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Asymptotic periodicity of densities and ergodic properties for nonsingular systems

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§ 0. Introduction

Each one dimensional piecewise smooth expanding transformation T on a finite interval has the following ergodic property (A), which is the result of Li and Yorke [11] and Wagner [15], (see also Morita [13]). In one dimensional case m denotes the Lebesgue measure.

(A) *There exists a sequence of m-absolutely continuous T-invariant probability measures* $\{\mu_1, \cdots, \mu_l\}$ *with* $L_i := \operatorname{supp} \mu_i$ *for* $i = 1, \cdots, l$, *which has the following properties.*

(1) $\mu_i(L_i) = 1$ *for* $i = 1, \cdots, l$.

(2) (T, μ_i) *is ergodic for* $i = 1, \cdots, l$.

(3) $m(L_i \cap L_j) = 0$ *if* $i \neq j$.

(4) $T^{-1}(L_i) \supset L_i$ $m-a.e.$ *for* $i = 1, \cdots, l$.

(5) *If* η *is an m-absolutely continuous T-invariant probability measure,* η *can be written as a convex combination of* μ_i*'s.*

(6) *Put* $C = \bigcup_{n=0}^{\infty} \{x;\ T^n(x) \notin \bigcup_{i=1}^{l} L_i\}$, *then* $m(C) = 0$.

(7) *For* $i = 1, \cdots, l$, *there exists a collection of sets* $L_{i1}, \cdots, L_{ir(i)}$ *with the following properties*:

(*a*) $L_i = \bigcup_{j=1}^{r(i)} L_{ij}$.

(*b*) $m(L_{ij} \cap L_{ik}) = 0$ *if* $j \neq k$.

(*c*) $T^{-1}(L_{i,j+1}) \supset L_{ij}$ $m-a.e.$ *for* $j = 1, \cdots, r(i) - 1$, and $T^{-1}(L_{i1}) \supset L_{i,r(i)}$ $m-a.e.$

(*d*) $(T^{r(i)}, \mu_{ij})$ *is exact, where* $\mu_{ij} = r(i) \cdot \mu_i|_{L_{ij}}$.

On the other hand, Lasota, Li and Yorke [8] pointed out that the behavior of the Frobenius-Perron operator P associated with T is asymptotically periodic. Namely it has the following property (B).

(B) *There exists a sequence of densities* $g_1, \cdots, g_r$ *and a sequence of bounded linear functionals* $\lambda_1, \cdots, \lambda_r$ *such that*

$$\lim_{n \to \infty} \| P^n(f - \textstyle\sum_{i=1}^{r} \lambda_i(f) g_i) \|_{L^1(m)} = 0 \qquad \text{for } f \in L^1(m),$$

the densities $\{g_i\}$ *have mutually disjoint supports* ($g_i g_j = 0$ *for* $i \neq j$) *and*

$$P g_i = g_{\alpha(i)}$$

where $\{\alpha(1),\cdots,\alpha(r)\}$ *is a permutation of the integers* $\{1,\cdots,r\}$.

In addition to the property (B), we interoduce the following property (B*) which is weaker than (B).

(B*) *There exists a sequence of densities* $g_1,\cdots,g_l$ *and a sequence of bounded linear functionals* $\lambda_1,\cdots,\lambda_l$ *such that*

$$\lim_{n\to\infty}\| A_n f - \sum_{i=1}^{l}\lambda_i(f)g_i\|_{L^1(m)} = 0 \qquad \text{for } f\in L^1(m),$$

where $A_n f = \frac{1}{n}\sum_{k=0}^{n-1}P^k f$, *the densities* $\{g_i\}$ *have mutually disjoint supports* ($g_i g_j = 0$ *for* $i\neq j$) *and* $Pg_i = g_i$.

In this paper we discuss the relation between (A) and (B) or (B*) for a nonsingular transformation on a σ-finite measure space $(X, \mathscr{F}, m)$, which is more general than a piecewise smooth expanding transformation on a finite interval. As a matter of fact, in §4 we prove that (B) implies (A) and that (B*) implies (A) except (7). Conversely, in §5, we show that (3), (6) and (7) of (A) implies (B) ((7) is a strong condition). And we also show that (A) except (7) implies (B*). Therefore (A) is equivalent to (B) and (A) except (7) is equivalent to (B*).

Further, for a nonsingular transformation on a σ-finite measure space with the property (B*), we discuss the ergodic decomposition (the corresponding result for piecewise smooth expanding transformations is studied in [7]) and prove "the individual ergodic theorem" in §6. The central limit problem is also discussed in §7.

§1. Preliminaries

In this section we give a definition of the Frobenius-Perron operator and state its basic properties. Let $(X, \mathscr{F}, m)$ be a σ-finite measure space and $T: X \to X$ be a nonsingular transformation, that is, a measurable transformation satisfying $m(T^{-1}(A))=0$ for all $A\in\mathscr{F}$ with $m(A)=0$.

DEFINITION 1.1. The operator $P: L^1 \to L^1$ defined by

$$(1.1) \qquad \int_A Pf(x)m(dx) = \int_{T^{-1}(A)} f(x)m(dx) \qquad \text{for } A\in\mathscr{F},\ f\in L^1(m)$$

is called the Frobenius-Perron operator associated with (T, m). Clearly P is a positive operator. We define the average A_n of the Frobenius-Perron operator by

$$A_n f = \frac{1}{n}\sum_{k=0}^{n-1}P^k f \qquad \text{for } f\in L^1(m).$$

By $D(m) = D(X, \mathscr{F}, m)$ we shall denote the set of all densities associated with m on X, that is,

$$D(m) := \{f \in L^1(m); f \geqq 0 \text{ and } \|f\|_{L^1(m)} = 1\}.$$

For an $f \in D(m)$ we define a probability measure m_f on $(X, \mathscr{F})$ by

$$m_f(A) = \int_A f dm, \quad A \in \mathscr{F}.$$

An $f \in D(m)$ is called a stationary density of P if $Pf = f$ m-a.e.

Here we state some basic properties of Frobenius-Perron operators, which are well known and are easily proved.

LEMMA 1.1. (1) *P is characterized by the following*:

$$\int g(x) Pf(x) dm = \int f(x) g(T(x)) dm$$

for $f \in L^1(m)$, $g \in L^\infty(m)$.

(2) *For every integer $n \geqq 1$, $P^n = P_{T^n}$, where P_{T^n} is the Frobenius-Perron operator associated with (T^n, m).*

(3) *For $g \in D(m)$, $Pg = g$ if and only if m_g is T-invariant, that is $m_g(A) = m_g(T^{-1}(A))$ for all $A \in \mathscr{F}$.*

(4) *Let g be a stationary density. Then,*

$$g \cdot P_g^n f = P^n(f \cdot g) \quad m\text{-a.e. for } f \in D(m_g),$$

where P_g is the Frobenius-Perron operator associated with (T, m_g).

(5) *For $h \in L^\infty(m)$ and $g \in L^1(m)$, we have $P((Uh)g) = hPg$ where $Uh(x) = h(T(x))$.*

In this paper $S(f)$ denotes the support of a nonnegative function f, that is,

$$S(f) := \{x; f(x) > 0\}.$$

The following lemmas play an important role.

LEMMA 1.2. (1) *$S(f) \subset T^{-1}(S(Pf))$ m-a.e. for every $f \in D(m)$. In particular, if g is a stationary density of P, then*

$$S(g) \subset T^{-1}(S(g)) \quad m\text{-a.e.}$$

(2) *For $A \in \mathscr{F}$ with $A \subset T^{-1}A$, $S(f) \subset A$ implies $S(Pf) \subset A$ m-a.e. for $f \in D(m)$.*

PROOF. (1) Put $A = S(Pf)$ in the equality (1.1). Then the left hand side of the equality (1.1) is equal to 1. Hence we have

$$S(f) \subset T^{-1}(S(Pf)) \ m-\text{a.e.}$$

(2) The assumption of (2) implies that the right hand side of the equality (1.1) is equal to 1. So,

$$S(Pf) \subset A \ m-\text{a.e.}$$

LEMMA 1.3. *Assume that P has a stationary density g, and put*

$$C(g) := \bigcap_{n=0}^{\infty} \{x;\ T^n(x) \notin S(g)\} = \bigcap_{n=0}^{\infty} T^{-n}(X \setminus S(g)).$$

Then $C(g)$ is an invariant set with respect to (T, m), that is, $C(g) = T^{-1}(C(g))$ $m-a.e.$

This lemma is easily obtained from Lemma 1.2.

LEMMA 1.4. *For $A \in \mathscr{F}$ with $A \subset T^{-1}A$ $m-a.e.$ and for a stationary density g, we have*

$$P(g \cdot I_A) = g \cdot I_A \ m-a.e.$$

where I_A denotes the indicator function of the set A.

PROOF. We get

$$P(g \cdot I_A) \leqq P(g \cdot I_{T^{-1}A}) = I_A P g = I_A \cdot g \ m-\text{a.e.}$$

from Lemma 1.1(5). This shows

$$P(g \cdot I_A) = g \cdot I_A \ m-\text{a.e.},$$

because P preserves integrals.

§2. Ergodicity and exactness

Here we define ergodicity and exactness of a nonsingular transformation, and we state some conditions for ergodicity and exactness using Frobenius-Perron operators.

DEFINITION 2.1. Let $(X, \mathscr{F}, m)$ be a σ-finite measure space, and $T: X \to X$ a nonsingular transformation. Then, (T, m) is called *ergodic* if $m(A) = 0$ or $m(X \setminus A) = 0$ for every $A \in \mathscr{F}$ with $T^{-1}A = A$ $m-$a.e.

PROPOSITION 2.1 ([9] Theorem 4.2.2). *If (T, m) is ergodic, then there is at most one stationary density of the Frobenius-Perron operator P.*

DEFINITION 2.2. Let $(X, \mathscr{F}, \mu)$ be a probability space and $T: X \to X$ a measure preserving transformation, that is, μ is T-invariant. If $\bigcap_{n=0}^{\infty} T^{-n}\mathscr{F}$ is trivial, then (T, μ) is called *exact.*

We have the following useful lemma concerned with a Frobenius-Perron operator and exactness.

LEMMA 2.2. *Let $(X, \mathscr{F}, \mu)$ be a probability space, $T: X \to X$ a measure preserving transformation and P the Frobenius-Perron operator associated with (T, μ). Then, (T, μ) is exact if and only if*

$$\lim_{n\to\infty} \| P^n f - 1 \|_{L^1(\mu)} = 0 \qquad \textit{for any } f \in D(\mu).$$

This lemma is proved in [12]. So we omit the proof. Using this lemma, we prove the following proposition which gives the condition for the existence of an exact invariant measure.

PROPOSITION 2.3. *Let $(X, \mathscr{F}, m)$ be a σ-finite measure space, $T: X \to X$ a nonsingular transformation and P the Frobenius-Perron operator corresponding to (T, m). If there exists $g \in D(m)$ such that*

$$\lim_{n\to\infty} \| P^n f - g \|_{L^1(m)} = 0 \qquad \textit{for } f \in D(m) \textit{ with } S(f) \subset S(g), \tag{2.1}$$

then T preserves the measure m_g and (T, m_g) is exact.

Conversely, if there exists a stationary density g such that (T, m_g) is exact, then (2.1) *holds.*

PROOF. To prove the first part of the proposition, assume that there exists $g \in D(m)$ with (2.1). Then it is clear that $Pg = g$. Thus T preserves the measure m_g. Now let P_g be the Frobenius-Perron operator associated with (T, m_g). Then, from Lemma 1.1(4), for any $f_0 \in D(m_g)$,

$$\int_X |P_g^n f_0 - 1| \, dm_g = \int_X g \, |P_g^n f_0 - 1| \, dm$$
$$= \int_X |P^n(f_0 \cdot g) - g| \, dm.$$

Clearly $f_0 \cdot g \in D(m)$ and $S(f_0 \cdot g) \subset S(g)$. Hence, from the assumption, the right hand side of the above equality converges to 0, as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \| P_g^n f_0 - 1 \|_{L^1(m_g)} = 0 \qquad \text{for any } f_0 \in D(m_g).$$

By Lemma 2.2, this implies that (T, m_g) is exact.

To prove the second part of the proposition, assume that g is a stationary density of P and (T, m_g) is exact. For any $f \in D(m)$ with $S(f) \subset S(g)$, set

$$\hat{f}(x) = \begin{cases} f(x)/g(x) & x \in S(g) \\ 0 & x \notin S(g). \end{cases}$$

Then it is easy to see that $\hat{f} \in D(m_g)$. Thus, from Lemma 1.1(4) we have

$$\|P^n f - g\|_{L^1(m)} = \int_X |P^n(\hat{f} \cdot g) - g| \, dm$$

$$= \int_X |g P_g^n \hat{f} - g| \, dm = \int_X |P_g^n \hat{f} - 1| \, dm_g.$$

Since (T, m_g) is exact, the right hand side of this equality converges to 0, as $n \to \infty$, which implies (2.1).

§ 3. Asymptotic periodicity

In this section we introduce the notion of asymptotic periodicity of Frobenius-Perron operators.

DEFINITION 3.1. Let $(X, \mathscr{F}, m)$ be a σ-finite measure space and $P: L^1(m) \to L^1(m)$ be a Frobenius-Perron operator. Then, $\{P^n\}$ is called *asymptotically periodic* if there exists a sequence of densities $g_1, \cdots, g_r$ and a sequence of bounded linear functionals $\lambda_1, \cdots, \lambda_r$ such that

(1) $\lim_{n\to\infty} \|P^n(f - \sum_{i=1}^r \lambda_i(f) g_i)\|_{L^1(m)} = 0$ for all $f \in D(m)$,

(2) the densities $\{g_i\}$ have mutually disjoint supports ($g_i g_j = 0$ for $i \neq j$),

(3) $P g_i = g_{\alpha(i)}$, where $\{\alpha(1), \cdots, \alpha(r)\}$ is a permutation of the integers $\{1, \cdots, r\}$.

If $r = 1$ namely there exists a unique density g such that

$$\lim_{n\to\infty} \|P^n f - g\|_{L^1(m)} = 0 \qquad \text{for all } f \in D(m),$$

then $\{P^n\}$ is called *asymptotically stable.*

REMARK 3.1. Taking the cycles from the permutation α, Definition 3.1 may be rewritten in the following form: $\{P^n\}$ is called asymptotically periodic if there exists a sequence of densities

$$g_{11}, \cdots, g_{1r(1)}, \cdots, g_{l1}, \cdots, g_{lr(l)}$$

and a sequence of bounded linear functionals

$$\lambda_{11}, \cdots, \lambda_{1r(1)}, \cdots, \lambda_{l1}, \cdots, \lambda_{lr(l)}$$

such that

(1) $\lim_{n\to\infty} \|P^n(f - \sum_{i=1}^l \sum_{j=1}^{r(i)} \lambda_{ij}(f) g_{ij})\|_{L^1(m)} = 0$ for all $f \in D(m)$,

(2) the densities $\{g_{ij}\}$ have mutually disjoint supports, and

(3) for each i, $P g_{ij} = g_{i,j+1}$ for $1 \leq j \leq r(i) - 1$, $P g_{ir(i)} = g_{i1}$.

REMARK 3.2 ([8], see also [6]). If P is a constrictive Frobenius-Perron operator, namely there exists a weakly precompact set $F \subset L^1(m)$ such that

$$\lim_{n\to\infty} d(P^n f, F) = 0 \qquad \text{for all } f \in D(m),$$

where $d(g, F) = \inf_{f\in F} \|g - f\|_{L^1(m)}$, then $\{P^n\}$ is asymptotically periodic.

EXAMPLE 3.1. A simple example of the constrictive Frobenius-Perron operator is the one corresponding to a transformation T: $[0, 1] \to [0, 1]$ satisfying the following two conditions: (1) There is a partition $0 = a_0 < a_1 < \cdots < a_n = 1$ of $[0, 1]$ such that for each integer $i = 1, \cdots, n$ the restriction of T to (a_{i-1}, a_i) is a C^2 function, (2) $\inf|T'(x)| > 1$, $x \neq a_i$, $i = 0, \cdots, n$. For the proof of this, see [8], [9] and [10].

§4. Ergodic structure

From now on, we research what asymptotic periodicity of Frobenius-Perron operators implies. In this section we consider the following situation. Let $(X, \mathscr{F}, m)$ be a σ-finite measure space, T: $X \to X$ a nonsingular transformation, and P the Frobenius-Perron operatoer associated with (T, m). The main result is the following theorem.

THEOREM 4.1. *If $\{P^n\}$ is asymptotically periodic, then T has the property* (A) *stated in* §0.

First we prove a weaker version of Theorem 4.1, which is

THEOREM 4.2. *Assume that there exists a sequence of stationary densities $g_1, \cdots, g_l$ with mutually disjoint supports and a sequence of bounded linear functionals $\lambda_1, \cdots, \lambda_l$ such that*

$$\lim_{n\to\infty} \|A_n f - \textstyle\sum_{i=1}^{l} \lambda_i(f) g_i\| = 0 \qquad \textit{for } f \in L^1(m),$$

where $A_n f = \frac{1}{n}\sum_{k=0}^{n-1} P^k f$. Let $\mu_i = m_{g_i}$ for $i = 1, \cdots, l$.

Then $\mu_1, \cdots, \mu_l$ are m-absolutely continuous T-invariant probability measures with the properties from (1) *to* (6) *in* (A).

PROOF. The properties (1) and (3) are clear from the definition of μ_i.

(2) Let E be a measurable set such that $\mu_i(E) > 0$ and $T^{-1}E = E$ μ_i-a.e. Pick an $f_E \in D(m)$ whose support is contained in E. Then Lemma 1.2 (2) shows that $S(Pf_E) \subset E$ $m-$a.e. Using inductive argument, we have $S(A_n f_E) \subset E$ $m-$a.e. for all n. Therefore, from the assumption of this theorem it follows that $S(g_i) \subset E$ $m-$a.e. This implies that $\mu_i(E) = 1$. Hence (T, μ_i) is ergodic.

(4) is a direct consequence of Lemma 1.2 (1).

(5) Let f be an arbitrary stationary density. Then we have $A_n f = f$, and

so $f=\sum_{i=1}^{l}\lambda_i(f)g_i$ $m-$a.e., which implies (5).

(6) Put $g(x)=\frac{1}{r}\sum_{i=1}^{l}g_i(x)$. Then g is a stationary density. So from Lemma 1.3, C is an invariant set. Let $\{X_k\}$ be an increasing sequence of measurable sets such that $m(X_k)<\infty$ and $\cup X_k=X$. Then we have

$$m(C\cap X_k)=\int_C I_{C\cap X_k}dm=\int_{T^{-n}C} I_{C\cap X_k}dm$$
$$=\int_C P^n I_{C\cap X_k}dm \qquad \text{for all } n,\ k.$$

And so,

$$m(C\cap X_k)=\int_C A_n I_{C\cap X_k}dm \qquad \text{for all } n,\ k.$$

From the assumption of the theorem it follows that

$$\lim_{n\to\infty}\int_C A_n I_{C\cap X_k}dm=\int_C \sum_{i=1}^{l}\lambda_i(I_{C\cap X_k})g_i\,dm=0.$$

Therefore, $m(C\cap X_k)=0$ for all k, which implies $m(C)=0$. This completes the proof.

Now we are going to prove Theorem 4.1. Below in this section, let g_{ij}'s and λ_{ij}'s be as in Remark 3.1 and f be a density on (X, m). In the proof of Theorem 4.1, the following lemma is essential.

LEMMA 4.3. *If* $S(f)\subset S(g_{i_0j_0})$ $m-a.e.$, *then* $\lambda_{ij}(f)=0$ *for* $(i,j)\neq(i_0,j_0)$.

PROOF. Put $r^*=r(1)\cdots r(l)$, and $G=S(g_{i_0j_0})$. Then, by Lemma 1.2, we have

$$S(P^{r^*}f)\subset G\ m-\text{a.e.}$$

From this and $P^{r^*}g_{ij}=g_{ij}$ for each i, j, it followws that

$$\int_X |P^{nr^*}(f-\sum_{i=1}^{l}\sum_{j=1}^{r(i)}\lambda_{ij}(f)g_{ij})|\,dm$$
$$=\int_G |P^{nr^*}f-\sum_{i=1}^{l}\sum_{j=1}^{r(i)}\lambda_{ij}(f)g_{ij}|\,dm+\int_{X\backslash G}\sum_{i=1}^{l}\sum_{j=1}^{r(i)}\lambda_{ij}(f)g_{ij}\,dm$$
$$=\int_G |P^{nr^*}f-\lambda_{i_0j_0}(f)g_{i_0j_0}|\,dm+\int_{X\backslash G}\sum_{(i,j)\neq(i_0,j_0)}\lambda_{ij}(f)g_{ij}\,dm.$$

The left hand side of this equality converges to 0, as $n \to \infty$. Hence,

$$\int_{X \setminus G} \sum\nolimits_{(i,j) \neq (i_0, j_0)} \lambda_{ij}(f) g_{ij} dm = 0.$$

Thus $\lambda_{ij}(f) = 0$ for $(i, j) \neq (i_0, j_0)$.

PROOF OF THEOREM 4.1. Put

$$g_i(x) := \frac{1}{r(i)} \sum\nolimits_{j=1}^{r(i)} g_{ij}(x) \text{ and } \mu_i = m_{g_i} \text{ for each } i.$$

Then from Remark 3.1 (3), g_i is a stationary density of P. Hence μ_i is an m-absolutely continuous T-invariant probability measure. Set

$$\lambda_i(f) := \sum\nolimits_{j=1}^{r(i)} \lambda_{ij}(f).$$

Then, it follows from Theorem 4.2 that μ_i's have the properties from (1) to (6). Because all assumptions of Theorem 4.2 are satisfied, which is easily checked. So, it remains to show that μ_i's have the property (7).

Put $L_{ij} := S(g_{ij})$. Then (a) and (b) are trivial and (c) follows from Lemma 1.2. So all we have to do is to prove (d). Let f be any density whose support is contained in L_{ij}. Then by Lemma 4.3, we have

$$\lim_{n \to \infty} \| P^n(f - g_{ij}) \| = 0.$$

Using the fact $P^{r(i)} g_{ij} = g_{ij}$, we have

$$\lim_{n \to \infty} \| P^{nr(i)} f - g_{ij} \| = 0.$$

Let $P_{T^{r(i)}}$ be the Frobenius-Perron operator associated with $(T^{r(i)}, m)$. Then the above equality may be rewritten as

$$\lim_{n \to \infty} \| P_{T^{r(i)n}} f - g_{ij} \| = 0.$$

Therefore Proposition 2.3 implies that $(T^{r(i)}, \mu_{ij})$ is exact. This completes the proof.

REMARK 4.4. Assume that T is a binonsingular transformation, that is, $m(T^{-1}A) = 0$ for $A \in \mathscr{F}$ with $m(A) = 0$, $TA \in \mathscr{F}$ for $A \in \mathscr{F}$ and $m(TA) = 0$ for $A \in \mathscr{F}$ with $m(A) = 0$. Then, the conclusion (4), (7) (c) of Theorem 4.1 may be rewritten in the following form.

(4) $T(L_i) = L_i$ $m-$a.e. for $i = 1, \cdots, l$.

(7) (c) $T(L_{i,j}) = L_{i,j+1}$ $m-$a.e. for $j = 1, \cdots, r(i)-1$.

$T(L_{i,r(i)}) = L_{i,1}$ $m -$ a.e.

REMARK 4.5. As an example of transformations of infinite measure space to which our result is applicable, we have the transformations on the real line which were studied by M. Jabłoński and A. Lasota [3] and so on.

§5. Converse theorems

In this section we study the converse of Theorems 4.1 and 4.2. Let $(X, \mathscr{F}, m)$, T and P be the same as in §4. First we prove the converse of Theorem 4.1.

THEOREM 5.1. *Assume that there exists a sequence of densities*

$$g_{11},\cdots,g_{1r(1)},\cdots,g_{l1},\cdots,g_{lr(l)}$$

with the following properties:

(i) $\{g_{ij}\}$ *have mutually disjoint supports.*

(ii) $m(C)=0$ *where* $C=\bigcap_{n=0}^{\infty}\{x;\ T^n(x)\notin\bigcup_{ij}S(g_{ij})\}$.

(iii) *For each* i

$S(g_{ij})\subset T^{-1}(S(g_{i,j+1}))$ $m-a.e.$ *for* $j=1,\cdots,r(i)-1$,

$S(g_{ir(i)})\subset T^{-1}(S(g_{i1}))$ $m-a.e.$

(iv) *For each* i, j $\mu_{ij}=m_{g_{ij}}$ *is a* $T^{r(i)}$*-invariant probability measure and* $(T^{r(i)}, \mu_{ij})$ *is exact.*

Then, we have

(1) $\lim_{n\to\infty}\|P^n(f-\sum_{i=1}^{l}\sum_{j=1}^{r(i)}\lambda_{ij}(f)g_{ij})\|_{L^1(m)}=0$ *for* $f\in L^1(m)$,

where $\lambda_{ij}(f)=\int_{B_{ij}} fdm$ *and* $B_{ij}=\bigcup_{n=0}^{\infty}\{x;\ T^{r(i)n}(x)\in S(g_{ij})\}$, *and*

(2) $Pg_{ij}=g_{i,j+1}$ $m-a.e.$ *for* $j=1,\cdots,r(i)-1$,

$Pg_{ir(i)}=g_{i1}$ $m-a.e.$

REMARK 5.2. From Theorems 4.1 and 5.1, if P is a Frobenius-Perron operator, the bounded linear functional λ_i in Definition 3.1 can be written as

$$\lambda_i(f)=\int I_{D_i}\cdot fdm$$

for some set D_i.

To prove this theorem, we first prove the following proposition which is a special case of Theorem 5.1.

PROPOSITION 5.3. *If there exists a stationary density* g *such that*

(ii)′ $m(\lim_{n\to\infty}(T^{-n}(X\setminus S(g))))=0$, *and*

(iv)′ (T, m_g) *is exact,*

then $\{P^n\}$ *is asymptotically stable, that is,*

$$\lim_{n\to\infty} \| P^n f - g \|_{L^1(m)} = 0 \qquad \textit{for all } f \in D(m).$$

We begin the proof of this proposition with the following lemma.

LEMMA 5.4. *If there exists a stationary density* g *such that*

$$m(\lim_{n\to\infty} (T^{-n}(X \setminus S(g)))) = 0,$$

then for each $\varepsilon > 0$ *and each* $f \in D(m)$, *there exists an integer* $N = N(\varepsilon, f)$ *such that for each* $k \geqq N$ *we can find a density* h_k *which satisfies the following conditions*:

$$S(h_k) \subset S(g) \textit{ and } \| P^k f - h_k \|_{L^1(m)} < \varepsilon.$$

PROOF. For simplicity we put $S = S(g)$. From the assumption, we have

$$\lim_{n\to\infty} \int_{X \setminus S} P^n f \, dm = \lim_{n\to\infty} \int_{T^{-n}(X \setminus S)} f \, dm = 0 \qquad \text{for } f \in D(m). \tag{5.1}$$

Thus

$$\lim_{n\to\infty} \int_S P^n f \, dm = 1 \qquad \text{for } f \in D(m). \tag{5.2}$$

From this fact, we may define

$$h_k := \frac{(P^k f) \cdot I_S}{\| (P^k f) \cdot I_S \|_{L^1(m)}} \qquad \text{for sufficiently large } k.$$

Clearly, $S(h_k) \subset S = S(g)$ and

$$\begin{aligned} \| P^k f - h_k \|_{L^1(m)} &= \int_X | P^k f - h_k | \, dm \\ &= \int_S | P^k f - h_k | \, dm + \int_{X \setminus S} P^k f \, dm. \end{aligned} \tag{5.3}$$

Since

$$\begin{aligned} \int_S | P^k f - h_k | \, dm &= \int_S \left| P^k f - \frac{P^k f}{\| (P^k f) \cdot I_S \|_{L^1(m)}} \right| dm \\ &= \left(\frac{1}{\| (P^k f) \cdot I_S \|_{L^1(m)}} - 1 \right) \int_S P^k f \, dm \\ &\leqq \frac{1}{\| (P^k f) \cdot I_S \|_{L^1(m)}} - 1, \end{aligned}$$

it follows from (5.1)–(5.3) that

$$\lim_{k\to\infty} \| P^k f - h_k \|_{L^1(m)} = 0.$$

This completes the proof.

PROOF OF PROPOSITION 5.3. Let h_k be defined in Lemma 5.4 and $n > k$. We shall denote $\| \cdot \|_{L^1(m)}$ by $\| \cdot \|$. Then we have

$$\| P^n f - g \| \leqq \| P^n f - P^{n-k} h_k \| + \| P^{n-k} h_k - g \|.$$

Since P is a contraction,

$$\| P^n f - P^{n-k} h_k \| \leqq \| P^k f - h_k \|.$$

By the condition (iv)′, Proposition 2.3 implies that

$$\lim_{n\to\infty} \| P^{n-k} h_k - g \| = 0.$$

Thus

$$\limsup_{n\to\infty} \| P^n f - g \| \leqq \| P^k f - h_k \|.$$

Therefore by Lemma 5.4, we obtain

$$\lim_{n\to\infty} \| P^n f - g \| = 0 \qquad \text{for } f \in D(m).$$

In the following discussion in this section, we use the notation below.

$$L_{ij} := S(g_{ij}), \quad L_i := \bigcup_{j=1}^{r(i)} L_{ij}, \quad L := \bigcup_{i=1}^{l} L_i$$

$$B_{ij} := \bigcup_{n=0}^{\infty} \{x;\, T^{r(i)n}(x) \in L_{ij}\}, \quad B_i := \bigcup_{n=0}^{\infty} \{x;\, T^n(x) \in L_i\}$$

$$r^* := r_1 \cdots r_l$$

To finish the proof of Theorem 5.1, we give the following lemmas, in which we assume the conditions in Theorem 5.1.

LEMMA 5.5. *We have the conclusion* (2) *of Theorem* 5.1.

PROOF. We shall estimate $\| P g_{ij} - g_{i,j+1} \|_{L^1(m)}$. We have

$$\| P g_{ij} - g_{i,j+1} \| = \| P^{r(i)n} P g_{ij} - g_{i,j+1} \|.$$

Since

$$\int_{L_{i,j+1}} P g_{ij} dm = \int_{T^{-1} L_{i,j+1}} g_{ij} dm = 1,$$

it follows that $S(P g_{ij}) \subset L_{i,j+1}$. From this and the condition (iv), Proposition 2.3 implies

$$\lim_{n\to\infty} \| P^{r(i)n} P g_{ij} - g_{i,j+1} \| = 0.$$

Therefore $\| P g_{ij} - g_{i,j+1} \| = 0$ and hence $P g_{ij} = g_{i,j+1}$ $m-$a.e.

LEMMA 5.6. *(a) B_i's are mutually disjoint $m-a.e.$*
(b) $m(X \setminus \bigcup_{i=1}^{l} B_i) = 0$.
(c) B_{ij}'s are mutually disjoint $m-a.e.$
(d) $B_i = \bigcup_{j=1}^{r(i)} B_{ij}$ for $i = 1, \cdots, l$.

PROOF. (a) Since $\{T^{-n}(L_i)\}$ is increasing in n, we have

$$\begin{aligned} m(B_i \cap B_j) &= m(\textstyle\bigcup_{n=0}^{\infty} T^{-n}(L_i) \cap \bigcup_{n=0}^{\infty} T^{-n}(L_j)) \\ &\leqq m(\textstyle\bigcup_{n=0}^{\infty} (T^{-n}(L_i) \cap T^{-n}(L_j))) \\ &= m(\textstyle\bigcup_{n=0}^{\infty} T^{-n}(L_i \cap L_j)) \\ &\leqq \textstyle\sum_{n=0}^{\infty} m(T^{-n}(L_i \cap L_j)). \end{aligned}$$

Since T is a nonsingular transformation, it follows from the condition (i) of Theorem 5.1 that

$$m(T^{-n}(L_i \cap L_j)) = 0 \qquad \text{for } n \geqq 0,\ i \neq j.$$

Thus $m(B_i \cap B_j) = 0$ for $i \neq j$.
(b), (c) and (d) are easily obtained.

The following two lemmas are easily proved.

LEMMA 5.7. *(a) B_i is an invariant set with respect to (T, m), that is, $T^{-1}(B_i) = B_i$ $m-a.e.$ for $i = 1, \cdots, l$.*
(b) B_{ij} is an invariant set with respect to $(T^{r(i)}, m)$, that is, $T^{-r(i)}(B_{ij}) = B_{ij}$ $m-a.e.$ for $j = 1, \cdots, r(i)$, $i = 1, \cdots, l$.

LEMMA 5.8. *(a) $\lim_{n\to\infty} T^{-n}(B_i \setminus L_i) = \emptyset$ for $i = 1, \cdots, l$.*
(b) $\lim_{n\to\infty} T^{-nr^*}(B_{ij} \setminus L_{ij}) = \emptyset$ for $j = 1, \cdots, r(i)$, $i = 1, \cdots, l$.

PROOF OF THEOREM 5.1. By Lemmas 5.7 and 5.8, Proposition 5.3 implies that

$$\begin{aligned} &\lim_{n\to\infty} \| P^{nr^*}(f \cdot I_{B_{ij}} - \lambda_{ij}(f) g_{ij}) \| \\ &\leqq \lim_{n\to\infty} (\| P^{nr^*}(f^+ \cdot I_{B_{ij}} - \lambda_{ij}(f^+) g_{ij}) \| + \| P^{nr^*}(f^- \cdot I_{B_{ij}} - \lambda_{ij}(f^-) g_{ij}) \|) \\ &= 0 \text{ for } f \in L^1(m). \end{aligned}$$

Thus, it follows from Lemma 5.6 that

$$\lim_{n\to\infty} \| P^{nr^*}(f - \textstyle\sum_{i=1}^{l} \sum_{j=1}^{r(i)} \lambda_{ij}(f) g_{ij}) \| = 0 \qquad \text{for } f \in L^1(m).$$

Since P is a contractive operator,

$$\lim_{n\to\infty} \|P^n(f-\textstyle\sum_{i=1}^{l}\sum_{j=1}^{r(i)}\lambda_{ij}(f)g_{ij})\| = 0 \qquad \text{for } f\in L^1(m).$$

From this and Lemma 5.5, we obtain the theorem.

Next, we discuss the converse of Theorem 4.2.

THEOREM 5.9. *Assume that there exists a sequence of stationary densities* $g_1,\cdots,g_l$ *with the following properties*:

(i)* $\{g_i\}$ *have mutually disjoint supports.*

(ii)* $m(C)=0$ *where* $C=\bigcap_{n=0}^{\infty}\{x;\ T^n(x)\notin\bigcup_{i=1}^{l}S(g_i)\}$.

(iv)* (T, m_{g_i}) *is ergodic for* $i=1,\cdots,l$.

Then, we have

$$\lim_{n\to\infty} \|A_nf-\textstyle\sum_{i=1}^{l}\lambda_i(f)g_i\|_{L^1(m)} = 0 \qquad \textit{for all } f\in L^1(m),$$

where $\lambda_i(f)=\int_{B_i} f dm$ *and* $B_i=\bigcup_{n=0}^{\infty}\{x;\ T^n(x)\in S(g_i)\}$.

We begin the proof of this theorem with the following proposition which is a special case of Theorem 5.9 and corresponds to Proposition 5.3.

PROPOSITION 5.10. *If there exists a stationary density* g *such that*

(ii)*′ $m(\lim_{n\to\infty}(T^{-n}(X\setminus S(g))))=0$

(iv)*′ (T, m_g) *is ergodic*,

then

$$\lim_{n\to\infty} \|A_nf-g\|_{L^1(m)} = 0 \qquad \textit{for all } f\in D(m).$$

This proposition is easily obtained from the following proposition.

PROPOSITION 5.11. *If there exists a stationary density* g *such that*

$$m(\lim_{n\to\infty}(T^{-n}(X\setminus S(g))))=0,$$

then $\{P^nf\}$ *is weakly precompact for all* $f\in D(m)$.

PROOF OF PROPOSITION 5.10. From Proposition 5.11, $\{P^nf\}$ is weakly precompact for $f\in D(m)$. Thus the Kakutani-Yosida abstract ergodic theorem implies that there exists a stationary density g_f such that

$$\lim_{n\to\infty} \|A_nf-g_f\|_{L^1(m)} = 0 \qquad \text{for } f\in D(m).$$

Since the ergodicity of (T, m_g) implies that g is a unique stationary density, we have $g=g_f$ for all $f\in D(m)$.

In order to prove Proposition 5.11, we use the following lemma.

LEMMA 5.12. *For* $f\in D(m)$ *with* $S(f)\subset S(g)$, $\{P^nf\}$ *is weakly precompact.*

PROOF. Set $\mu = m_g$. For any $f \in D(\mu)$ and $\varepsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\int_A f d\mu < \varepsilon \text{ if } \mu(A) < \delta_1 \qquad \text{for all } A \in \mathscr{F}.$$

Since μ is T-invariant and m-absolutely continuous, there exists a $\delta > 0$ such that

$$\mu(T^{-n}A) = \mu(A) < \delta_1 \text{ if } m(A) < \delta \qquad \text{for all } n.$$

Therefore, for P_μ, the Frobenius-Perron operator corresponding to (T, μ), we have

$$\int_A P_\mu^n f d\mu = \int_{T^{-n}A} f d\mu < \varepsilon \quad \text{if } m(A) < \delta \qquad \text{for all } n.$$

Thus the criterion of weakly precompactness (see [1] VI.8.9) shows that $\{P_\mu^n f\}$ is weakly precompact with respect to the measure μ.

Now for any $f \in D(m)$ with $S(f) \subset S(g)$, set

$$f_g := \begin{cases} f(x)/g(x) & x \in S(g) \\ 0 & x \notin S(g). \end{cases}$$

Then it is clear that $f_g \in D(\mu)$. Hence the above discussion implies that there exists an $h \in L^1(\mu)$ such that

$$\lim_{n_k \to \infty} \langle P_\mu^{n_k} f_g, e \rangle_\mu = \langle h, e \rangle_\mu \qquad \text{for } e \in L^\infty(\mu).$$

So

$$\lim_{n_k \to \infty} \langle g \cdot P_\mu^{n_k} f_g, e \rangle_m = \langle g \cdot h, e \rangle_m \qquad \text{for } e \in L^\infty(\mu).$$

From Lemma 1.1 (4) and $L^\infty(m) \subset L^\infty(\mu)$, it follows that

$$\lim_{n_k \to \infty} \langle P^{n_k}(f_g \cdot g), e \rangle_m = \langle g \cdot h, e \rangle_m \qquad \text{for } e \in L^\infty(m).$$

Thus

$$\lim_{n_k \to \infty} \langle P^{n_k} f, e \rangle_m = \langle g \cdot h, e \rangle_m \qquad \text{for } e \in L^\infty(m).$$

This completes the proof.

PROOF OF PROPOSITION 5.11. Let $\varepsilon > 0$ and $f \in D(m)$ be arbitrary and h_k be as in Lemma 5.4. Then it follows from Lemma 5.12 that $\{P^{n-k} h_k\}$ is weakly precompact, that is, there exists a density g_0 and a subsequence $\{n_j\}$ of $\{n\}$ such that

$$\lim_{n_j \to \infty} \langle P^{n_j - k} h_k, e \rangle_m = \langle g_0, e \rangle_m \qquad \text{for } e \in L^\infty(m).$$

For $n_j > k$, we have

$$\begin{aligned}&|\langle P^{n_j}f, e\rangle_m - \langle g_0, e\rangle_m|\\ &\leqq |\langle P^{n_j}f, e\rangle_m - \langle P^{n_j-k}h_k, e\rangle_m| + |\langle P^{n_j-k}h_k, e\rangle_m - \langle g_0, e\rangle_m|.\end{aligned}$$

Since P is a contraction,

$$\|P^{n_j}f - P^{n_j-k}h_k\|_{L^1(m)} \leqq \|P^k f - h_k\|_{L^1(m)}.$$

Therefore

$$\lim_{n_j\to\infty} |\langle P^{n_j}f, e\rangle_m - \langle g_0, e\rangle_m| \leqq \|e\|_{L^\infty(m)} \cdot \|P^k f - h_k\|_{L^1(m)} \leqq \varepsilon \|e\|_{L^\infty(m)}.$$

Thus $\{P^n f\}$ is weakly precompact for $f \in D(m)$.

Now we return to the proof of Theorem 5.9. First we remark that Lemma 5.6 (a) and (b), Lemma 5.7 (a) and Lemma 5.8 (a) still hold for B_i's in Theorem 5.9.

PROOF OF THEOREM 5.9. By Lemmas 5.7 and 5.8, Proposition 5.10 implies that

$$\begin{aligned}&\lim_{n\to\infty} \|A_n(f\cdot I_{B_i}) - \lambda_i(f)\cdot g_i\|\\ &\leqq \lim_{n\to\infty} (\|A_n(f^+\cdot I_{B_i}) - \lambda_i(f^+)\cdot g_i\| + \|A_n(f^-\cdot I_{B_i}) - \lambda_i(f^-)\cdot g_i\|)\\ &= 0 \qquad \text{for } f \in L^1(m).\end{aligned}$$

Thus it follows from Lemma 5.6 that

$$\lim_{n\to\infty} \|A_n f - \textstyle\sum_{i=1}^{l} \lambda_i(f)\cdot g_i\| = 0 \qquad \text{for } f \in L^1(m).$$

REMARK 5.13. G. Keller [5] proved the following result.
Keller's result. If $T: [0, 1] \to [0, 1]$ is S-unimodal and if

$$\limsup_{n\to\infty} n^{-1} \log |(T^n)'(x)| > 0$$

on a set of x's of positive Lebesgue measure, then there is a unique m-absolutely continuous ergodic T-invariant probaility measure μ with the property that $m(\bigcup_{n=0}^{\infty} T^{-n}(\operatorname{supp}\mu)) = 1$ and for some power T^p of T the measure μ can be decomposed into p components each of which is exact for T^p.

From this result and Theorem 5.1 it follows that the Frobenius-Perron operator P associated with T has asymptotic periodicity with only one cycle, that is, there exists a sequence of densities $g_1, \cdots, g_p$ with the following properties:

(1) $\lim_{n\to\infty} \|P^n(f - \sum_{j=1}^{p} \lambda_j(f) g_j)\|_{L^1(m)} = 0 \qquad \text{for } f \in L^1(m),$

where $\lambda_j(f) = \int_{B_j} f dm$ and $B_j = \bigcup_{n=0}^{\infty} \{x;\ T^{pn}(x) \in S(g_j)\}$, and

(2) $P g_j = g_{j+1}$ $m-$a.e. for $j = 1, \cdots p-1$,

$P g_p = g_1$ $m-$a.e.

§6. Ergodic decomposition

In this section we discuss the ergodic decomposition of nonsingular transformations and prove the "individual ergodic theorem". The first main result in the present section is the following theorem.

THEOREM 6.1 (*Ergodic decomposition theorem*). *Under the assumption of Theorem* 4.2, *put*

$$B_i := \bigcup_{n=0}^{\infty} \{x;\ T^n(x) \in S(g_i)\} \qquad \textit{for } i = 1, \cdots, l.$$

Then $\{B_1, \cdots, B_l\}$ *is a measurable partition of* X *such that* $B_i = T^{-1} B_i$ $m-a.e.$ *and* $(T, m|_{B_i})$ *is ergodic.*

PROOF. It was already proved in Lemmas 5.6 and 5.7 that $\{B_1, \cdots, B_l\}$ is a measurable partition and $B_i = T^{-1} B_i$ $m -$ a.e. So all we have to do is to show that $(T, m|_{B_i})$ is ergodic. In the following proof, B denotes B_i for simplicity. Let A be a measurable set such that $T^{-1} A = A$ $m|_B-$a.e.

First suppose $m(S(g_i) \cap A) > 0$. Lemma 1.4 shows that $P(g_i \cdot I_A) = g_i \cdot I_A$ $m|_B-$a.e. Hence, applying Proposition 2.1 to $(T, m|_B)$, we get $g_i \cdot I_A = g_i$ $m-$a.e. and so

$$S(g_i) \subset A \ \ m-\text{a.e.} \tag{6.1}$$

Let $\{X_k\}$ $X_k \in \mathscr{F}$ be an increasing sequence such that $m(X_k) < \infty$ and $\cup X_k = B$. From the fact $T^{-1}(A^C) = A^C$ $m|_B-$a.e. and (6.1), we have

$$m|_{B \cap X_k}(A^C) = m|_{B \cap X_k}(T^{-n}(A^C)) \leqq m|_{B \cap X_k}(T^{-n}(B \setminus S(g_i))).$$

Since the right hand side of the above inequality converges to 0 as $n \to \infty$,

$$m|_{B \cap X_k}(A^C) = 0.$$

Therefore

$$m|_B(A^C) = 0.$$

If $m(S(g_i) \cap A) = 0$, then $S(g_i) \subset A^C$ $m-$a.e. Substituting A^C for A in the above discussion, we get

$$m|_B(A) = 0.$$

Thus $(T, m|_B)$ is ergodic.

Related to this theorem, we have the following theorem, which is obtained from the Birkhoff Individual Ergodic Theorem.

THEOREM 6.2. *Under the assumption of the above theorem, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f(x)g_i(x)dm \quad m|_{B_i}-a.e.$$

for $f\in L^1(\mu_i)$, *where* $\mu_i = m_{g_i}$. *Consequently,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \sum_{i=1}^{l} I_{B_i}(x)\int_X f(x)g_i(x)dm \quad m-a.e.$$

for every $f\in \bigcap_{i=1}^{l} L^1(\mu_i)$.

For the proof of Theorem 6.2 it is sufficient to prove the following lemma.

LEMMA 6.3. *If there exists a unique stationary density g and*

$$m(C) = 0 \text{ where } C = \bigcap_{n=0}^{\infty}\{x;\ T^n(x)\notin S(g)\}, \tag{6.3}$$

then, for $f\in L^1(m_g)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f(x)g(x)dm \quad m-a.e.$$

PROOF. Let $f\in L^1(m_g)$. By the virtue of the Birkhoff Individual Ergodic Theorem, we may choose a set $N\subset S(g)$ with m_g-measure 0 such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f(x)g(x)dm \qquad \text{for each } x\in S(g)\backslash N.$$

Put $S_j := \bigcup_{n=0}^{j} T^{-n}(S(g))$, $S_\infty := \lim_{j\to\infty} S_j$ and $N_\infty := \bigcup_{n=0}^{\infty} T^{-n}(N)$. Then, for each $x\in S_\infty\backslash N_\infty$, there exists an integer j such that $x\in S_j$ and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = \int_X f(x)g(x)dm.$$

Thus, all we have to do is to prove that

$$m(X\backslash(S_\infty\backslash N_\infty)) = 0.$$

Since $m(X\backslash(S_\infty\backslash N_\infty)) \leqq m(C) + m(N_\infty)$ and T is a nonsingular transformation, it is sufficient to prove $m(N) = 0$.

First we assume $m(X) < \infty$. Since $m_g(N) = 0$, we have

$$\begin{aligned} m(N) &= m(\{x\in X;\, g(x)I_N(x) > 0\}) \\ &= m(\{x\in X;\, g(x)I_N(x) \geqq k^{-1}\}) \\ &\quad + m(\{x\in X;\, 0 < g(x)I_N(x) < k^{-1}\}) \\ &\leqq k\int_X g(x)I_N(x)\,dm + m(\{x\in X;\, 0 < g(x)I_N(x) < k^{-1}\}) \\ &= m\{x\in X;\, 0 < g(x)I_N(x) < k^{-1}\}. \end{aligned}$$

Letting $k\to\infty$, we obtain $m(N)=0$.

In the case $m(X)=\infty$, there exists a sequence of measurable sets $\{X_n\}$ such that $m(X_n)<\infty$ and $X_n\uparrow X$. From the discussion in the case $m(X)<\infty$, we have $m(N\cap X_n)=0$ for all n. Therefore $m(N)=0$.

REMARK 6.4. In the above lemma, the assumption $m(C)=0$ may be replaced by the ergodicity of (T, m). In fact, since C^{C} is an invariant set and has positive m-measure, the ergodicity of (T, m) implies $m(C)=0$.

Conversely we have:

REMARK 6.5. If there exists a stationary density g and (6.3) holds, then (T, m) is ergodic. In fact, suppose that there is a measurable set E such that $m(E)>0$, $m(E^{C})>0$ and $T^{-1}E=E$ $m-$a.e. And put $f=I_E$. Then there exists an $x\in E$ such that

$$1=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} I_E(T^k x)=\int_X I_E(x)g(x)dm.$$

Thus we have $E\supset S(g)$ $m-$a.e. Similarly we have $E^{C}\supset S(g)$ $m-$a.e. This is a contradiction. Therefore (T, m) is ergodic.

§7. Central limit theorems

The aim of this section is to give central limit theorems of mixed-type under the assumption of Theorem 4.2 by means of the one for stationary processes (Lemma 7.2) proved by Gordin. For the simplicity of notation, we restrict ourselves to the case that T is not invertible with respect to each invariant measure.

In the present section we use the following notation

$$G(\sigma, z):=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{z}\exp\left(\frac{-x^2}{2\sigma^2}\right)dx \qquad \text{for } \sigma>0,$$

$$G(0, z):=\begin{cases} 1 & z>0 \\ 0 & z\leqq 0. \end{cases}$$

LEMMA 7.1. *In addition to the assumption of Theorem 4.2, assume that for each i, $f \in L^2(\mu_i)$, the limit*

$$\sigma_i^2 = \lim_{n \to \infty} \int \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(T^k x) \right)^2 d\mu_i$$

exists and

$$\lim_{n \to \infty} \mu_i \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(T^k x) < z \right\} = G(\sigma_i, z) \tag{7.1}$$

at every continuity point of the right hand side, where $d\mu_i = g_i dm$.

Then, for an m-absolutely continuous (not necessarily T-invariant) probability measure ν, we have

$$\lim_{n \to \infty} \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(T^k x) < z \right\} = \sum_{i=1}^{l} \lambda_i(h) G(\sigma_i, z)$$

at every continuity point of the right hand side, where $h\,dm = d\nu$.

PROOF. We first prove the following equality:

$$\lim_{n \to \infty} \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x)) \right) h(x) dm \tag{7.2}$$

$$= \lim_{n \to \infty} \sum_{i=1}^{l} \lambda_i(h) \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x)) \right) g_i(x) dm.$$

For any $\varepsilon > 0$, choose an integer p such that

$$\| A_p h - \sum_{i=1}^{l} \lambda_i(h) g_i \|_{L^1(m)} < \varepsilon. \tag{7.3}$$

From Lemma 1.1 (1), it follows that

$$\lim_{n \to \infty} \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x)) \right) h(x) dm$$

$$= \lim_{n \to \infty} \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=q}^{n-1} (f(T^k x)) \right) h(x) dm$$

$$= \lim_{n \to \infty} \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=0}^{n-1-q} (f(T^k x)) \right) P^q h(x) dm$$

$$= \lim_{n \to \infty} \int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x)) \right) P^q h(x) dm \qquad \text{for } q < p.$$

As a consequence, we have

$$\text{(7.4)} \qquad \lim_{n\to\infty}\int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}}\sum_{k=0}^{n-1}(f(T^k x))\right)h(x)dm$$

$$= \lim_{n\to\infty}\int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}}\sum_{k=0}^{n-1}(f(T^k x))\right)A_p h(x)dm.$$

(7.3) and (7.4) imply (7.2).

From (7.1) and (7.2), it follows that

$$\lim_{n\to\infty}\int \exp\left(\frac{\sqrt{-1}\,\theta}{\sqrt{n}}\sum_{k=0}^{n-1}(f(T^k x))\right)h(x)dm = \sum_{i=1}^{l}\lambda_i(h)\exp\left(-\frac{\theta^2\sigma_i^2}{2}\right).$$

This completes the proof.

In the above lemma, we assume that a central limit theorem holds for each invariant measure. Next, we quote a central limit theorem for stationary processes by Gordin.

LEMMA 7.2 ([2]). *Let $(X, \mathcal{F}, \mu)$ be a probability space, T be an ergodic not invertible measure preserving transformation. Assume that f belongs to $L^2(\mu)$ with $\int f d\mu = 0$ and that*

$$\sum_{k=0}^{\infty}\|E_\mu(f\,|\,T^{-k}\mathcal{F})\|_{L^2(\mu)} < \infty.$$

Then

$$\sigma^2 = \lim_{n\to\infty}\int\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}f(T^k x)\right)^2 d\mu$$

exists and

$$\lim_{n\to\infty}\mu\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}f(T^k x) < z\right\} = G(\sigma, z)$$

at every continuity point of the right hand side.

The following Theorem follows from Lemma 7.1 and Lemma 7.2.

THEOREM 7.3. *Suppose that the assumption of Theorem 4.2 is fulfilled. Assume*

$$f \in \bigcap_{i=1}^{l} L^2(\mu_i)$$

and

$$\text{(7.5)} \qquad \sum_{k=0}^{\infty}\|E_{\mu_i}(f\,|\,T^{-k}\mathcal{F}) - f^*\|_{L^2(\mu_i)} < \infty,$$

where $d\mu_i = g_i dm$,

$$f^* = \sum_{i=1}^{l} I_{B_i}(x) \int_X f(x) g_i(x) dm \quad \textit{and} \quad B_i = \bigcup_{n=0}^{\infty} T^{-n} S(g_i).$$

Then the limit

$$\sigma_i^2 = \lim_{n\to\infty} \int \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x) - f^*(x)) \right)^2 d\mu_i$$

exists for each i. Further, if ν *is an m-absolutely continuous probability measure, then*

$$\lim_{n\to\infty} \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (f(T^k x) - f^*(x)) < z \right\} = \sum_{i=1}^{l} \lambda_i(h) G(\sigma_i, z)$$

at every continuity point of the right hand side, where $h\,dm = d\nu$.

PROOF. Substitute $f - f^*$ for f in Lemma 7.2. Then for all i, the assumptions of Lemma 7.2 are satisfied. Hence Lemma 7.1 immediately shows our result.

REMARK 7.4. If the Frobenius-Perron operator P associated with (T, m) is asymptotically periodic, the condition (7.5) can be replaced by

$$\sum_{k=0}^{\infty} \| P_{\mu_i}^k (f - E_{\mu_i}(f | \mathscr{F}_\infty)) \|_{L^2(\mu_i)} < \infty. \tag{7.6}$$

In fact, we have

$$\begin{aligned} & \| P_{\mu_i}^k (f - E_{\mu_i}(f | \mathscr{F}_\infty)) \|_{L^2(\mu_i)} \\ &= \| U^k P_{\mu_i}^k (f - E_{\mu_i}(f | \mathscr{F}_\infty)) \|_{L^2(\mu_i)} \\ &= \| E_{\mu_i}(f | T^{-k}\mathscr{F}) - E_{\mu_i}(f | \mathscr{F}_\infty) \|_{L^2(\mu_i)}. \end{aligned}$$

Remark that $E_{\mu_i}(U^k f | T^{-n}\mathscr{F}) = U^k E_{\mu_i}(f | T^{-(n-k)}\mathscr{F})$ for all $0 \leqq k \leqq n$. Then it is clear that condition (7.6) implies

$$\sum_{k=0}^{\infty} \| E_{\mu_i}(F | T^{-kr(i)}\mathscr{F}) - E_{\mu_i}(F | \mathscr{F}_\infty) \|_{L^2(\mu)} < +\infty,$$

where $F(x) = \{f(x) + f(Tx) + \cdots + f(T^{r(i)-1}x)\}/(r(i))^{1/2}$. Since $E_{\mu_i}(F | \mathscr{F}_\infty) = F^*(x)$ μ_i-a.e., it follows from Theorem 7.3 that

$$\lim_{n\to\infty} \nu \left\{ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (F(T^{kr(i)} x) - F^*(x)) < z \right\} = \sum_{i=1}^{l} \lambda_i(h) G(\sigma_i, z)$$

at every continuity point of the right hand side. This implies that (7.5) can be replaced by (7.6) in this case.

Next we give the examples which satisfy the condition (7.5).

EXAMPLE 7.1. Let T be a transformation as in Example 3.1 and f: $[0, 1] \to R$ be of bounded variation. Then the condition (7.5) is satisfied. For the proof of this fact, see [5], in which a simple proof is given by means of a spectral theorem of I. Tulcea and Marinescu.

EXAMPLE 7.2. Let $(X, \mathscr{F}, \mu)$ be a probability space and T be a measure preserving transformation on it. Assume that there exists a family of sub-σ-fields

$$\{\mathscr{F}_n^m : 0 \leqq n \leqq m \leqq \infty\}$$

satisfying

(i) $\mathscr{F}_n^m \subset \mathscr{F}_{n'}^{m'}$: for $n' \leqq n \leqq m \leqq m'$,

(ii) $\mathscr{F}_0^\infty = \mathscr{F}$,

(iii) $T^{-1}\mathscr{F}_n^m = \mathscr{F}_{n+1}^{m+1}$.

Define $\phi(n)$ by

$$\phi(n) := \sup|\mu(A \cap B) - \mu(A)\mu(B)|/\mu(A),$$

where the supremum is taken over all $A \in \mathscr{F}_0^k$ with $\mu(A) \neq 0$, $B \in \mathscr{F}_{k+n}^\infty$ and $k \geqq 0$.

It is not difficult to verify that if

$$\sum_{k=1}^\infty (\phi(k))^{1/2} < \infty$$

and if $f \in L^\infty$ satisfies

$$\sum_{k=0}^\infty \| E(f|\mathscr{F}_0^k) - f \|_{L^2(\mu)} < \infty,$$

then we have

$$\sum_{k=1}^\infty \| E(f|\mathscr{F}_k^\infty) - E(f) \|_{L^2(\mu)} < \infty.$$

Hence the condition (7.5) is valid for such a function f.

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