overline{z_n^*}w_n^*$$

defined for $z^*, w^* \in D^{-1}(V)$. We can regard the linear transformation D^{-1} as a mapping of complex unit ball B^n to the domain $\tilde{H}^n = \{z \in C^n | \mathrm{Re}(z_1) > (1/2)\Sigma_{k=2}^n |z_k|^2\}$. The action of $U(1, n; C)$ in B^n is converted by D^{-1} into the action of $\tilde{U}(1, n; C)$ in $\tilde{H}^n$. The distance $\tilde{d}(z, w)$ for $z, w \in \tilde{H}^n$ is defined by

$$\tilde{d}(z, w) = \cosh^{-1} [|\tilde{\Phi}(z^*, w^*)|\{\tilde{\Phi}(z^*, z^*)\tilde{\Phi}(w^*, w^*)\}^{-1/2}] ,$$

where $z^* \in \pi^{-1}(z)$ and $w^* \in \pi^{-1}(w)$. We note that $\tilde{d}(z, w) = d(D(z), D(w))$ for $z, w \in \tilde{H}^n$.

Let $g = (a_{ij})_{i,j=1,2,\ldots,n+1}$ be an element of $\tilde{U}(1, n; C)$. Noting that

$$\bar{g}^T \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} g = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},$$

we see that

(8) $$-2\,\mathrm{Re}\,(\overline{a_{11}}a_{12}) + \textstyle\sum_{k=3}^{n+1} |a_{1k}|^2 = 0 ,$$

(9) $$-2\,\mathrm{Re}\,(\overline{a_{21}}a_{22}) + \textstyle\sum_{k=3}^{n+1} |a_{2k}|^2 = 0 ,$$

(10) $$-2\,\mathrm{Re}\,(\overline{a_{i1}}a_{i2}) + \textstyle\sum_{k=3}^{n+1} |a_{ik}|^2 = 1 \qquad \text{for } i = 3, 4, \ldots, n+1 ,$$

$$-2\,\mathrm{Re}\,(\overline{a_{11}}a_{21}) + \textstyle\sum_{k=3}^{n+1} |a_{k1}|^2 = 0\,, \tag{11}$$

$$-2\,\mathrm{Re}\,(\overline{a_{12}}a_{22}) + \textstyle\sum_{k=3}^{n+1} |a_{k2}|^2 = 0\,, \tag{12}$$

$$-(\overline{a_{11}}a_{22} + \overline{a_{21}}a_{12}) + \textstyle\sum_{k=3}^{n+1} \overline{a_{k1}}a_{k2} = -1\,. \tag{13}$$

DEFINITION 1.9. Let g be an element of $\tilde{U}(1, n; \boldsymbol{C})$ which is not the identity. We shall call g *elliptic* if it has a fixed point in $\tilde{H}^n$ and g *parabolic* if it has exactly one fixed point and this lies on $\partial\tilde{H}^n$. A unipotent parabolic element will be called *strictly parabolic* and in particular the element which is conjugate to an element having the form

$$\begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \qquad (s \neq 0 \text{ and } \mathrm{Re}\,(s) = 0)\,,$$

will be called a *translation*. An element g will be called *loxodromic* if it has exactly two fixed points and they lie on $\partial\tilde{H}^n$. If g is conjugate to an element having the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \qquad (t \in \boldsymbol{R} - \{0, 1\})\,,$$

then g is called *hyperbolic*.

PROPOSITION 1.10 ([3; Proposition 3.4.1]). *Let g be a parabolic element in $\tilde{U}(1, n; \boldsymbol{C})$.*

(a) *There exist a unique strictly parabolic element p and a unique elliptic element e such that $g = pe = ep$.*

(b) *Any element of $\tilde{U}(1, n; \boldsymbol{C})$ which commutes with g also commutes with p and e.*

(c) *g is not semisimple. All absolute values of the eigenvalues of g are* 1.

PROPOSITION 1.11. *Let f_1 and f_2 be elements of $\tilde{U}(1, n; \boldsymbol{C})$. Assume that these two elements have one and only one common fixed point and it lies on $\partial\tilde{H}^n$. Then the commutator g of f_1 and f_2 is either elliptic, parabolic or the identity. However, if both elements f_1 and f_2 are elliptic, or, if at least one element of f_1 and f_2 is loxodromic, then g can not be the identity.*

PROOF. We may assume that the common fixed point is ∞. Then the forms of $f_i (i = 1, 2)$ are as follows:

$$f_i = \begin{bmatrix} \xi_i & 0 & 0 \\ s_i & \eta_i & b_i \\ a_i & 0 & A_i \end{bmatrix},$$

where a_i, b_i, A_i are $(n-1)\times 1$, $1\times(n-1)$, $(n-1)\times(n-1)$ matrices respectively, $\bar{\xi}_i\eta_i = 1$, $\mathrm{Re}(\bar{\xi}_i s_i) = (1/2)\|a_i\|^2$, $b_i = \eta_i \bar{a}_i^T A_i$ and $A_i \in U(n-1; C)$. The commutator g of f_1 and f_2 is of the form

$$g = f_1 f_2 f_1^{-1} f_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & F \\ G & 0 & A_1 A_2 A_1^{-1} A_2^{-1} \end{bmatrix}.$$

This implies that all absolute values of the eigenvalues of g equal 1. By (b) in Proposition 1.7, (c) in Proposition 1.8 and (c) in Proposition 1.10, g is either elliptic, parabolic or the identity.

Next let f_1 and f_2 be elliptic elements of $\tilde{U}(1, n; C)$. We may assume that the common fixed point is ∞ and another fixed point of f_2 is 0. Then in the form of f_1, $\xi_1 = \eta_1$ and $a_1 \neq 0$. In the element f_2, $\xi_2 = \eta_2$, $s_2 = 0$, $a_2 = 0$ and $b_2 = 0$. Therefore we see that $a_{21} = \bar{a}_1^T(I_{n-1} - \xi_2^{-1}A_1A_2A_1^{-1})a_1$ in the commutator g. Suppose that g is the identity. Then $a_{21} = \bar{a}_1^T(I_{n-1} - \xi_2^{-1}A_1A_2A_1^{-1})a_1 = 0$ and $A_1A_2A_1^{-1}A_2^{-1} = I_{n-1}$. It follows that $A_2 = \xi_2 I_{n-1}$. This implies that f_2 is the identity. This is a contradiction. Thus g is not the identity.

Lastly let f_1 be a loxodromic element with fixed points α and ∞. If the commutator g is the identity, then $f_1 f_2 = f_2 f_1$. We see that $f_1 f_2(\alpha) = f_2 f_1(\alpha) = f_2(\alpha)$ and $f_2(\alpha)$ is a fixed point of f_1. Then either $f_2(\alpha) = \infty$ or $f_2(\alpha) = \alpha$. The former does not occur. In the latter case f_1 and f_2 have two fixed points in common. This contradicts our assumption. Hence if f_1 is loxodromic, then the commutator g of f_1 and f_2 is not the identity.

REMARK 1.12. The following table describes all the possible type for $g = f_1 f_2 f_1^{-1} f_2^{-1}$. There exist examples that demonstrate the table.

f_2 \ f_1	E	P	L
E	E, P	E, P, I	E, P
P	E, P, I	E, P, I	E, P
L	E, P	E, P	E, P

(The symbols E, P, L and I denote elliptic, parabolic, loxodromic type and the identity, respectively.)

We shall consider the displacement function $z \to \sinh^2 \tilde{d}(z, g(z))$ for an element g of $\tilde{U}(1, n; C)$. Before stating our proposition, we distinguish between the fixed points α, β of a loxodromic element g in $\tilde{U}(1, n; C)$. If $\lim_{k\to\infty} g^k(z) = \alpha$ for a point $z \in \tilde{H}^n$, then α is called an *attracting fixed point* of g. This definition does not depend on the choice of z. For a loxodromic element g we can define the *axis* $\tilde{A}_g$ and the *translation length* $\tilde{T}(g)$ in the same manner as in Proposition 1.8.

PROPOSITION 1.13 (cf. [1; Theorem 7.35.1]).

(a) *Suppose that g is a hyperbolic element of $\tilde{U}(1, n; \boldsymbol{C})$. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be the attracting fixed point of g. We denote the shortest distance from a point z in $\tilde{H}^n$ to the axis $\tilde{A}_g$ by $\tilde{d}(z, \tilde{A}_g)$. Let $z^* = (1, z_1, \dots, z_n) \in \pi^{-1}(z)$ and $\alpha^* = (1, \alpha_1, \alpha_2, \dots, \alpha_n) \in \pi^{-1}(\alpha)$. Set $k = -\mathrm{Re}\,(\tilde{\Phi}(\alpha^*, z^*))/|\tilde{\Phi}(\alpha^*, z^*)|$. Then*

$$\begin{aligned}\sinh^2 \tilde{d}(z, g(z)) = 8(1+k)^{-2} \cosh^2 \tilde{d}(z, \tilde{A}_g) \sinh^2 (1/2)\tilde{T}(g) \\ \times \{2\cosh^2 (1/2)\tilde{T}(g) \cosh^2 \tilde{d}(z, \tilde{A}_g) \\ - 2k^2 \cosh^2 \tilde{d}(z, \tilde{A}_g) + k + k^2\}\,.\end{aligned}$$

(b) *If g is a translation with a fixed point ζ, then $\sinh^2 \tilde{d}(z, g(z)) \{\tilde{P}(z, \zeta)\}^{2/n}$ is constant, where $\tilde{P}(z, \zeta)$ is the Poisson kernel defined by*

$$\tilde{P}(z, \zeta) = \begin{cases} |\tilde{\Phi}(z^*, z^*)|^n & \text{if } \zeta = \infty\,,\ z^* = (1, z_1, z_2, \dots, z_n) \in \pi^{-1}(z)\,; \\ \{|\tilde{\Phi}(z^*, z^*)|\,|\tilde{\Phi}(z^*, \zeta^*)|^{-2}\}^n & \text{if } \zeta \neq \infty\,,\quad z^* = (1, z_1, z_2, \dots, z_n) \in \\ & \pi^{-1}(z),\ \zeta^* = (1, \zeta_1, \zeta_2, \dots, \zeta_n) \in \pi^{-1}(\zeta)\,. \end{cases}$$

PROOF. (a) Without loss of generality, we may assume that

$$g = \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},$$

where $a > 1$. Then it is seen that $\tilde{A}_g = \{w = (t, 0, \dots, 0) \in \tilde{H}^n \,|\, t > 0\}$ and $\tilde{T}(g) = \log a$. We shall compute $\tilde{d}(z, \tilde{A}_g)$. Let $w^* \in \pi^{-1}(w) \subset \pi^{-1}(\tilde{A}_g)$. We see that

$$\begin{aligned}\tilde{d}(z, \tilde{A}_g) &= \min_{\pi(w^*) \in \tilde{A}_g} \{\cosh^{-1} [|\tilde{\Phi}(z^*, w^*)| \{\tilde{\Phi}(z^*, z^*) \tilde{\Phi}(w^*, w^*)\}^{-1/2}]\} \\ &= \min_{t > 0} \{\cosh^{-1} [|t + z_1| \{2t(2\,\mathrm{Re}\,(z_1) - \textstyle\sum_{j=2}^n |z_j|^2)\}^{-1/2}]\} \\ &= \cosh^{-1} [\{(|z_1| + \mathrm{Re}\,(z_1))(2\,\mathrm{Re}\,(z_1) - \textstyle\sum_{j=2}^n |z_j|^2)^{-1}\}^{1/2}]\,.\end{aligned}$$

Write $W = 2\,\mathrm{Re}\,(z_1) - \Sigma_{j=2}^n |z_j|^2$ and let $z_1 = |z_1| e^{i\theta}$. We note that

$$|z_1| = (1 + \cos\theta)^{-1} W \cosh^2 \tilde{d}(z, \tilde{A}_g)\,,$$

$$\textstyle\sum_{j=2}^n |z_j|^2 = W \sinh^2 \tilde{d}(z, \tilde{A}_g) - (1 - \cos\theta)(1 + \cos\theta)^{-1} W \cosh^2 \tilde{d}(z, \tilde{A}_g)\,,$$

$$4 \sinh^2 \tilde{T}(g) = (a - 1/a)^2\,,$$

$$4 \sinh^2 (1/2)\tilde{T}(g) = a + 1/a - 2\,.$$

From the above equalities it follows that

$$
\begin{aligned}
&\sinh^2 \tilde{d}(z, g(z)) \\
&\quad = W^{-2}|-(a\overline{z_1} + (1/a)z_1) + \textstyle\sum_{j=2}^n |z_j|^2|^2 - 1 \\
&\quad = W^{-2}\{|z_1|^2(a - 1/a)^2 - 2|z_1|(\textstyle\sum_{j=2}^n |z_j|^2)(a + 1/a - 2)\cos\theta\} \\
&\quad = 8(1 + \cos\theta)^{-2}\cosh^2 \tilde{d}(z, \tilde{A}_g)\sinh^2 (1/2)\tilde{T}(g)\{2\cosh^2 (1/2)\tilde{T}(g)\cosh^2 \tilde{d}(z, \tilde{A}_g) \\
&\qquad - 2\cos^2\theta\cosh^2 \tilde{d}(z, \tilde{A}_g) + \cos\theta + \cos^2\theta\}\,.
\end{aligned}
$$

Noting that $\cos\theta = k$, we have the desired equality.

(b) Let g be a translation with a fixed point $\zeta \in \partial\tilde{H}^n$. There is an element $f = (a_{ij})_{i,j=1,2,\dots,n+1}$ of $\tilde{U}(1, n; C)$ such that $h = fgf^{-1}$ has the form

$$
h = \begin{bmatrix} 1 & 0 & 0 \\ t\sqrt{-1} & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},
$$

where $t \in \mathbf{R} - \{0\}$. It follows that $gf^{-1} = f^{-1}h$ and hence $gf^{-1}(\infty) = f^{-1}h(\infty) = f^{-1}(\infty)$. Thus $f^{-1}(\infty)$ is a fixed point of g. Since ζ is the only fixed point of g, $f^{-1}(\infty) = \zeta$ so that $f(\zeta) = \infty$. Hence $a_{k1} + a_{k2}\zeta_1 + \cdots + a_{k,n+1}\zeta_n = 0$ for $k \neq 2$ and $\neq 0$ for $k = 2$. Write $f(z) = w = (w_1, w_2, \dots, w_n)$ and let $f(z)^* = (1, w_1, w_2, \dots, w_n) \in \pi^{-1}(f(z))$. Then

$$
\begin{aligned}
|\tilde{\Phi}(f(z^*), f(z^*))| &= |(f(z^*))_0|^2|\tilde{\Phi}(f(z)^*, f(z)^*)|\,, \\
|\tilde{\Phi}(f(z^*), f(\zeta^*))| &= |(f(z^*))_0||a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\{\tilde{P}(z, \zeta)\}^{2/n} &= |\tilde{\Phi}(z^*, z^*)|^2|\tilde{\Phi}(z^*, \zeta^*)|^{-4} \\
&= |\tilde{\Phi}(f(z^*), f(z^*))|^2|\tilde{\Phi}(f(z^*), f(\zeta^*))|^{-4} \\
&= |\tilde{\Phi}(f(z)^*, f(z)^*)|^2|a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|^{-4} \\
&= \{\tilde{P}(w, \infty)\}^{2/n}|a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|^{-4}\,.
\end{aligned}
$$

By using this equality, we have

$$
\begin{aligned}
&\sinh^2 \tilde{d}(z, g(z))\{\tilde{P}(z, \zeta)\}^{2/n} \\
&\quad = \sinh^2 \tilde{d}(z, f^{-1}hf(z))\{\tilde{P}(z, \zeta)\}^{2/n} \\
&\quad = \sinh^2 \tilde{d}(f(z), hf(z))\{\tilde{P}(z, \zeta)\}^{2/n} \\
&\quad = \sinh^2 \tilde{d}(w, h(w))\{\tilde{P}(w, \infty)\}^{2/n}|a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|^{-4}
\end{aligned}
$$

$$= \{2 \operatorname{Re}(w_1) - \textstyle\sum_{k=2}^{n} |w_k|^2\}^{-2} t^2 \{2 \operatorname{Re}(w_1) - \textstyle\sum_{k=2}^{n} |w_k|^2\}^2$$
$$\times |a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|^{-4}$$
$$= t^2 |a_{21} + a_{22}\zeta_1 + \cdots + a_{2,n+1}\zeta_n|^{-4} .$$

Thus $\sinh^2 \tilde{d}(z, g(z)) \{\tilde{P}(z, \zeta)\}^{2/n}$ is equal to a constant which does not depend on z.

REMARK 1.14. If g is a hyperbolic element of $\tilde{U}(1, 1; \boldsymbol{C})$, then we have

$$\sinh \tilde{d}(z, g(z)) = \cosh 2\tilde{d}(z, \tilde{A}_g) \sinh \tilde{T}(g) .$$

If g is strictly parabolic and of the form

$$g = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a}^T \\ a & 0 & I_{n-1} \end{bmatrix},$$

where $\operatorname{Re}(s) = (1/2)\|a\|^2$, then $\sinh^2 \tilde{d}(z, g(z)) \{\tilde{P}(z, \zeta)\}^{2/n}$ is not necessarily constant.

2. Elements of discrete subgroups of $U(1, n; \boldsymbol{C})$ and $\tilde{U}(1, n; \boldsymbol{C})$

First we quote one theorem from [5].

THEOREM 2.1 ([5; Theorem 3.2]). *Let G be a discrete subgroup of $\tilde{U}(1, n; \boldsymbol{C})$. Assume that g is a translation of G having the form*

$$g = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},$$

where $s \neq 0$ and $\operatorname{Re}(s) = 0$. If $f = (a_{ij})_{i,j=1,2,\dots,n+1}$ is an element of G, then either $a_{12} = 0$ or $|a_{12}| \geqq |s|^{-1}$.

Using this theorem, we shall show the existence of a domain where the action of G is equal to the action of the cyclic group generated by g.

THEOREM 2.2. *Let G, g and s be the same as in Theorem 2.1. Assume that the stabilizer $G_\infty = \{h \in G \mid h(\infty) = \infty\}$ is generated by g. Let Σ be the set $\{z \in \tilde{H}^n \mid \operatorname{Re}(z_1) > (1/2) \Sigma_{k=2}^{n} |z_k|^2 + |s|\}$. Then*

$$f(\Sigma) = \Sigma \quad \text{if} \quad f \in G_\infty ,$$
$$f(\Sigma) \cap \Sigma = \varnothing \quad \text{if} \quad f \in G - G_\infty .$$

PROOF. Assume that f is an element of G_∞ and that z is a point in Σ. Noting that f has the form

$$\begin{bmatrix} 1 & 0 & 0 \\ ms & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix} \qquad (m \in \boldsymbol{Z}),$$

we see that $f(z) = (z_1 + ms, z_2, \dots, z_n)$. It follows that

$$\operatorname{Re}(z_1 + ms) = \operatorname{Re}(z_1) > (1/2) \textstyle\sum_{k=2}^n |z_k|^2 + |s| .$$

Thus $\Sigma \supset f(\Sigma)$. If we replace f by f^{-1}, then we have $f(\Sigma) = \Sigma$.

Next we suppose that $f = (a_{ij})_{i,j=1,2,\dots,n+1}$ is an element of $G - G_\infty$. Then $a_{12} \neq 0$ so that $|a_{12}| \geqq |s|^{-1}$ by Theorem 2.1. Take $z \in \Sigma$ and $z^* = (1, z_1, z_2, \dots, z_n) \in \pi^{-1}(z)$. Write $f(z^*) = (w_0^*, w_1^*, \dots, w_n^*)$ and let $t = \operatorname{Re}(z_1) - (1/2)\Sigma_{k=2}^n |z_k|^2$. Then $t > |s|$. Noting that $\tilde{\Phi}(z^*, z^*) = \tilde{\Phi}(f(z^*), f(z^*))$, we have

$$\operatorname{Re}(w_1^*/w_0^*) = |w_0^*|^{-2} \operatorname{Re}(\overline{w_0^*} w_1^*) = (1/2) \textstyle\sum_{k=2}^n |w_k^*/w_0^*|^2 + |w_0^*|^{-2} t .$$

It follows from (8) in Section 1 that

$$\begin{aligned}
&|a_{11}a_{12}{}^{-1} + z_1 + a_{13}a_{12}{}^{-1}z_2 + \cdots + a_{1,n+1}a_{12}{}^{-1}z_n| - t \\
&\quad \geqq \operatorname{Re}(a_{11}a_{12}{}^{-1}) + \operatorname{Re}(z_1) + \operatorname{Re}(a_{13}a_{12}{}^{-1}z_2) + \cdots \\
&\qquad + \operatorname{Re}(a_{1,n+1}a_{12}{}^{-1}z_n) - \operatorname{Re}(z_1) + (1/2) \textstyle\sum_{k=2}^n |z_k|^2 \\
&\quad = |a_{12}|^{-2} \operatorname{Re}(a_{11}\overline{a_{12}}) + \operatorname{Re}(a_{13}a_{12}{}^{-1}z_2) + \cdots + \operatorname{Re}(a_{1,n+1}a_{12}{}^{-1}z_n) \\
&\qquad + (1/2) \textstyle\sum_{k=2}^n |z_k|^2 \\
&\quad = |a_{12}|^{-2}\{(1/2)(|a_{13}|^2 + |a_{14}|^2 + \cdots + |a_{1,n+1}|^2)\} \\
&\qquad + \operatorname{Re}(a_{13}a_{12}{}^{-1}z_2) + \cdots + \operatorname{Re}(a_{1,n+1}a_{12}{}^{-1}z_n) + (1/2) \textstyle\sum_{k=2}^n |z_k|^2 \\
&\quad = (1/2)(|\overline{z_2} + a_{13}a_{12}{}^{-1}|^2 + |\overline{z_3} + a_{14}a_{12}{}^{-1}|^2 + \cdots + |\overline{z_n} + a_{1,n+1}a_{12}{}^{-1}|^2) \geqq 0 .
\end{aligned}$$

Therefore

$$\begin{aligned}
|w_0^*|^2 &= |a_{12}|^2 |a_{11}a_{12}{}^{-1} + \cdots + a_{1,n+1}a_{12}{}^{-1}z_n|^2 \\
&\geqq |a_{12}|^2 t^2 \geqq |s|^{-2} t^2 \geqq |s|^{-1} t .
\end{aligned}$$

It follows that $\operatorname{Re}(w_1^*/w_0^*) \leqq (1/2)\Sigma_{k=2}^n |w_k^*/w_0^*|^2 + |s|$, which shows $f(z) \notin \Sigma$. Thus, if $f \in G - G_\infty$, then $f(\Sigma) \cap \Sigma = \varnothing$.

PROPOSITION 2.3 (cf. [1; Theorem 5.4.3]). *Let G, g and s be the same as in Theorem 2.1. Let $f = (a_{ij})_{i,j=1,2,\dots,n+1}$ be an element of $\tilde{U}(1, n; \boldsymbol{C})$ such that $f(\infty) \neq \infty$. Suppose that the group $\langle f, g \rangle$ generated by f and g is discrete.*

Then:

(a) $\|f - I\| \|g - I\| \geqq 1$.

(b) *If f is strictly parabolic, then* $\sinh \tilde{d}(e, f(e)) \sinh \tilde{d}(e, g(e)) \geqq 1/4$, *where* $e = (1, 0, \ldots, 0) \in \tilde{H}^n$.

If f is of the form

$$\begin{bmatrix} 1 & s^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix},$$

then the equalities are satisfied in (a) *and* (b).

To prove Proposition 2.3 we need a lemma.

LEMMA 2.4 (cf. [Proposition 1.1]). *For* $g \in \tilde{U}(1, n; \boldsymbol{C})$,

$$\|g\|^2 = \|I\|^2 + 4 \sinh^2 \tilde{d}(e, g(e)).$$

PROOF. Let $g = (a_{ij})_{i,j=1,2,\ldots,n+1} \in \tilde{U}(1, n; \boldsymbol{C})$. By making use of (11), (12) and (13) in Section 1, we obtain

$$\tilde{d}(e, g(e)) = \cosh^{-1} (1/2)|a_{11} + a_{12} + a_{21} + a_{22}|.$$

From this it follows that

$$4 \sinh^2 \tilde{d}(e, g(e)) = |a_{11} + a_{12} + a_{21} + a_{22}|^2 - 4.$$

Using (8), (9), (10), (11), (12) and (13), we see that

$$\begin{aligned} \|g\|^2 &= |a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 + 2 \operatorname{Re} (\overline{a_{11}} a_{12}) \\ &\quad + 2 \operatorname{Re} (\overline{a_{11}} a_{21}) + 2 \operatorname{Re} (\overline{a_{11}} a_{22}) + 2 \operatorname{Re} (\overline{a_{12}} a_{21}) \\ &\quad + 2 \operatorname{Re} (\overline{a_{12}} a_{22}) + 2 \operatorname{Re} (\overline{a_{21}} a_{22}) + n - 3 \\ &= 4 \sinh^2 \tilde{d}(e, g(e)) + 4 + n - 3 \\ &= 4 \sinh^2 \tilde{d}(e, g(e)) + \|I\|^2. \end{aligned}$$

PROOF OF PROPOSITION 2.3. Since $\|g - I\| = |s|$ and $\|f - I\|^2 \geqq |a_{12}|^2 \neq 0$, it follows from Theorem 2.1 that $\|f - I\| \|g - I\| \geqq 1$. Assume that all eigenvalues of the element f are 1. By Lemma 2.4,

$$\begin{aligned} \|f - I\|^2 &= \|f\|^2 + \|I\|^2 - 2 \textstyle\sum_{i=1}^{n+1} \operatorname{Re} (a_{ii}) \\ &= \|f\|^2 + \|I\|^2 - 2(n + 1) = \|f\|^2 - \|I\|^2 \\ &= 4 \sinh^2 \tilde{d}(e, f(e)). \end{aligned}$$

Therefore $\|g - I\| = |s| = 2 \sinh \tilde{d}(e, g(e))$ implies $\sinh \tilde{d}(e, f(e)) \sinh \tilde{d}(e, g(e)) \geqq 1/4$.

It is easy to show that the equalities are satisfied if

$$f = \begin{bmatrix} 1 & s^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{bmatrix}.$$

Next we shall consider loxodromic elements of a discrete subgroup.

THEOREM 2.5. *Let G be a discrete subgroup of $U(1, n; \boldsymbol{C})$. Let f and g be elements of G. Suppose that f is loxodromic and that f and g have fixed point sets $\{x, y\}$ and $\{x', y'\}$, respectively in ∂B^n. Then either these sets are disjoint or they are identical. Moreover, if the latter occurs, then there is an integer m such that $f^m g = g f^m$.*

PROOF. Assume that f and g have only one fixed point, say $x \in \partial B^n$, in common. It follows from [5; Theorem 3.1] that the subgroup $\langle f, g\rangle$ generated by f and g is not discrete. Hence $\{x, y\} = \{x', y'\}$ or $\{x, y\} \cap \{x', y'\} = \varnothing$. Without loss of generality, we may assume that $\{x, y\} = \{(1, 0, \dots, 0), (-1, 0, \dots, 0)\}$. If $\{x, y\} = \{x', y'\}$, then it follows from (1), (2), (3), (4) and (5) in Section 1 that f and g are of the form

$$f = \begin{bmatrix} \lambda \cosh t & \lambda \sinh t & 0 \\ \lambda \sinh t & \lambda \cosh t & 0 \\ 0 & 0 & A \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} \mu \cosh s & \mu \sinh s & 0 \\ \mu \sinh s & \mu \cosh s & 0 \\ 0 & 0 & B \end{bmatrix},$$

where $|\lambda| = 1$, $|\mu| = 1$, $t, s \in \boldsymbol{R}$ and $A, B \in U(n-1; \boldsymbol{C})$. Therefore

$$f^j g f^{-j} g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A^j B A^{-j} B^{-1} \end{bmatrix} \qquad (j \in \boldsymbol{Z}).$$

Let $F = \{f^j g f^{-j} g^{-1} \mid j \in \boldsymbol{Z}\}$. Assume that F is an infinite set. Noting that $U(n-1; \boldsymbol{C})$ is compact, we see that there exists a sequence $\{h_k\}$ of different elements of F which converges to some element h of $U(1, n; \boldsymbol{C})$. Since $\lim_{k\to\infty} h_k(z) = h(z)$ for $z \in B^n$, G is not discontinuous at $h(z)$ in B^n. This is a contradiction. Hence F is a finite set, so $A^m B A^{-m} B^{-1} = I_{n-1}$ for some integer m. Thus $f^m g = g f^m$.

For the remainder of this section G denotes a discrete subgroup of $U(1, n; \boldsymbol{C})$. We shall give the definition of G-duality.

DEFINITION 2.6. Let x and y be any two not necessarily distinct points in ∂B^n. If there exists a sequence $\{g_k\}$ of elements of G such that $\lim_{k\to\infty} g_k(p) = x$

and $\lim_{k\to\infty} g_k^{-1}(p) = y$ for any point p in B^n, then we say that x and y are G-dual and denote this duality by $x \sim y$.

PROPOSITION 2.7. *Two points x and y in ∂B^n are G-dual if and only if there exists an element g of G such that $g(\overline{B^n} - V) \subset U$, where U (resp. V) is any open neighborhood of x (resp. y) in $\overline{B^n}$.*

We need the following lemma for the proof.

LEMMA 2.8. *Let ε be any positive number. If $d^*(z, w)^2 < \varepsilon$ for $z = (z_1, z_2, \dots, z_n)$, $w = (w_1, w_2, \dots, w_n) \in B^n$, then $\|z - w\|^2 = \Sigma_{i=1}^n |z_i - w_i|^2 < 2\varepsilon$.*

PROOF. If $z = w = 0$, then $\|z - w\|^2 = 0$. Hence we may assume that one of z and w, say z, is not zero. Without loss of generality, we may assume that $z = (r, 0, \dots, 0)$, where $0 < r < 1$. Noting that $\inf\{\mathrm{Re}(w_1) | d^*(z, w)^2 < \varepsilon\} = (1 - \varepsilon)r^{-1}$, we see that

$$\begin{aligned}\|z - w\|^2 &= |r - w_1|^2 + \textstyle\sum_{i=2}^n |w_i|^2 = r^2 - 2r\,\mathrm{Re}(w_1) + \|w\|^2 \\ &\leqq r^2 - 2r\,\mathrm{Re}(w_1) + 1 < r^2 - 2r(1 - \varepsilon)r^{-1} + 1 \\ &= 2\varepsilon - (1 - r^2) < 2\varepsilon\,.\end{aligned}$$

Let us go back to the proof of Proposition 2.7. We shall prove that if part first. Let U_k (resp. V_k) be a sequence of open neighborhoods of x (resp. y) in $\overline{B^n}$ such that $U_k \supset \overline{U_{k+1}}$ and $\bigcap_{k\geqq 1} U_k = \{x\}$ (resp. $V_k \supset \overline{V_{k+1}}$ and $\bigcap_{k\geqq 1} V_k = \{y\}$). By our assumption, there exists a sequence $\{g_k\}$ of elements in G such that $g_k(\overline{B^n} - V_k) \subset U_k$ and $g_k^{-1}(\overline{B^n} - U_k) \subset V_k$ for each k. Let p be a point in B^n. If k is sufficiently large, then $p \in (\overline{B^n} - U_k) \cap (\overline{B^n} - V_k)$. Therefore we see that $g_k(p) \in U_k$ and $g_k^{-1}(p) \in V_k$. Thus $g_k(p) \to x$ and $g_k^{-1}(p) \to y$.

Conversely we assume that x and y are G-dual. Let U (resp. V) be an open neighborhood of x (resp. y) in $\overline{B^n}$. By our assumption, there is a sequence $\{g_k\} \subset G$ such that $g_k(0) \to x$ and $g_k^{-1}(0) \to y$ as $k \to \infty$. Since $\lim_{k\to\infty} g_k^{-1}(0) = y$, there exist $\delta > 0$ and an integer $N > 0$ such that $\|g_k^{-1}(0) - z\| > \delta$ for all $z \in \overline{B^n} - V$ and all $k \geqq N$. Fix $z \in \overline{B^n} - V$. Then $d^*(g_k^{-1}(0), z) \geqq \delta/2$ for all $k \geqq N$ by Lemma 2.8.

We can find an element v_k of $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$ which carries $g_k^{-1}(0)$ to $(a_k, 0, \dots, 0)$, where $|a_k| = \|g_k^{-1}(0)\|$. Set $p_k = (a_k, 0, \dots, 0)$. By Theorem 1.2, we have two elements u_k and f_{p_k} which satisfy the following conditions:

1) $g_k v_k^{-1} = u_k f_{p_k}$;
2) $f_{p_k}(p_k) = 0$, $f_{p_k}(0) = p_k$ and $f_{p_k}{}^2 =$ identity:
3) $u_k \in U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$,

where f_{p_k} is defined in the same manner as in the proof of Theorem 1.2.

Write $v_k(z) = (v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)})$. Using $u_k^{-1}(0) = 0$ and the fact that d^* is invariant under $U(1; \boldsymbol{C}) \times U(n; \boldsymbol{C})$, we see that

$$\begin{aligned} d^*(g_k^{-1}(0), z)^2 &= d^*(v_k^{-1} f_{p_k}^{-1} u_k^{-1}(0), z)^2 = d^*(v_k^{-1} f_{p_k}^{-1}(0), z)^2 \\ &= d^*(f_{p_k}^{-1}(0), v_k(z))^2 = d^*(p_k, v_k(z))^2 = |1 - \bar{a}_k v_1^{(k)}| . \end{aligned}$$

Therefore

$$|1 - \bar{a}_k v_1^{(k)}| \geqq \delta^2/4 \qquad \text{for all } k \geqq N . \tag{14}$$

It follows from the proof of Lemma 1.4 that

$$\begin{aligned} d^*(g_k(0), g_k(z))^2 &= d^*(u_k f_{p_k} v_k(0), u_k f_{p_k} v_k(z))^2 \\ &= d^*(f_{p_k}(v_k(0)), f_{p_k}(v_k(z)))^2 \\ &= (1 - |a_k|^2)|1 - \bar{a}_k v_1^{(k)}|^{-1} d^*(v_k(0), v_k(z))^2 \\ &= (1 - |a_k|^2)|1 - \bar{a}_k v_1^{(k)}|^{-1} d^*(0, z)^2 \\ &= (1 - |a_k|^2)|1 - \bar{a}_k v_1^{(k)}|^{-1} . \end{aligned}$$

Using (14), we see that

$$d^*(g_k(0), g_k(z))^2 \leqq 4(1 - \|g_k^{-1}(0)\|^2)\delta^{-2}$$

for all $k \geqq N$. Let $\varepsilon > 0$ be given. Since $\lim_{k \to \infty} g_k^{-1}(0) = y \in \partial B^n$, there exists an integer $M > 0$ such that

$$4(1 - \|g_k^{-1}(0)\|^2)\delta^{-2} < \varepsilon \qquad \text{for all } k \geqq M .$$

Lemma 2.8 implies that

$$\|g_k(0) - g_k(z)\|^2 < 2\varepsilon \qquad \text{for all } k \geqq \max\{N, M\} .$$

Since $\lim_{k \to \infty} g_k(0) = x$, $\{g_k\}$ uniformly converges to x on $\overline{B^n} - V$. Thus $g_k(\overline{B^n} - V) \subset U$ for sufficiently large k.

PROPOSITION 2.9. *Suppose that two points x and y are G-dual. Let U and V be open neighborhoods in $\overline{B^n}$ of x and y, respectively. If $\bar{U} \cap \bar{V} = \varnothing$, then there exists a loxodromic element of G that has one fixed point in U and another fixed point in V.*

PROOF. We may take U and V to be convex. It follows from Proposition 2.7 that there exists an element g of G such that $g(\overline{B^n} - V) \subset U$. Therefore $g(\bar{U} \cap \partial B^n) \subset U \cap \partial B^n$. By the Brouwer fixed point theorem, we see that g has a fixed point in $U \cap \partial B^n$. Similarily we have that $g^{-1}(\bar{V} \cap \partial B^n) \subset V \cap \partial B^n$. Therefore g has another fixed point in V. Assume that g is elliptic. It follows from [3; Lemma 3.3.2] that g must fix any point in the geodesic connecting x to y. This is impossible, because $g(\overline{B^n} - V) \subset U$. Thus g is a loxodromic element of G.

We shall derive some properties of G-dual points. Before stating our theorem, we give the definition of the limit set. Let $G(p) = \{g(p) | g \in G\}$ for a point $p \in B^n$. Define the *limit set* $L(G)$ of G by $L(G) = \overline{G(p)} \cap \partial B^n$. Note that $L(G)$ does not depend on the choice of p (see [3; Lemma 4.3.1]). By definition, $L(G)$ is a G-invariant closed set.

THEOREM 2.10. *Let G be a discrete subgroup of $U(1, n; \boldsymbol{C})$.*

(a) *G-dual points x and y belong to the limit set $L(G)$.*

(b) *If $x \in L(G)$, then there is some point $y \in L(G)$ such that $x \sim y$.*

(c) *Denote $\{y \in L(G) | x \sim y\}$ by $D(x)$. The set $D(x)$ is closed and G-invariant. If $\#(D(x)) \geqq 2$, then $D(x) = L(G)$.*

(d) *The set $D(x)$ is contained in the derived set $dG(y)$ of $G(y)$ for any $y \in \partial B^n - \{x\}$.*

(e) *If $\#(L(G)) = 1$, then the point in $L(G)$ is G-dual to itself. If $\#(L(G)) \geqq 2$, then any two points in $L(G)$ are G-dual.*

PROOF. (a) This is immediate.

(b) If $x \in L(G)$, then there exists a sequence $\{g_j\} \subset G$ such that $g_j(p) \to x$ as $j \to \infty$ for any point p. If we take a subsequence $\{g_{j_k}{}^{-1}(p)\}$, then there is a point y such that $g_{j_k}{}^{-1}(p) \to y$ as $j_k \to \infty$.

(c) Suppose that there is a sequence $\{y_j\}$ in $D(x)$ such that $y_j \to y$ as $j \to \infty$. Since $L(G)$ is closed, $y \in L(G)$. We shall show that y is G-dual to x. For each j, there is a sequence $\{g_m{}^{(j)}\} \subset G$ such that $g_m{}^{(j)}(p) \to x$ and $(g_m{}^{(j)})^{-1}(p) \to y_j$ as $m \to \infty$ for any point p. There exists a sequence $\{g^{(m)}\} \subset G$ such that $g^{(m)}(p) \to x$ and $(g^{(m)})^{-1}(p) \to y$. Hence y is G-dual to x. If $y \in D(x)$, then there is a sequence $\{g_m\}$ in G such that $g_m(p) \to x$ and $g_m{}^{-1}(p) \to y$. Let g be an element of G. Replace p by $g^{-1}(p)$. By [3; Lemma 4.3.1], $g_m(g^{-1}(p)) \to x$. Consider the sequence $\{g_m g^{-1}\}$ in G. Since $g_m g^{-1}(p) \to x$ and $(g_m g^{-1})^{-1}(p) \to g(y)$, $g(y)$ is contained in $D(x)$. Assume that $D(x)$ contains more than one point. Then it follows from [3; Lemma 4.3.3] that $D(x) \supset L(G)$. Thus we conclude that $D(x) = L(G)$.

(d) Before showing this, we define an angle and prove a lemma.

Let $x, y \in \overline{\boldsymbol{B}^n}$ and $p \in \boldsymbol{B}^n$. Set

$$\Psi_p(x^*, y^*) = -\mathrm{Re}\,[\Phi(p^*, p^*)^{-2}\{\Phi(x^*, y^*)\Phi(p^*, p^*) - \Phi(x^*, p^*)\Phi(p^*, y^*)\}]\,,$$

where $p^* \in \pi^{-1}(p)$, $x^* \in \pi^{-1}(x)$ and $y^* \in \pi^{-1}(y)$. We define the angle $\measuredangle_p(x, y)$ $(0 \leqq \measuredangle_p(x, y) \leqq \pi)$ at p between two geodesics $\widehat{xp}$ and $\widehat{yp}$ by

$$\cos \sphericalangle_p(x, y) = \Psi_p(x^*, y^*)\{\Psi_p(x^*, x^*)\Psi_p(y^*, y^*)\}^{-1/2} .$$

We note that $\cos \sphericalangle_p(x, y)$ is invariant under $U(1, n; \boldsymbol{C})$.

LEMMA 2.11. *Let p be a point in the geodesic γ having the end points x, y. Then*

$$\sphericalangle_z(p, y) \leqq \sphericalangle_p(z, x) \qquad \text{for any point } z \in \boldsymbol{B}^n .$$

PROOF. Without loss of generality, we may assume that $x = (1, 0, \ldots, 0)$, $y = (-1, 0, \ldots, 0)$, $p = (t, 0, \ldots, 0)$, where $t \in (-1, 1)$. Write $z = (z_1, \ldots, z_n)$. Setting $s = 1 - \Sigma_{i=1}^n |z_i|^2$ and $x_1 = \operatorname{Re}(z_1)$, we see that

$$\begin{aligned} \cos \sphericalangle_z(p, y) = & -[\operatorname{Re}\{(1+t)s - (1 - tz_1)(1 + \overline{z_1})\}] \\ & \times [\{-(1-t^2)s + |1 - tz_1|^2\}^{1/2}|1 + z_1|]^{-1} \end{aligned}$$

and

$$\cos \sphericalangle_p(z, x) = (\operatorname{Re}(z_1) - t)\{-(1-t^2)s + |1 - tz_1|^2\}^{-1/2} .$$

Let

$$F(t) = 1 - s - st + x_1 - tx_1 - t|z_1|^2 - |1 + z_1|(x_1 - t)$$

for $t \in [-1, 1]$. We observe that

$$\begin{aligned} F(-1) &= 1 - s + s + x_1 + x_1 + |z_1|^2 - |1 + z_1|(x_1 + 1) \\ &= 1 + 2x_1 + |z_1|^2 - |1 + z_1|(1 + x_1) \\ &= |1 + z_1|^2 - |1 + z_1|(1 + x_1) \\ &= |1 + z_1|\{|1 + z_1| - (1 + x_1)\} \geqq 0 \end{aligned}$$

and

$$F'(t) = -s - x_1 - |z_1|^2 + |1 + z_1| \geqq -1 - x_1 + |1 + x_1| \geqq 0 .$$

These facts imply that $F(t) \geqq 0$ in $[-1, 1]$. Therefore

$$\cos \sphericalangle_z(p, y) \geqq \cos \sphericalangle_p(z, x) .$$

Thus we have

$$\sphericalangle_z(p, y) \leqq \sphericalangle_p(z, x) .$$

Now we are ready to prove (d).

PROOF OF (d). Take $y \in \partial B^n - \{x\}$. Let γ be the geodesic with the end points x and y, and let p be a point on γ. Suppose that z is a point in

$D(x)$. Then there is a sequence $\{g_k\}$ in G such that $g_k(p) \to z$ and $g_k^{-1}(p) \to x$ as $k \to \infty$. Using Lemma 2.11, we have

$$\measuredangle_p(g_k(p), g_k(y)) = \measuredangle_{g_k^{-1}(p)}(p, y)$$
$$\leqq \measuredangle_p(g_k^{-1}(p), x) \to 0 .$$

Hence $\measuredangle_p(g_k(p), g_k(y)) \to 0$ as $k \to \infty$. Therefore $g_k(y) \to z$ as $k \to \infty$. Thus $D(x) \subset dG(y)$.

(e) To prove this, we prepare two lemmas.

LEMMA 2.12. *Let x and y be two points of ∂B^n. Let G be a discrete subgroup of $U(1, n; \boldsymbol{C})$ consisting only of elliptic elements all of which leave the set $\{x, y\}$ invariant. Then G is a finite group.*

PROOF. By [3; Proposition 2.1.3], we may assume that $x = (1, 0, \dots, 0)$ and $y = (-1, 0, \dots, 0)$. We write $U_{x,y}$ for the subgroup of all elliptic elements g in $U(1, n; \boldsymbol{C})$ such that g fixes x and y. It follows that an element in $U_{x,y}$ is of the form

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & A \end{bmatrix},$$

where $|\alpha| = 1$ and $A \in U(n-1; \boldsymbol{C})$. Let $G_{x,y} = \{g \in G | g(x) = x \text{ and } g(y) = y\}$. Since $U_{x,y}$ is compact, we see that $G_{x,y}$ is a finite group in the same manner as in the proof of Theorem 2.5. Therefore we have only to prove that $G - G_{x,y}$ is a finite set. Assume that $G - G_{x,y}$ is not a finite set, say, $G - G_{x,y} = \{h_1, h_2, \dots, h_k, \dots\}$. Since each element of $G - G_{x,y}$ interchanges x and y, the set $\{h_1 h_1, h_1 h_2, \dots, h_1 h_k, \dots\}$ is contained in $G_{x,y}$. Hence $\{h_1 h_1, h_1 h_2, \dots, h_1 h_k, \dots\}$ is a finite set. This is a contradiction. Therefore there exist at most a finite number of elements in $G - G_{x,y}$. Thus G is a finite group.

LEMMA 2.13. *If $\#(L(G)) \geqq 3$, then there exists a point in $L(G)$ which is not fixed by some element of G.*

PROOF. It is easy to show our statement in the case where G contains a loxodromic or parabolic element. Therefore we have only to consider the case where all elements of G except the identity are elliptic. Assume that any point in $L(G)$ is fixed by all elements of G. Using Lemma 2.12, we see that G is a finite group. This contradicts our assumption that $L(G) \neq \emptyset$. Thus our lemma is proved.

We now come to the proof of (e).

PROOF OF (e). If $L(G) = \{x\}$, then x is G-dual to itself by (b).

Next assume that $L(G) = \{x, y\}$. Since all elements of G leave the set $\{x, y\}$ invariant, there are no parabolic elements in G. Suppose that G contains only the identity and elliptic elements which leave $\{x, y\}$ invariant. Lemma 2.12 implies that G is a finite group. This contradicts our assumption that $L(G) \neq \varnothing$. Therefore G contains a loxodromic element with fixed points x and y. Thus x and y are G-dual.

Lastly assume that $\#(L(G)) \geqq 3$. By Lemma 2.13, there exists a point ζ in $L(G)$ such that some element f of G does not fix ζ. By (b), ζ has a dual point η in $L(G)$. It follows from (c) that ζ and $f(\zeta)$ are contained in $D(\eta)$ and hence that $D(\eta) = L(G)$, that is, η is G-dual to every point in $L(G)$.

Choose x and y in $L(G)$ such that η, x and y are all distinct. Let W, U and V be disjoint open neighborhoods of η, x and y, respectively. Using Proposition 2.9, we can find two elements g and h in G such that g has fixed points in U and W and h has fixed points in V and W. Two elements g and h do not have a common fixed point in W, otherwise G would be non-discrete by [5; Theorem 3.1]. Therefore either g or h does not fix η. Let $g(\eta) \neq \eta$. This implies that $D(x)$ contains at least two points η and $g(\eta)$. Using (c) again, we see that $D(x) = L(G)$. Hence any two points in $L(G)$ are G-dual.

REMARK 2.14. (1) According to Theorem 2.10 (c), $D(x) = L(G)$ for any $x \in L(G)$ in case $\#(D(x)) \geqq 2$. In case $\#(D(x)) = 1$ it may happen that $D(x) \neq L(G)$. In fact, let g be a loxodromic element with fixed points x and y, and let G be a cyclic group generated by g. Then $D(x) = \{y\}$, $D(y) = \{x\}$ but $L(G) = \{x, y\}$.

(2) The argument in the proof of (e) shows that, if $\#(L(G)) \geqq 3$, then there exist at least two loxodromic elements in G without a common fixed point.

Next we shall state the properties of the limit set $L(G)$.

THEOREM 2.15. *Let G be a discrete subgroup of $U(1, n; \boldsymbol{C})$.*

(a) *$L(G) = dG(y)$ for any $y \in \overline{B^n}$ if $\#(L(G)) \geqq 3$.*

(b) *Either $L(G) = \partial B^n$ or $L(G)$ is nowhere dense on ∂B^n.*

(c) *$L(G)$ is a perfect set if $\#(L(G)) \geqq 3$.*

(d) *$L(G)$ is the closure of the set of points fixed by some loxodromic elements of G if $\#(L(G)) \geqq 3$.*

PROOF. (a) Since G is discontinuous in B^n, $L(G) = dG(y)$ for any $y \in B^n$. Therefore we have only to show that $dG(0) = dG(y)$ for any $y \in \partial B^n$.

First we shall prove that $dG(0) \subset dG(y)$. By (2) in Remark 2.14, there exists a loxodromic element h of G such that $h(y) \neq y$. Let ζ be a point of

$dG(0)$. Take a point z of B^n in the geodesic connecting y to $h(y)$. Then there exists a sequence $\{g_k\}$ of elements of G such that $g_k(z) \to \zeta$ as $k \to \infty$. By taking a subsequence, if necessary, we may assume that $g_k(y) \to \zeta_1$ and $g_k(h(y)) \to \zeta_2$ as $k \to \infty$. If $\zeta_1 = \zeta_2$, then $\zeta = \zeta_1 = \zeta_2$. Hence $dG(0) \subset dG(y)$. On the other hand, assume that $\zeta_1 \neq \zeta_2$. It follows from [3; Lemma 4.3.3] that $dG(0) \subset dG(y)$.

Next we shall show that $dG(0) \supset dG(y)$. Let ζ be a point of $dG(y)$. Then there exists a sequence $\{g_k\}$ of elements of G such that $g_k(y) \to \zeta$, $g_k(0) \to \zeta_1$ and $g_k^{-1}(0) \to \zeta_2$. Suppose that $y \neq \zeta_2$. It follows from the proof of Proposition 2.7 that $g_k(y) \to \zeta_1$. Hence $\zeta = \zeta_1$, so ζ belongs to $dG(0)$. Assume therefore that $y = \zeta_2$. Since $dG(0)$ is invariant under G, $g_k(\zeta_2)$ is included in $dG(0)$, hence $g_k(y) \in dG(0)$. As $dG(0)$ is closed, ζ belongs to $dG(0)$. Thus $dG(0) \supset dG(y)$.

(b) If $\#(L(G)) \leqq 2$, then $L(G)$ is nowhere dense on ∂B^n. Therefore we may assume that $\#(L(G)) \geqq 3$. Let ζ be a point in $L(G)$. Suppose that there is a point z in $\partial B^n - L(G)$. By (a), there exists a sequence $\{g_k\}$ of elements of G such that $g_k(z) \to \zeta$ as $k \to \infty$. Every neighborhood of ζ contains points in the complement of $L(G)$, so $L(G)$ is nowhere dense on ∂B^n.

(c) This statement is an immediate consequence of (a).

(d) By (2) in Remark 2.14, there is a loxodromic element in G with fixed points ζ_1, ζ_2. Let M be the set of points fixed by some loxodromic elements of G. It follows from (a) that $L(G) = dG(\zeta_1)$. Since $L(G)$ is closed and $L(G) \supset M \supset G(\zeta_1)$, $L(G) \supset \overline{M} \supset \overline{G(\zeta_1)} \supset dG(\zeta_1)$. Thus $L(G) = \overline{M}$.

The groups for which $L(G) = \partial B^n$ are called groups of the *first kind*; those for which $L(G) \neq \partial B^n$ are called groups of the *second kind*.

Proposition 2.9 and (e) in Theorem 2.10 lead to

THEOREM 2.16 (cf. [4; Proposition 12]). *If $\#(L(G)) \geqq 3$, then the fixed points of the loxodromic elements of G are dense in $L(G) \times L(G)$, that is, for any points $x, y \in L(G)$ and open neighborhoods U, V of these points in ∂B^n, there is a loxodromic element in G which has one fixed point in U and the other in V.*

COROLLARY 2.17. *If G is of the first kind, then the fixed points of the loxodromic elements of G are dense in $\partial B^n \times \partial B^n$.*

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