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Algebraic L^2 decay for Navier-Stokes flows in exterior domains, II

Dedicated to Professor R. Rautmann on his 60th birthday

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1. Statement of results

Consider the Navier-Stokes equations in an exterior domain $D \subset R^n$, $n \geq 3$:

$$
\text{(NS)} \qquad \begin{aligned} &\frac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - \nabla p && (x \in D,\ t > 0) \\ &\nabla \cdot u = 0 && (x \in D,\ t \geq 0) \\ &u|_S = 0;\ u|_{t=0} = a. \end{aligned}
$$

Here S is the (smooth) boundary of D, $x = (x_1, \cdots, x_n)$ is a point in R^n, $u = (u_j)_{j=1}^n$ and p denote, respectively, unknown velocity and pressure, a is a given initial velocity; and $u \cdot \nabla u = \sum_j u_j \partial_j u_j$, $\nabla \cdot u = \sum_j \partial_j u_j$, $\nabla p = (\partial_j p)_{j=1}^n$, $\partial_j = \partial / \partial x_j$.

In this paper we discuss large time behavior of the L^2-norm of weak solutions of problem (NS) and improve our previous results in [1]. To state our results, we use the following notation: $C_{0,\sigma}^\infty(D)$ denotes the set of smooth solenoidal vector fields with compact support in D, and X_r, $1 < r < \infty$, its L^r-closure. Using the Helmholtz decomposition [7]:

$$
L^r(D)^n = X_r \oplus G_r, \quad G_r = \{\nabla p \in L^r(D)^n;\ p \in L^r_{loc}(\bar{D})\}
$$

and the associated projector $P = P_r$ onto X_r, we define the Stokes operator $A = A_r$ in X_r as

$$
Au = A_r u = -P_r \Delta u, \quad u \in D(A_r) = X_r \cap \{u \in W^{2,r}(D)^n;\ u|_S = 0\}.
$$

As is well known [3], A_r is a closed linear operator in X_r with dense domain $D(A_r)$ and $-A_r$ generates a bounded analytic semigroup $\{e^{-tA_r};\ t \geq 0\}$ on X_r. Given $a \in X_r$, the function $u(t) = e^{-tA}a$ uniquely solves the nonstationary Stokes system:

$$
\frac{\partial u}{\partial t} = \Delta u - \nabla p \qquad (x \in D,\ t > 0)
$$

$$\text{(S)} \qquad \begin{aligned} &\nabla \cdot u = 0 \qquad (x \in D,\ t \geq 0) \\ &u|_S = 0;\ u|_{t=0} = a \end{aligned}$$

together with some function p. Let V be the H^1-closure of $C_{0,\sigma}^{\infty}(D)$. We then easily see that A_2 is the positive self-adjoint operator in X_2 associated to the bilinear form $\langle \nabla u, \nabla v \rangle$ on $V \times V$ and therefore satisfies $\| A^{1/2} u \|_2 = \| \nabla u \|_2$. Given $a \in X_2$, a weakly continuous function $u: [0, \infty) \to X_2$ is called a *weak solution* of (NS) if

$$u \in L^{\infty}(0, T; X_2) \cap L^2(0, T; V)$$

for all $T > 0$, $u(0) = a$, and the identity:

$$\text{(W)} \quad \langle u, \phi \rangle (t) + \int_s^t [\langle \nabla u, \nabla \phi \rangle + \langle u \cdot \nabla u, \phi \rangle]\, d\tau = \langle u, \phi \rangle (s) + \int_s^t \langle u, \phi' \rangle\, d\tau$$

holds for all $\phi \in C^0([0, \infty); V \cap L^n) \cap C^1([0, \infty); X_2)$, where $\phi' = \partial \phi / \partial t$. Here and hereafter $\langle \cdot, \cdot \rangle$ denotes various duality pairings. The existence of a weak solution is now well known [6]. The uniqueness and the regularity of weak solutions are still open questions. All the weak solutions constructed so far satisfy the *energy inequality*:

$$\text{(E)} \qquad \| u(t) \|_2^2 + 2 \int_0^t \| \nabla u \|_2^2\, ds \leq \| a \|_2^2$$

for all $t \geq 0$. Furthermore, in case $n = 3, 4$, there always exists at least one weak solution u satisfying the *strong energy inequality*:

$$\text{(SE)} \qquad \| u(t) \|_2^2 + 2 \int_s^t \| \nabla u \|_2^2\, d\tau \leq \| u(s) \|_2^2$$

for $s = 0$, a.e. $s > 0$, and all $t \geq s$; see [8].

In this paper we improve the results obtained in our previous work [1]. Namely, we shall establish the following three theorems.

THEOREM 1. *Given $a \in X_2$, there exists a weak solution u such that, as $t \to \infty$,*

(i) $\| u(t) \|_2 \to 0$.

(ii) *If $\| e^{-tA} a \|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then*

$$\| u(t) \|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4; \\ O(t^{-n/4}) & \text{if } \alpha \geq n/4. \end{cases}$$

(iii) $\| u(t) - e^{-tA} a \|_2 = o(t^{1/2 - n/4})$ *as $t \to \infty$.*

(iv) *If $\| e^{-tA} a \|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then*

$$\|u(t)-e^{-tA}a\|_2=\begin{cases} O(t^{1/2-n/4-\alpha}) & \text{if } \alpha<1/2;\\ O(t^{-n/4}) & \text{if } \alpha>1/2;\\ O(t^{-n/4}[\log t]^{1/2}) & \text{if } \alpha=1/2.\end{cases}$$

THEOREM 2. *Given $a\in X_2$, suppose there exists a weak solution u satisfying the strong energy inequality* (SE). *Then u possesses the properties* (i)–(iv) *of Theorem* 1.

The next result concerns only the three-dimensional case, but deals with general weak solutions satisfying only the energy inequality (E).

THEOREM 3. *Let $n=3$ and let u be any weak solution with $u(0)=a$ satisfying the energy inequality* (E). *Then, as $t\to\infty$,*

(v) $\int_t^{t+1}\|u\|_2\,ds\to 0$.

(vi) *If $\|e^{-tA}a\|_2=O(t^{-\alpha})$ for some $\alpha>0$, then*

$$\frac{1}{t}\int_0^t\|u\|_2\,ds=\begin{cases} O(t^{-\alpha}) & \text{if } \alpha<3/4;\\ O(t^{-3/4}) & \text{if } \alpha\ge 3/4.\end{cases}$$

(vii) $t^{-1}\int_0^t\|u(s)-e^{-sA}a\|_2\,ds=o(t^{-1/4})$.

(viii) *If $\|e^{-tA}a\|_2=O(t^{-\alpha})$ for some $\alpha>0$, then*

$$\frac{1}{t}\int_0^t\|u(s)-e^{-sA}a\|_2\,ds=\begin{cases} O(t^{-1/4-\alpha}) & \text{if } \alpha<1/2;\\ O(t^{-3/4}) & \text{if } \alpha>1/2;\\ O(t^{-3/4}[\log t]^{1/2}) & \text{if } \alpha=1/2.\end{cases}$$

Our method for proving Theorem 3 reproduces a result of Masuda [6]:

COROLLARY 4. *Let $n\ge 3$ and let a weak solution u satisfy the energy inequality* (E). *Then,*

$$\lim_{t\to\infty}\int_t^{t+1}\|u\|_2\,ds=0.$$

Theorems 1 and 2 together improve Theorem A of [1], which asserts, among others, the existence of a weak solution u such that if $\|e^{-tA}a\|_2=O(t^{-\alpha})$, then

$$\|u(t)\|_2=\begin{cases} O(t^{-\alpha}) & \text{for } \alpha<n/4;\\ O(t^{\varepsilon-n/4}) & \text{for } \alpha\ge n/4,\end{cases}$$

with an arbitrary $0<\varepsilon<1/4$, and

$$\|u(t)-e^{-tA}a\|_2=O(t^{-\gamma})$$

where $\gamma=n/4-1/2+\alpha$ if $\alpha<1/2$ and $0<\gamma<n/4$ is arbitrary in case $\alpha\ge 1/2$.

Theorems 1, 2, 3 will be proved in Sections 2, 3, 4, respectively, and Corollary 4 in Section 4. Our basic tools are the $L^p - L^q$ estimates of Iwashita [4] for the bounded analytic semigroup $\{e^{-tA}; t \geq 0\}$, the estimates of [1] for fractional powers of A_r over exterior domains, the idea of taking the time-average of various functions, and the use of the weak version of the Hölder and Young inequalities in order to deduce explicit decay rates. It is the use of the L^p-L^q estimates for the semigroup that provides us with the above-mentioned improvement; and by applying the weak version of the Hölder and Young inequalities, we can treat general weak solutions satisfying only the energy inequality (E) to deduce Theorem 3, which was not discussed in the previous work [1].

2. Proof of Theorem 1

First we recall that the positive self-adjoint operator A_2 in X_2 admits the spectral decomposition

$$A_2 = \int_0^\infty \lambda dE_\lambda.$$

Moreover, since $\|A^{1/2}v\|_2 = \|\nabla v\|_2$ for $v \in V$, the operator A_2 is injective; so we get

$$\lim_{t\to\infty} \|e^{-tA}a\|_2 = 0 \qquad \text{for all } a \in X_2.$$

We begin by establishing the following, which improves [1, Lemma 5.1].

LEMMA 1. *Let $n \geq 3$, and $u, w \in V$. Then for all $\rho > 0$,*

$$\|E_\rho e^{-tA}P(w\cdot\nabla)u\|_2 \leq Ct^{-3/4}\rho^{(n-3)/4}(\|w\|_2\|u\|_2)^{1/2}(\|\nabla w\|_2\|\nabla u\|_2)^{1/2}. \tag{1}$$

PROOF. We use the estimates ([1, 4])

$$\|e^{-tA}a\|_q \leq Ct^{-(n/p-n/q)/2}\|a\|_p \qquad (1 < p \leq q < \infty); \tag{2}$$

and

$$\|\nabla e^{-tA}a\|_q \leq Ct^{-1/2-(n/p-n/q)/2}\|a\|_p \qquad (1 < p \leq q \leq n). \tag{3}$$

Estimate (3) with $q = n$ is due to [4]. Since $P = P_2$ is the orthogonal projector onto X_2, it follows from Hölder's inequality and the condition $\nabla\cdot w = 0$ that, for $\psi \in X_2$,

$$\begin{aligned} |\langle E_\rho e^{-tA}P(w\cdot\nabla)u, \psi\rangle| &= |\langle u, w\cdot\nabla e^{-tA}E_\rho\psi\rangle| \\ &\leq \|u\|_{2n'}\|w\|_{2n'}\|\nabla e^{-tA}E_\rho\psi\|_n, \end{aligned} \tag{4}$$

where $1/n' = 1 - 1/n$. By the Hölder and Sobolev inequalities we get

$$\|u\|_{2n'} \le C(\|u\|_2 \|\nabla u\|_2)^{1/2}; \quad \|w\|_{2n'} \le C(\|w\|_2 \|\nabla w\|_2)^{1/2}. \tag{5}$$

On the other hand, we already know that (see [1])

$$\|v\|_r \le C\|A^\alpha v\|_q$$

provided $\alpha \ge 0$, $1 < q < n$ and $1/r = 1/q - 2\alpha/n$. This, together with (3), implies that

$$\begin{aligned}\|\nabla e^{-tA} E_\rho \psi\|_n &\le Ct^{-3/4}\|E_\rho \psi\|_{2n/3} \\ &\le Ct^{-3/4}\|A^{(n-3)/4}E_\rho\psi\|_2 \le Ct^{-3/4}\rho^{(n-3)/4}\|\psi\|_2.\end{aligned}$$

Combining this with (4) and (5) yields (1).

PROOF OF THEOREM 1. We use the approximate solutions $u = u_k$, which are obtained by solving the problem

$$\frac{du_k}{dt} + Au_k + P(\bar{u}_k \cdot \nabla)u_k = 0; \quad u_k(0) = \bar{a}_k$$

where

$$\bar{u}_k = (I + k^{-1}A)^{-1-[n/4]}u_k; \quad \bar{a}_k = (I + k^{-1}A)^{-1-[n/4]}a,$$

and $[b]$ is the integral part of the real number b. As shown in [8], the functions u_k are found in the space $L^2_{loc}([0, \infty); D(A_2))$ and the time-derivatives u_k' belong to $L^2_{loc}([0, \infty); X_2)$. Convergence of u_k to a weak solution u is proved in [8] only when $n = 3, 4$. But, the argument given there applies also to the higher-dimensional case with slight modification. Since the mode of convergence given in [8] implies

$$\|u(t)\|_2 \le \liminf_{k\to\infty}\|u_k(t)\|_2; \quad \|u(t) - e^{-tA}a\|_2 \le \liminf_{k\to\infty}\|u_k(t) - e^{-tA}\bar{a}_k\|_2$$

for a.e. $t > 0$, we need only show that the results of Theorem 1 hold for u_k uniformly in k. So, in what follows we omit the subscript k for simplicity in notation. First observe that

$$\|\bar{u}\|_2 \le \|u\|_2; \quad \|\nabla\bar{u}\|_2 = \|A^{1/2}\bar{u}\|_2 \le \|A^{1/2}u\|_2 = \|\nabla u\|_2. \tag{6}$$

By the standard energy method we get

$$\frac{d}{dt}\|u\|_2^2 + 2\|A^{1/2}u\|_2^2 = 0.$$

Integrating this gives

$$\|u(t)\|_2 \le \|\bar{a}\|_2 \le \|a\|_2; \quad \int_0^\infty \|\nabla u\|_2^2 dt \le \frac{1}{2}\|\bar{a}\|_2^2 \le \frac{1}{2}\|a\|_2^2. \tag{7}$$

Moreover, using the estimate

$$\|A^{1/2}u\|_2^2 \ge \int_\rho^\infty \lambda d\|E_\lambda u\|_2^2 \ge \rho\int_\rho^\infty d\|E_\lambda u\|_2^2 = \rho(\|u\|_2^2 - \|E_\rho u\|_2^2)$$

for any $\rho > 0$, we obtain

$$2\|u\|_2 \frac{d}{dt}\|u\|_2 + 2\rho\|u\|_2^2 \le 2\rho\|E_\rho u\|_2^2.$$

Since $\|E_\rho u\|_2 \le \|u\|_2$,

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \le \rho\|E_\rho u\|_2. \tag{8}$$

On the other hand, from the integral equation

$$u(t) = e^{-tA}\bar{a} - \int_0^t e^{-(t-s)A}P(\bar{u}\cdot\nabla)u(s)ds$$

we obtain

$$\|E_\rho u(t)\|_2 \le \|e^{-tA}a\|_2 + \int_0^t \|E_\rho e^{-(t-s)A}P(\bar{u}\cdot\nabla)u(s)\|_2 ds,$$

since $\|e^{-tA}\bar{a}\|_2 \le \|e^{-tA}a\|_2$. Applying Lemma 1 and (6) then yields

$$\begin{aligned} \|E_\rho u(t)\|_2 &\le \|e^{-tA}a\|_2 + C\rho^{(n-3)/4}\int_0^t (t-s)^{-3/4}\|u\|_2\|\nabla u\|_2(s)ds \\ &\equiv \|e^{-tA}a\|_2 + C\rho^{(n-3)/4}F(t). \end{aligned} \tag{9}$$

By the Schwarz inequality, we obtain

$$\begin{aligned} F(t) &\le \left[\int_0^t (t-s)^{-3/4}\|u(s)\|_2^2 ds\right]^{1/2}\left[\int_0^t (t-s)^{-3/4}\|\nabla u(s)\|_2^2 ds\right]^{1/2} \\ &\equiv F_1(t)^{1/2}F_2(t)^{1/2} \end{aligned} \tag{10}$$

so that

$$\|E_\rho u(t)\|_2 \le \|e^{-tA}a\|_2 + C\rho^{(n-3)/4}F_1(t)^{1/2}F_2(t)^{1/2}. \tag{11}$$

From (8) and (11) we have

$$\frac{d}{dt}\|u\|_2 + \rho\|u\|_2 \le \rho\|e^{-tA}a\|_2 + C\rho^{(n+1)/4}F_1(t)^{1/2}F_2(t)^{1/2}. \tag{12}$$

Now, take in (12) $\rho = m/t$ with large $m > 0$, multiply both sides by t^m, and then integrate in t, to get

(13)
$$\|u(t)\|_2 \le \frac{1}{t^m}\int_0^t ms^{m-1}\|e^{-sA}a\|_2 ds + Ct^{(3-n)/4}\left[\frac{1}{t}\int_0^t F_1 ds\right]^{1/2}\left[\frac{1}{t}\int_0^t F_2 ds\right]^{1/2}.$$

We now prove (i) and (ii) of Theorem 1. First observe that (7) implies

(14) $$\frac{1}{t}\int_0^t F_2(s)ds \le Ct^{-3/4}; \quad \frac{1}{t}\int_0^t F_1(s)ds \le Ct^{-3/4}\int_0^t \|u\|_2^2 ds.$$

From (13) and (14) we obtain

(15) $$\|u(t)\|_2 \le \frac{1}{t^m}\int_0^t ms^{m-1}\|e^{-sA}a\|_2 ds + Ct^{-n/4}\left(\int_0^t \|u\|_2^2 ds\right)^{1/2}.$$

The last term is $O(t^{1/2-n/4})$; so assertion (i) follows. If $\|e^{-tA}a\|_2 = O(t^{-\alpha})$, then (15) yields

(16) $$\|u(t)\|_2 \le C(t^{-\alpha} + t^{1/2-n/4}).$$

This proves (ii) in case $\alpha < n/4 - 1/2$. When $\alpha \ge n/4 - 1/2$, (16) implies $\|u(t)\|_2 = O(t^{-1/4})$. Thus, (15) yields

(17) $$\|u(t)\|_2 \le C(t^{-\alpha} + t^{(1-n)/4})$$

which shows (ii) in case $\alpha < n/4 - 1/4$. When $\alpha \ge n/4 - 1/4$, (17) implies $\|u(t)\|_2 = O((t+1)^{-1/2})$; so (15) gives

(18) $$\|u(t)\|_2 \le C(t^{-\alpha} + t^{-n/4}\log(t+1))$$

and this shows (ii) for $\alpha < n/4$. When $\alpha \ge n/4$, (18) implies $\|u\|_2 \in L^2$. So, (15) yields

$$\|u(t)\|_2 \le C(t^{-\alpha} + t^{-n/4}).$$

This completes the proof of (ii). We next prove (iii) and (iv). Let

$$w(t) = u(t) - u^0(t); \qquad u^0(t) = e^{-tA}\bar{a}.$$

Then,

$$\frac{dw}{dt} + Aw + P(\bar{u}\cdot\nabla)u = 0,$$

and so

(19) $$\frac{d}{dt}\|w\|_2^2 + 2\|\nabla w\|_2^2 = -2\langle \bar{u}\cdot\nabla u, w\rangle.$$

Since

$$\langle \bar{u}\cdot\nabla u, w\rangle = \langle \bar{w}\cdot\nabla u^0, w\rangle + \langle \overline{u^0}\cdot\nabla u^0, w\rangle,$$

the Hölder and Sobolev inequalities yield, with $1/2^* = 1/2 - 1/n$,

$$\begin{aligned} 2|\langle \bar{u}\cdot\nabla u, w\rangle| &\le 2(\|\bar{w}\|_{2^*}\|\nabla u^0\|_n\|w\|_2 + \|\nabla u^0\|_n\|\overline{u^0}\|_{2^*}\|w\|_2) \\ &\le C\|w\|_2(\|\nabla w\|_2\|\nabla u^0\|_n + \|\nabla u^0\|_2\|\nabla u^0\|_n) \\ &\le \|\nabla w\|_2^2 + C\|w\|_2^2\|\nabla u^0\|_n^2 + C\|w\|_2\|\nabla u^0\|_2\|\nabla u^0\|_n. \end{aligned}$$

Substituting this into (19) and applying (2) and (3) gives

$$\frac{d}{dt}\|w\|_2^2 + \|\nabla w\|_2^2 \le Ct^{-n/2}\|w\|_2^2 + Ct^{-(n+2)/4}\|u^0(t/2)\|_2\|w\|_2.$$

As in the proof of (i), we obtain

$$\frac{d}{dt}\|w\|_2 + \rho\|w\|_2 \le \rho\|E_\rho w\|_2 + Ct^{-n/2}\|w\|_2 + Ct^{-(n+2)/4}\|u^0(t/2)\|_2. \tag{20}$$

On the other hand, as in the proof of (i),

$$\|E_\rho w\|_2 \le C\rho^{(n-3)/4}\int_0^t (t-s)^{-3/4}\|u\|_2\|\nabla u\|_2 ds \le C\rho^{(n-3)/4}F_1(t)^{1/2}F_2(t)^{1/2}.$$

We thus obtain from (20)

$$\frac{d}{dt}\|w\|_2 + \rho\|w\|_2 \le C[\rho^{(n+1)/4}F_1^{1/2}F_2^{1/2} + t^{-n/2}\|w\|_2 + t^{-(n+2)/4}\|u^0(t/2)\|_2].$$

Now take $\rho = m/t$ with large $m > 0$, apply (14) and proceed as in the proof of (i), to get

$$\|w(t)\|_2 \le Ct^{-n/4}\left(\int_0^t \|u\|_2^2 ds\right)^{1/2} + \frac{C}{t^m}\int_0^t (s^{m-n/2}\|w\|_2 + s^{m-(n+2)/4}\|u^0(s/2)\|_2)ds.$$

We thus have

$$\|w(t)\|_2 \le I_1(t) + I_2(t) + I_3(t), \tag{21}$$

where $I_j(t)$, $(j = 1, 2, 3)$, are written via the Schwarz inequality as

$$I_1(t) = Ct^{1/2-n/4}\left[\frac{1}{t}\int_0^t \|u\|_2^2 ds\right]^{1/2};$$

$$I_2(t) = Ct^{1-n/2}\left[\frac{1}{t}\int_0^t \|w\|_2^2 ds\right]^{1/2};$$

$$I_3(t) = Ct^{1/2-n/4}\left[\frac{1}{t}\int_0^t \|u^0(s/2)\|_2^2 ds\right]^{1/2}.$$

Since $\|u(t)\|_2 \to 0$ as $t \to \infty$, we obtain

$$\|w(t)\|_2 = o(t^{1/2-n/4}) + o(t^{1-n/2}) + o(t^{1/2-n/4}) = o(t^{1/2-n/4})$$

and this proves (iii). Suppose now that $\|u^0(t)\|_2 = O(t^{-\alpha})$, for some $0 < \alpha < 1/2$. Since $\alpha < n/4$, (ii) shows that $\|u(t)\|_2 = O(t^{-\alpha})$. It follows from (21) that

$$\|w(t)\|_2 = O(t^{1/2-n/4-\alpha}).$$

Suppose next that $\alpha > 1/2$. Then (ii) shows $\|u\|_2 \in L^2$. Hence (21) yields

$$\|w(t)\|_2 = O(t^{-n/4}).$$

Finally, if $\alpha = 1/2$, then (ii) shows that $\|u(t)\|_2 = O((t+1)^{-1/2})$, because $1/2 < n/4$. From (21) we get

$$\|w(t)\|_2 = O(t^{-n/4}[\log t]^{1/2}).$$

This completes the proof of (iv).

3. Decay results for turbulent solutions

This section establishes Theorem 2. Following Leray [5], we call a weak solution satisfying (SE) a *turbulent solution*. The existence of a turbulent solution was established in [5] for the Cauchy problem in R^3, and in [8] for the case of exterior domains of dimensions $n = 3,4$. Observe that (SE) implies

$$\|u(t)\|_2 \le \|a\|_2; \quad \int_0^\infty \|\nabla u\|_2^2 ds \le \frac{1}{2}\|a\|_2^2. \tag{22}$$

In this section we need also the following

LEMMA 2. *Let* $0 < \alpha < 1$, $\beta \ge 0$. *Then, as* $t \to \infty$,

$$\int_0^t (t-s)^{-\alpha}(1+s)^{-\beta} ds = \begin{cases} O(t^{1-\alpha-\beta}) & \text{if } \beta < 1; \\ O(t^{-\alpha}) & \text{if } \beta > 1; \\ O(t^{-\alpha}\log(t+1)) & \text{if } \beta = 1. \end{cases}$$

PROOF. In case $0 \le \beta < 1$ we have

$$\int_0^t (t-s)^{-\alpha}(1+s)^{-\beta} ds \le \int_0^t (t-s)^{-\alpha} s^{-\beta} ds = Ct^{1-\alpha-\beta}$$

with $C = C(\alpha, \beta)$. When $\beta \geq 1$, we split the domain of integration $[0, t]$ into $[0, t/2]$ and $[t/2, t]$ to get

$$\int_0^t (t-s)^{-\alpha}(1+s)^{-\beta} ds \leq (t/2)^{-\alpha} \int_0^{t/2} (1+s)^{-\beta} ds + (1+t/2)^{-\beta} \int_{t/2}^t (t-s)^{-\alpha} ds.$$

The result now follows immediately.

PROOF OF THEOREM 2. Since (i) and (iii) are already proved in [1], we need only establish (ii) and (iv). To this end, we shall use the following decay result, which is obtained in [1, Theorem A]: if $\|e^{-tA}a\|_2 = O(t^{-\alpha})$, then for any $0 < \varepsilon < 1/4$,

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < n/4; \\ O(t^{\varepsilon - n/4}) & \text{if } \alpha \geq 4. \end{cases} \tag{23}$$

First observe that the application of the estimate

$$2\|\nabla u\|_2^2 \geq \lambda(\|u\|_2^2 - \|E_\lambda u\|_2^2)$$

to (SE) yields

$$\|u(t)\|_2^2 + \int_s^t \rho(\tau)\|u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2 + \int_s^t \rho(\tau)\|E_{\rho(\tau)}u(\tau)\|_2^2 d\tau, \tag{24}$$

for a.e. $s > 0$ and all $t \geq s$, where $\rho(\tau)$ is a positive smooth function of τ to be fixed later. On the other hand, the identity (W) with $\phi(\tau) = e^{-(s-\tau)A}E_\lambda \psi$ and estimate (3) together yield, for $\psi \in C_{0,\sigma}^\infty(D)$,

$$\begin{aligned} |\langle E_\lambda u(s), \psi\rangle| &\leq |\langle e^{-sA}a, E_\lambda \psi\rangle| + \int_0^s |\langle u, u\cdot\nabla e^{-(s-\tau)A}E_\lambda\psi\rangle| d\tau \\ &\leq \|\psi\|_2 \left[\|e^{-sA}a\|_2 + C\lambda^{(n-3)/4} \int_0^s (s-\tau)^{-3/4}\|u\|_2\|\nabla u\|_2 d\tau \right]; \end{aligned}$$

whence

$$\|E_\lambda u(s)\|_2 \leq \|e^{-sA}a\|_2 + C\lambda^{(n-3)/4} F_1(s)^{1/2} F_2(s)^{1/2} \tag{25}$$

for all $s \geq 0$. From (24) and (25) we obtain

$$\|u(t)\|_2^2 + \int_s^t \rho(\tau)\|u(\tau)\|_2^2 d\tau \leq \|u(s)\|_2^2 + g(t, s)$$

for a.e $s > 0$ and all $t \geq s$, where

$$g(t, s) = C \int_s^t [\rho(\tau)\|e^{-\tau A}a\|_2^2 + \rho(\tau)^{(n-1)/2} F_1 F_2(\tau)] d\tau.$$

We now take $\rho(\tau) = m/\tau$ with large $m > 0$, multiply τ^m and let $s \to 0$ to get, as in [1, Sect. 5],

$$\|u(t)\|_2^2 \leq \frac{C}{t^m}\int_0^t m s^{m-1}\|e^{-sA}a\|_2^2 ds + H(t),$$

where

$$H(t) = \frac{C}{t^m}\int_0^t s^{m+(1-n)/2} F_1 F_2(s) ds.$$

Now, Lemma 2 and the decay result (23) together show that if $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$, then

$$(26) \qquad F_1(t) = \begin{cases} O(t^{1/4-2\alpha}) & \text{if } 2\alpha < 1; \\ O(t^{-3/4}) & \text{if } 2\alpha > 1; \\ O(t^{-3/4}\log(t+1)) & \text{if } 2\alpha = 1. \end{cases}$$

Hence, estimate (14) for $F_2(t)$ yields

$$H(t) = \begin{cases} O(t^{1-n/2-2\alpha}) & \text{if } 2\alpha < 1; \\ O(t^{-n/2}) & \text{if } 2\alpha > 1; \\ O(t^{-n/2}\log(t+1)) & \text{if } 2\alpha = 1. \end{cases}$$

We thus obtain

$$\|u(t)\|_2^2 \leq \begin{cases} C(t^{-2\alpha} + t^{1-n/2-2\alpha}) & \text{if } 2\alpha < 1; \\ C(t^{-2\alpha} + t^{-n/2}) & \text{if } 2\alpha > 1; \\ C(t^{-2\alpha} + t^{-n/2}\log(t+1)) & \text{if } 2\alpha = 1. \end{cases}$$

This proves assertion (ii). To prove (iv), recall that an argument in [1, Sect. 5] gives

$$\|w(t)\|_2^2 + 2\int_s^t \|\nabla w\|_2^2 d\tau \leq \|w(s)\|_2^2 + 2\int_s^t \langle u\cdot\nabla u, u^0\rangle d\tau$$

for a.e. $s > 0$ and all $t \geq s$, where $w(t) = u(t) - u^0(t)$. We estimate the last term to get

$$\begin{aligned} 2|\langle u\cdot\nabla u, u^0\rangle| &= 2|\langle w\cdot\nabla u^0, w\rangle + \langle u^0\cdot\nabla u^0, w\rangle| \\ &\leq 2\|\nabla u^0\|_n(\|w\|_{2n'}^2 + \|u^0\|_2\|w\|_{2*}) \\ &\leq C\|\nabla u^0\|_n(\|\nabla w\|_2\|w\|_2 + \|u^0\|_2\|\nabla w\|_2) \\ &\leq \|\nabla w\|_2^2 + C\|\nabla u^0\|_n^2(\|w\|_2^2 + \|u^0\|_2^2). \end{aligned}$$

Using (3) to treat $\|\nabla u^0\|_n$ and applying the spectral decomposition to estimate $\|\nabla w\|_2$, we obtain

$$\|w(t)\|_2^2 + \int_s^t \rho(\tau)\|w(\tau)\|_2^2 d\tau \leq \|w(s)\|_2^2 + C\int_s^t \rho(\tau)\|E_{\rho(\tau)}w(\tau)\|_2^2 d\tau$$
$$+ C\int_s^t \tau^{-n/2}(\|w\|_2^2 + \|u^0\|_2^2)(\tau)d\tau$$

for a.e. $s > 0$ and all $t \geq s$. Since

$$\|E_\rho w\|_2 \leq C\rho^{(n-3)/4}F_1^{1/2}F_2^{1/2},$$

this yields

$$\|w(t)\|_2^2 + \int_s^t \rho(\tau)\|w(\tau)\|_2^2 d\tau \leq \|w(s)\|_2^2 + h(t, s)$$

for a.e. $s > 0$ and all $t \geq s$, where

$$h(t, s) = C\int_s^t [\rho(\tau)^{(n-1)/2}F_1F_2(\tau) + \tau^{-n/2}(\|w\|_2^2 + \|u^0\|_2^2)(\tau)]\, d\tau.$$

Taking $\rho(\tau) = m/\tau$ with large $m > 0$ and letting $s \to 0$, we obtain as in [1, Sect. 5]

$$\|w(t)\|_2^2 \leq Ct^{-m}\int_0^t [s^{m-n/2}(\|w\|_2^2 + \|u^0\|_2^2) + s^{m-(n-1)/2}F_1F_2]ds$$
$$\leq Ct^{1-n/2}\left[t^{-1}\int_0^t (\|w\|_2^2 + \|u^0\|_2^2)ds\right] + Ct^{-m}\int_0^t s^{m-(n-1)/2}F_1F_2 ds.$$

Assertion (iv) now follows from (14) and (26). The proof is complete.

4. General weak solutions

In this section we restrict ourselves to the case $n = 3$ and prove Theorem 3 for general weak solutions satisfying only the energy inequality (E). Our basic tools in this section are presented in the following Lemma 3 and Lemma 4.

LEMMA 3. *Let L_w^p, $1 < p < \infty$ denote the Banach space of measurable functions f on the real line with norm*

$$\|f\|_{p,w} \equiv \sup_E |E|^{-1+1/p}\int_E |f|dt < +\infty$$

where $|E|$ is the Lebesgue measure of a measurable set E.

(i) *If $f \in L_w^p$, $g \in L_w^q$ and $1/p + 1/q = 1/r$, then $fg \in L_w^r$ and we have*

$$\|fg\|_{r,w} \leq C\|f\|_{p,w}\|g\|_{q,w}$$

with $C>0$ *depending only on* p *and* q.

(ii) *If* $f\in L_w^p$, $g\in L_w^q$ *and* $1/p+1/q=1+1/r$, *then the convolution* $f*g$ *is in* L_w^r *and there is a constant* $C>0$ *depending only on* p *and* q *so that*

$$\|f*g\|_{r,w}\leq C\|f\|_{p,w}\|g\|_{q,w}.$$

(iii) *If* $f\in L_w^p$ *and* $g\in L^1$, *then* $f*g\in L_w^p$ *and we have*

$$\|f*g\|_{p,w}\leq\|f\|_{p,w}\|g\|_1.$$

It is easy to see that $L^p\subset L_w^p$ with continuous injection. A typical example of L_w^p functions that we need below is

$$f(t)=\begin{cases}t^{-1/p} & (t>0)\\ 0 & (t\leq 0).\end{cases}$$

Also, notice that $f\in L_w^p$ implies the estimate

$$\frac{1}{t}\int_0^t|f|ds\leq Ct^{-1/p}.$$

This property of L_w^p-functions will be frequently used in order to deduce the desired decay rates (see the proof of (vi) below).

Lemma 3 (i) is the weak version of Hölder's inequality, while (ii) and (iii) are the weak version of Young's inequality. Although these inequalities seem to be well known (see [9, p. 32] for (ii)), we give here an elementary proof for the reader's convenience.

PROOF. First we show that f is in L_w^p if and only if

$$\|f\|_{p,w}^*\equiv\sup_{t>0}t|R(|f|>t)|^{1/p}<+\infty,$$

where $R(|f|>t)=\{s\in R;|f(s)|>t\}$, and that

$$\|f\|_{p,w}^*\leq\|f\|_{p,w}\leq\frac{p}{p-1}\|f\|_{p,w}^*. \tag{27}$$

Let $E_r=R(|f|>t)\cap[-r,r]$ for any $r>0$. Then Chebyshev's inequality yields

$$t|E_r|\leq\int_{E_r}|f|ds\leq\|f\|_{p,w}|E_r|^{1-1/p}$$

so that $t|E_r|^{1/p}\leq\|f\|_{p,w}$. Letting $r\to\infty$, we obtain the first inequality in (27). To show the second, we recall the formula [10]:

$$\int_E |f|ds = \int_0^\infty |E(|f| > t)|dt \tag{28}$$

for any measurable E. The definition of $\|f\|_{p,w}^*$ then gives, with $\beta = \|f\|_{p,w}^* |E|^{-1/p}$,

$$|E(|f| > t)| \leq \begin{cases} |E| & (0 < t \leq \beta), \\ (\|f\|_{p,w}^*)^p t^{-p} & (t \geq \beta). \end{cases}$$

Direct calculation thus yields

$$\int_0^\infty |E(|f| > t)|dt = \int_0^\beta |E(|f| > t)|dt + \int_\beta^\infty |E(|f| > t)|dt$$

$$\leq |E|\beta + \frac{1}{p-1} \|f\|_{p,w}^* |E|^{1-1/p} = \frac{p}{p-1} \|f\|_{p,w}^* |E|^{1-1/p},$$

which completes the proof of (27).

We shall now prove Lemma 3 (i). Applying the classical Young's inequality:

$$|fg| \leq \frac{r}{p} \varepsilon^{p/r} |f|^{p/r} + \frac{r}{q} \varepsilon^{-q/r} |g|^{q/r}$$

for any $\varepsilon > 0$, we obtain

$$R(|fg| > t) \subset R(|f| > c_1 \varepsilon^{-1} t^{r/p}) \cup R(|g| > c_2 \varepsilon t^{r/q})$$

with c_1 and c_2 depending only on p and q. Direct calculation thus gives

$$(\|fg\|_{r,w}^*)^r \leq C_1 \varepsilon^p (\|f\|_{p,w}^*)^p + C_2 \varepsilon^{-q} (\|g\|_{q,w}^*)^q$$

for all $\varepsilon > 0$, where C_1 and C_2 depend only on p and q. The result (i) follows by taking the minimum with respect to $\varepsilon > 0$.

(ii) We fix $\alpha > 0$ and write $|f(\tau)| = K_1 + K_2$, where

$$K_1(\tau) = \begin{cases} |f(\tau)| & (|f(\tau)| \leq \alpha) \\ 0 & (|f(\tau)| > \alpha). \end{cases}$$

Then we have

$$|f*g|(s) \leq K_1 * |g|(s) + K_2 * |g|(s) \equiv I_1(s) + I_2(s). \tag{29}$$

By the definition of the Lebesgue integral [10] we obtain

$$I_1(s) = \int_0^\infty dt \int_{E_t} |g(\tau)| d\tau \leq \|g\|_{q,w} \int_0^\infty |E_t|^{1-1/q} dt,$$

where $E_t = \{\tau; K_1(s-\tau) > t\}$, so $|E_t| \leq (\|f\|^*_{p,w})^p t^{-p}$, and $|E_t| = 0$ if $t > \alpha$. Since $p(1-1/q) < 1$ by assumption, we obtain, with $1/q' = 1 - 1/q$,

$$I_1(s) \leq \|g\|_{q,w}(\|f\|^*_{p,w})^{p/q'} \int_0^\alpha t^{-p/q'} dt = C\alpha^{1-p/q'} \|g\|_{q,w}(\|f\|^*_{p,w})^{p/q'},$$

with C depending only on p and q. We thus have

$$\int_E I_1(s) ds \leq C\alpha^{1-p/q'} |E| \cdot \|g\|_{q,w}(\|f\|^*_{p,w})^{p/q'}. \tag{30}$$

On the other hand, denoting by 1_E the indicator function of the set E, we have

$$\begin{aligned}\int_E I_2(s) ds &= \iint 1_E(s) K_2(s-\tau)|g(\tau)| ds d\tau \\ &= \int |g(\tau)| d\tau \int 1_E(s+\tau) K_2(s) ds = \int K_2(s) ds \int 1_E(s+\tau)|g(\tau)| d\tau \\ &\leq \|g\|_{q,w} |E|^{1-1/q} \int K_2(s) ds.\end{aligned}$$

Since $|E(|f| > t)| \leq (\|f\|^*_{p,w})^p t^{-p}$, (28) yields

$$\begin{aligned}\int K_2(s)\, ds &= \alpha |E(|f| > \alpha)| + \int_\alpha^\infty |E(|f| > t)|\, dt \\ &\leq C(\|f\|^*_{p,w})^p \alpha^{1-p}\end{aligned}$$

with C depending only on p and q. Hence,

$$\int_E I_2(s)\, ds \leq C\alpha^{1-p} |E|^{1-1/q} \|g\|_{q,w}(\|f\|^*_{p,w})^p. \tag{31}$$

Combining (29), (30) and (31) gives

$$\int_E |f*g|\, ds \leq C\|g\|_{q,w}[\alpha^{1-p/q'}(\|f\|^*_{p,w})^{p/q'}|E| + \alpha^{1-p}(\|f\|^*_{p,w})^p |E|^{1-1/q}].$$

Taking the minimum with respect to $\alpha > 0$ yields

$$\int_E |f*g|\, ds \leq C\|f\|^*_{p,w} \|g\|_{q,w} |E|^{1-1/r}$$

and this proves (ii).

(iii) Direct calculation gives

$$\int_E |f*g|\, ds \leq \iint 1_E(s)|f(s-\tau)| \cdot |g(\tau)|\, ds d\tau$$

$$= \int |g(\tau)| d\tau \int 1_E(s+\tau)|f(s)| ds \leq \|g\|_1 \|f\|_{p,w} |E|^{1-1/p}.$$

This proves (iii).

LEMMA 4. *Let f be a measurable function on the real line, and suppose there exist constants $M > 0$, $C > 0$ and $p > q > 1$ so that $0 \leq f \leq M$ and*

$$\int_E f ds \leq C(|E|^{1-1/p} + |E|^{1-1/q})$$

for all measurable subsets E. Then, there is another constant $C' > 0$ such that

$$\int_E f ds \leq C' |E|^{1-1/p}$$

for all measurable E.

PROOF. Since $1 - 1/p > 1 - 1/q$, the result is obvious for E with $|E| \geq 1$. So we may assume $|E| < 1$. Then, since

$$\int_E f ds \leq 2C|E|^{1-1/q},$$

Hölder's inequality yields, with $\theta = 1 - q/p$,

$$\begin{aligned} \int_E f ds &\leq M^\theta \int_E f^{1-\theta} ds \\ &\leq M^\theta |E|^\theta \left[\int_e f ds \right]^{1-\theta} \leq M^\theta (2C)^{1-\theta} |E|^{1-1/p}, \end{aligned}$$

which completes the proof.

The next lemma is needed in proving Theorem 3 (viii).

LEMMA 5. *Let $L^1_w(R)$ denote the set of all measurable functions f such that*

$$\|f\|^*_{1,w} \equiv \sup_{\tau > 0} \tau |R(|f| > \tau)| < +\infty.$$

If $f \in L^1_w \cap L^\infty$, then there is a constant $C > 0$ with

$$\int_0^t |f| ds \leq C \log(t + e), \qquad \textit{for all } t > 0.$$

PROOF. The definition of the Lebesgue integral [10] implies that if we write

$$\lambda_t(\tau) = |[0, t] \cap R(|f| > \tau)| \quad \text{and} \quad \alpha = \|f\|_\infty,$$

then

$$\int_0^t |f|\, ds = \int_0^\alpha \lambda_t(\tau)\, d\tau.$$

By assumption we have

$$\lambda_t(\tau) \le \begin{cases} t & \text{if } \tau \le t^{-1}\|f\|_{1,w}^* \\ \tau^{-1}\|f\|_{1,w}^* & \text{if } \tau \ge t^{-1}\|f\|_{1,w}^*. \end{cases}$$

Denoting $\beta = t^{-1}\|f\|_{1,w}^*$, we obtain

$$\int_0^t |f|\, ds \le \int_0^\beta t\, d\tau + \|f\|_{1,w}^* \int_\beta^\alpha \tau^{-1}\, d\tau \le C\log(t+e)$$

which completes the proof.

PROOF OF THEOREM 3. First, the energy inequality (E) implies (22), so we see that $\|u\|_2 \in L^\infty$ and $\|\nabla u\|_2 \in L^2$. Second, estimate (25) takes the form

$$\|u(s)\|_2 \le \|e^{-sA}a\|_2 + CF(s) \tag{32}$$

because the right-hand side of (25) is independent of $\lambda > 0$ and so we can let $\lambda \to \infty$ in the left-hand side. Since $\|u\|_2 \|\nabla u\|_2 \in L^2$, the Hardy-Littlewood-Sobolev inequality [9, p. 31] implies $F \in L^4$. So we get

$$\int_t^{t+1} F\, ds \le \left[\int_t^{t+1} F^4\, ds\right]^{1/4} \longrightarrow 0$$

as $t \to \infty$. Hence (32) implies that

$$\int_t^{t+1} \|u\|_2\, ds \le \int_t^{t+1} \|e^{-sA}a\|_2\, ds + C\int_t^{t+1} F\, ds \longrightarrow 0.$$

This shows assertion (v). We now prove (vi). Without loss of generality we may assume $0 < \alpha < 1$. From (32) and the assumption, we get

$$\|u(s)\|_2 \le C(s^{-\alpha} + F(s)). \tag{33}$$

From now on we regard $\|u\|_2$ and $\|\nabla u\|_2$ as defined on R by defining them to be zero on the negative real axis, and systematically apply Lemmas 3 and 4. First, (33) shows $\|u\|_2 \in L_w^{1/\alpha} + L^4 \subset L_w^{1/\alpha} + L_w^4$; so Lemma 4 implies $\|u\|_2 \in L^{1/\alpha}$ provided $\alpha < 1/4$. Hence we get (vi) for $\alpha < 1/4$. When $\alpha \ge 1/4$, Lemma 4 and (33) together imply $\|u\|_2 \in L_w^4$, and so Lemma 3 (i) gives $\|u\|_2 \|\nabla u\|_2 \in L_w^{4/3}$. Since $3/4 + 3/4 = 1 + 1/2$, Lemma 3 (ii) implies $F \in L_w^2$, so we obtain (vi) for $\alpha < 1/2$. When $\alpha \ge 1/2$, Lemma 4, (33) and the boundedness of $\|u\|_2$ together yield $\|u\|_2 \in L^2_w \cap L^\infty \subset L^q_w$ for all $2 < q < \infty$. Hence, Lemma 3 (i) implies $\|u\|_2 \|\nabla u\|_2 \in L^r_w$ with $1/r = 1 - \varepsilon$ for any $\varepsilon > 0$. Lemma 3 (ii) then shows $F \in L^s_w$ with $1/s = 3/4 - \varepsilon$, and this proves (vi) for $\alpha < 3/4$. When $\alpha \geq 3/4$, (33) and the foregoing result together show that $\|u\|_2 \in L^{1/\alpha}_w + L^p_w$ with $1/p = 3/4 - \varepsilon$. So, Lemma 4 yields $\|u\|_2 \in L^p_w$. But, the definition of the Lebesgue integral and (22) then yield

$$\int_0^\infty \|u\|_2^2 \, ds = 2 \int_0^\gamma t|E_t| \, dt, \tag{34}$$

where $\gamma = \|a\|_2$ and $E_t = R(\|u\|_2 > t)$. Since we can choose $p > 4/3$ arbitrarily close to $4/3$, we may assume that $-1 < 1 - p < -1/3$. It then follows from (34) and the estimate $t|E_t| \leq Ct^{1-p}$ that $\|u\|_2 \in L^2$ and therefore $\|u\|_2 \|\nabla u\|_2 \in L^1$. Hence Lemma 3 (iii) mplies $F \in L^{4/3}_w$. This completes the proof of (vi).

We next prove (vii). Let $w(t) = u(t) - u^0(t)$ with $u^0 = e^{-tA}a$. Since

$$\|w(s)\|_2 \leq CF(s) \leq F_1(s)^{1/2} F_2(s)^{1/2} \tag{35}$$

because $n = 3$, integrating this and applying (14) gives

$$\frac{1}{t}\int_0^t \|w\|_2 \, ds \leq Ct^{-1/4}\left[\frac{1}{t}\int_0^t \|u\|_2^2 \, ds\right]^{1/2}. \tag{36}$$

Since (v) implies

$$\frac{1}{t}\int_0^t \|u\|_2^2 \, ds \longrightarrow 0,$$

it follows from (36) that

$$\frac{1}{t}\int_0^t \|w\|_2 \, ds = o(t^{-1/4}).$$

This proves (vii). We now prove (viii). If $\alpha < 1/2$, then $\|u\|_2 \in L^{1/\alpha}_w$ by (vi); so we get $F \in L^p_w$ with $1/p = 1/4 + \alpha$ by Lemma 3. This, together with (35), shows the first part of (viii). When $\alpha > 1/2$, the proof of (vi) shows $F \in L^q_w$ with $1/q = 3/4 - \varepsilon$. Thus, Lemma 4 and (33) imply $\|u\|_2 \in L^q_w$, and so applying (34) yields $\|u\|_2 \in L^2$. Hence we obtain $F \in L^{4/3}_w$ by Lemma 3 (iii). This shows the second part of (viii). Suppose finally $\alpha = 1/2$. The proof of (vi) shows in this case $\|u\|_2 \in L^2_w \cap L^\infty$, so $\|u\|_2^2 \in L^1_w \cap L^\infty$. Hence Lemma 5 applies to deduce

$$\int_0^t \|u\|_2^2 \, ds \leq C \log(t + e),$$

and therefore (36) yields

$$\frac{1}{t}\int_0^t \|w\|_2\,ds \le Ct^{-3/4}\left(\int_0^t \|u\|_2^2\,ds\right)^{1/2} = O(t^{-3/4}[\log t]^{1/2}).$$

This completes the proof of Theorem 3.

We finally prove Corollary 4. From (25) with fixed $\lambda > 0$ we get, as $t \to \infty$,

$$\int_t^{t+1} \|E_\lambda u\|_2\,ds \le \int_t^{t+1} \|e^{-sA}a\|_2\,ds + C\lambda^{(n-3)/4}\left[\int_t^{t+1} F^4\,ds\right]^{1/4} \longrightarrow 0.$$

Since $\|\nabla u\|_2 \in L^2$, the estimate

$$\|\nabla u\|_2^2 \ge \lambda(\|u\|_2^2 - \|E_\lambda u\|_2^2)$$

implies

$$\begin{aligned}\int_t^{t+1} \|u\|_2\,ds &\le C_\lambda \int_t^{t+1} (\|E_\lambda u\|_2 + \|\nabla u\|_2)\,ds\\ &\le C_\lambda\left[\int_t^{t+1} \|E_\lambda u\|_2\,ds + \left(\int_t^{t+1} \|\nabla u\|_2^2\,ds\right)^{1/2}\right]\\ &\longrightarrow 0\end{aligned}$$

as $t \to \infty$. This proves Corollary 4.

REMARKS. (i) In case $n = 3$, Theorem 2 is directly obtained from Theorem 3. For example, the strong energy inequality (SE) implies $\|u(t)\|_2 \le \|u(s)\|_2$ for a.e. $s \in (0, t)$. Integrating this gives

$$\|u(t)\|_2 \le \frac{1}{t}\int_0^t \|u\|_2\,ds.$$

Assertions (i) and (ii) immediately follow from (v) and (vi), respectively. Assertions (iii) and (iv) are similarly obtained from (vii) and (viii), respectively, but the details are omitted here.

(ii) Employing Lemmas 3 and 4, we can also discuss L^2 decay for Navier-Stokes flows in arbitrary unbounded domains of space dimensions $n \le 4$, only with the aid of the L^2-theory for the Stokes operator A. Since in this case no L^p-theory for A is available, except in the case of halfspaces and exterior domains, we cannot obtain so good decay rates as deduced in this paper. However, this approach enables us to treat the stability problem for exterior stationary flows in R^3. The details are given in [2] and will be published elsewhere.

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