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# A spectrum whose $BP_*$-homology is $(BP_*/I_5)[t_1]$

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## §1. Introduction

For each prime $p$, we have the Brown-Peterson spectrum $BP$ whose coefficient is the polynomial ring $BP_* = Z_{(p)}[v_1, v_2, \cdots]$ over Hazewinkel's generators $v_i$ with $|v_i| = 2p^i - 2$. This has the invariant prime ideals $I_n = (p, v_1, \cdots, v_{n-1})$ for $n \geq -1$, where $I_{-1} = (0)$ and $I_0 = (p)$. Then the Toda-Smith spectrum $V(n)$ is the finite ring spectrum characterized by

$$BP_* V(n) = BP_*/I_{n+1}$$

for $n \geq -1$. Once we know the existence of this spectrum, we can construct a family of nontrivial elements of the homotopy groups $\pi_* S$ of the sphere spectrum $S$, which are known as the Greek letter elements. The existence of the spectrum $V(n)$ is known only for $n < 4$. In this case $V(n)$ exists if and only if the prime $p$ is greater than $2n$. It seems that $V(4)$ exists for a large prime $p$, but still now we have no way to prove it. We so consider a similar spectrum $W_k(n)$ defined by

$$BP_* W_k(n) = (BP_*/I_{n+1})[t_1, \cdots, t_k]$$

as a $BP_*BP$-comodule subalgebra of $BP_*BP/I_{n+1} = (BP_*/I_{n+1})[t_1, t_2, \cdots]$. Then $V(4) = W_0(4)$. If $W_k(n)$ does not exist for some $k$, neither does $V(n)$. However by computing obstructions we obtain the existence of $W_k(4)$ for $k > 1$ at a prime $p > 7$ in [6], and in this paper we prove the following

THEOREM. *Let $p$ be a prime number greater than 7. Then $W_1(4)$ exists.*

In §2 we recall Ravenel's ring spectra $T(k)$ and show the following

PROPOSITION. *Let $p$ be any prime and $k$ and $n$ non-negative integers with $k \geq n$. Then there exists a $T(k)$-module spectrum $W_k(n)$.*

In §§3–4 we compute the differentials of the Adams-Novikov spectral sequence for the spectrum $W_1(3)$ and show the above theorem.

## §2. $W_k(n)$

Let $p$ denote an odd prime number and $S$ be the sphere spectrum. The

Brown-Peterson spectrum $BP$ is a commutative ring spectrum with the structure maps $\iota: S \to BP$ and $\mu: BP \wedge BP \to BP$ and gives rise to the homology theory with the coefficient ring

$$BP_*(S) = BP_* = \mathbf{Z}_{(p)}[v_1, v_2, \cdots]$$

with $|v_i| = 2p^i - 2$. The $BP_*$-homology of $BP$ is the polynomial

$$BP_*BP = BP_*[t_1, t_2, \cdots]$$

with $|t_i| = 2p^i - 2$. Besides $BP_*BP$ becomes a Hopf algebroid over $BP_*$ from the ring spectrum $BP$ by a standard argument (*cf.* [1]).

In [3, p. 369], Ravenel gives a spectrum $T(k)$ for each $k \geq 0$ with

$$BP_* T(k) = BP_*[t_1, t_2, \cdots, t_k]$$

as a comodule algebra over $BP_*BP$ ([2]). Since there is a $(2p^{k+1} - 3)$-equivalence $T(k) \to BP$, we see that

$$T(k)_* = \mathbf{Z}_{(p)}[v_1, \cdots, v_k] \oplus \operatorname{Ker}(T(k)_* \longrightarrow BP_*). \tag{2.1}$$

Let $I_n = (p, v_1, \cdots, v_{n-1})$ denote the invariant prime ideal of $BP_*$. Then we consider the Toda-Smith spectrum $V(n)$ for each $n \geq -1$ defined by

$$BP_* V(n) = BP_*/I_{n+1}.$$

On the existence of the spectrum $V(n)$, we have results only for the cases $n \leq 3$, which state that $V(n)$ exists if and only if the prime $p \geq 2n + 1$ (*cf.* [7], [8], [5]). We define a spectrum $W_k(n)$ for $k \geq 0$ and $n \geq -1$ to be the one with

$$BP_* W_k(n) = (BP_*/I_{n+1})[t_1, \cdots, t_k]$$

as a comodule subalgebra of $BP_*BP/I_{n+1}$. Note that $W_k(-1) = T(k)$. The spectrum $W_k(n)$ exists if $n \leq 3$ and the prime $p \geq 2n + 1$. In fact, put $W_k(n) = T(k) \wedge V(n)$. In [2, Prop. 1.4.3] Hopkins shows that $T(k) \wedge T(k)$ is homotopic to $T(k) \wedge B(k)$ for the Moore spectrum $B(k)$ for the ring $\mathbf{Z}[t_1, \cdots, t_k]$. Similar results hold for $W_k(n)$:

LEMMA 2.2. *Let $k$ and $l$ be fixed non-negative integers and suppose that there exist spectra $W_k(n)$ for integers $k$ and $n$ with $l \geq n$ and maps $\eta_{n+1}: W_k(n) \to W_k(n)$ for $l > n$ such that $W_k(n+1)$ is a cofiber of $\eta_{n+1}$. Then $T(k) \wedge W_k(n)$ is homotopic to $W_k(n) \wedge B(k)$ for $l \geq n$. Furthermore $W_k(n)$ for each $k \geq n$ is a $T(k)$-module spectrum.*

PROOF. Let $s_k: T(k) \wedge T(k) \to T(k) \wedge B(k)$ be the homotopy equivalence. Then we define a map $s_{k,n}: W_k(n) \wedge B(k) \to T(k) \wedge W_k(n)$ by the composition

$(\mu_k \wedge 1)(s_k^{-1} \wedge 1)(\iota_k \wedge 1 \wedge 1)(1 \wedge T)$, where $T\colon W_k(n) \wedge B(k) \to B(k) \wedge W_k(n)$ is the switching map and $\mu_k\colon T(k) \wedge T(k) \to T(k)$ and $\iota_k\colon S \to T(k)$ are the structure maps of the ring spectrum $T(k)$. Then we have the commutative diagram

$$\begin{array}{ccccc} W_k(n) \wedge B(k) & \xrightarrow{\eta_{n+1} \wedge 1} & W_k(n) \wedge B(k) & \longrightarrow & W_k(n+1) \wedge B(k) \\ \downarrow{s_{k,n}} & & \downarrow{s_{k,n}} & & \downarrow{s_{k,n+1}} \\ T(k) \wedge W_k(n) & \xrightarrow{1 \wedge \eta_{n+1}} & T(k) \wedge W_k(n) & \longrightarrow & T(k) \wedge W_k(n+1) \end{array}$$

by the definition of the map $s_{k,n}$. Notice that $s_{k,-1} = s_k^{-1}$. Then we inductively obtain from the five lemma that $s_{k,n}$ for $n \le l$ are all homotopy equivalences. We denote the inverse of $s_{k,n}$ by $t_{k,n}$. Note here that there exist maps $i\colon S \to B(k)$ and $j\colon B(k) \to S$ of degree 0 such that $ji = 1$. Suppose next that $\varphi_n = (1 \wedge j)t_{k,n}(\iota_k \wedge 1)\colon W_k(n) \to W_k(n)$ is a homotopy equivalence, and we also see that $\varphi_{n+1} = (1 \wedge j)t_{k,n+1}(\iota_k \wedge 1)$ is a homotopy equivalence. Since $\varphi_{-1} = (1 \wedge j)s_k(\iota_k \wedge 1) = 1$, the induction shows that every $\varphi_n$ for $n \le l$ is a homotopy equivalence. Define $\nu_{k,n}\colon T(k) \wedge W_k(n) \to W_k(n)$ by the composition $(1 \wedge j)(\varphi_n^{-1} \wedge 1)t_{k,n}$. Then we see that $\nu_{k,n}(\iota_k \wedge 1) = 1$ and both $\nu_{k,n}(\mu_k \wedge 1)$ and $\nu_{k,n}(1 \wedge \nu_{k,n})$ turn out to be the same map $1 \wedge j \wedge j$. These imply that $W_k(n)$ is a $T(k)$-module spectrum with structure map $\nu_{k,n}$. q.e.d.

Suppose that integers $k$ and $n$ satisfy the inequality $k > n$. Then $v_{n+1} \in T(k)_*$ by (2.1), and so we define the map $\eta_{n+1}\colon W_k(n) \to W_k(n)$ of Lemma 2.2 inductively by the composition $\eta_{n+1} = \nu_{k,n}(v_{n+1} \wedge 1)\colon W_k(n) = S \wedge W_k(n) \to T(k) \wedge W_k(n) \to W_k(n)$. Hence we have

PROPOSITION 2.3. *Let $n$ and $k$ be non-negative integers such that $n \le k$. Then there exists a $T(k)$-module spectrum $W_k(n)$.*

## §3. Cobar complexes

Let $(A, \Gamma)$ denote a Hopf algebroid over a commutative ring $K$. Then it is a pair of $K$-algebras $A$ and $\Gamma$ provided with structure maps, which are a left and a right units $\eta_L, \eta_R\colon A \to \Gamma$, a coproduct $\Delta\colon \Gamma \to \Gamma \otimes_A \Gamma$, a counit $\varepsilon\colon \Gamma \to A$, and a conjugation $c\colon \Gamma \to \Gamma$, with the relations $\varepsilon\eta_L = \varepsilon\eta_R = 1_A$, $(1_\Gamma \otimes \varepsilon)\Delta = (\varepsilon \otimes 1_\Gamma)\Delta = 1_\Gamma$, $(1_\Gamma \otimes \Delta)\Delta = (\Delta \otimes 1_\Gamma)\Delta$, $c\eta_R = \eta_L$, $c\eta_L = \eta_R$, and $cc = 1_\Gamma$. A *lefl $\Gamma$-comodule* $M$ is defined to be a left $A$-module together with a left $A$-linear map $\psi_M\colon \to \Gamma \otimes_A M$ such that $(\varepsilon \otimes 1_M)\psi_M = 1_M$ and $(\Delta \otimes 1_M)\psi_M = (1_\Gamma \otimes \psi_M)\psi_M$. A *right $\Gamma$-comodule* is similarly defined. The *cotensor product* $M \square_\Gamma N$ of a right and a left $\Gamma$-comodules $M$ and $N$ is the kernel of the $K$-module map $\psi_M \otimes 1_N - 1_M \otimes \psi_N\colon M \otimes_A N \to M \otimes_A \Gamma \otimes_A N$. For a left

$A$-module $N$, consider the map $\psi = (\Delta \otimes 1_N): \Gamma \otimes_A N \to \Gamma \otimes_A (\Gamma \otimes_A N)$, and we obtain a left $\Gamma$-comodule $\Gamma \otimes_A N$ with the structure map $\psi$. We call this an *extended comodule* (*cf.* [5, Appendix A]).

From here on we assume that $\Gamma$ is $A$-flat. Then it is well known that the category of $\Gamma$-comodules has enough injectives. We denote the $s$th right derived functor of $\mathrm{Hom}_\Gamma(M, \ )$(resp. $M \square_\Gamma \ $) for a left (resp. right) $\Gamma$-comodule $M$ by $\mathrm{Ext}^s_\Gamma(M, \ )$ (resp. $\mathrm{Cotor}^s_\Gamma(M, \ )$). We note here that $\mathrm{Ext}^s_\Gamma(A, M) = \mathrm{Cotor}^s_\Gamma(A, M)$, since we see that $\mathrm{Hom}_\Gamma(A, M) = A \square_\Gamma M$ by definition. By virtue of this we shall not distinguish these groups hereafter. We call $I$ *weak* (*$\Gamma$-*)*injective* if $\mathrm{Ext}^s_\Gamma(A, I) = 0$ for $s > 0$. Let $0 \to M \to I^0 \to I^1 \to \cdots$ be an exact sequence with $I^i$ weak injective for $i \geq 0$. This is said to be a *weak* (*$\Gamma$-*)*injective resolution.* Then this sequence splits into short ones $0 \to K^i \to I^i \to K^{i+1} \to 0$ and the Ext group satisfies $\mathrm{Ext}^s_\Gamma(A, K^{i+1}) = \mathrm{Ext}^{s+1}_\Gamma(A, K^i)$ for $s > 0$ and $0 \to \mathrm{Ext}^0_\Gamma(A, K^i) \to \mathrm{Ext}^0_\Gamma(A, I^i) \to \mathrm{Ext}^0_\Gamma(A, K^{i+1}) \to \mathrm{Ext}^1_\Gamma(A, K^i) \to 0$ to be exact. Therefore we compute $\mathrm{Ext}^*_\Gamma(A, M)$ for a $\Gamma$-comodule $M$ from a weak injective resolution as well as a injective one. For an $A$-free $\Gamma$-comodule $M$, we call a resolution $0 \to M \to I^0 \to I^1 \to \cdots$ *good* if $I^i$ is an $A$-free extended comodule. Since an extended comodule $E$ is weak injective, a good resolution is a weak injective resolution.

As an example of a good resolution for an $A$-free comodule $M$, we have the *cobar resolution* $0 \to M \to D^0_\Gamma M \to D^1_\Gamma M \to \cdots$ defined by $D^s_\Gamma M = \Gamma^{\otimes s+1} \otimes_A M$ with differential $d_s: D^s_\Gamma M \to D^{s+1}_\Gamma M$ such that $d_s(x \otimes m) = \sum_{i=0}^s (-1)^i \Delta_i x \otimes m - (-1)^s x \otimes \psi_M m$ for $m \in M$ and $x \in \Gamma^{\otimes s+1}$, where $\Delta_i = 1_i \otimes \Delta \otimes 1_{s-i}$ for $i \geq 0$ and for the identity map $1_n: \Gamma^{\otimes n} \to \Gamma^{\otimes n}$.

If $i: I \to J$ is a monomorphism of $A$-free comodules, then any map $f$ from $I$ to an extended comodule $\Gamma \otimes_A L$ extends to $J$. In fact, we get the extension $\tilde{f} = (1_\Gamma \otimes \varepsilon \otimes 1_L)(1_\Gamma \otimes f)(1_\Gamma \otimes j)\psi_J$, for a map $j: J \to I$ such that $ji = 1_I$. This fact implies

LEMMA 3.1 (*cf.* [5, Lemma A.1.2.9]). *Let $M$ and $N$ be $A$-free comodules and let sequences $0 \to M \to I^0 \to I^1 \to \cdots$ and $0 \to N \to J^0 \to J^1 \to \cdots$ be good resolutions. Then a map $f: M \to N$ of comodules extends to a map of resolutions and these extended maps induce a unique map on* Ext *groups.*

Let $\pi: (A, \Gamma) \to (A, \Sigma)$ be a map of Hopf algebroids over $A$. Then we regard $\Gamma$ as a $\Sigma$-comodule by the structure map $\psi_\Gamma = (1_\Gamma \otimes \pi)\Delta: \Gamma \to \Gamma \otimes_A \Sigma$. In this situation, we have

LEMMA 3.2. *Let $M$ be an $A$-free $\Sigma$-comodule and let a sequence $S: 0 \to M \to I^0 \xrightarrow{d_0} I^1 \to \cdots$ be a good $\Sigma$-resolution. If $\Gamma$ is a weak injective $\Sigma$-comodule, then the sequence $\Gamma \square_\Sigma S: 0 \to \Gamma \square_\Sigma M \to \Gamma \square_\Sigma I^0 \to \cdots$ is also a good $\Gamma$-resolution.*

PROOF. Since $\Gamma$ is $A$-flat, we have the exact sequences $0 \to \Gamma \otimes_A \mathrm{Ker}\, d_i \to \Gamma \otimes_A I^i \to \Gamma \otimes_A \mathrm{Im}\, d_i \to 0$ and $0 \to \Gamma \otimes_A \Sigma \otimes_A \mathrm{Ker}\, d_i \to \Gamma \otimes_A \Sigma \otimes_A I^i \to \Gamma \otimes_A \Sigma \otimes_A \mathrm{Im}\, d_i \to 0$, which give the exact sequence $0 \to \Gamma \square_\Sigma \mathrm{Ker}\, d_i \to \Gamma \square_\Sigma I^i \to \Gamma \square_\Sigma \mathrm{Im}\, d_i \to \mathrm{Cotor}^1_\Sigma(\Gamma, \mathrm{Ker}\, d_i)$. By the hypothesis, $\mathrm{Cotor}^k_\Sigma(\Gamma, M) = 0 = \mathrm{Cotor}^k_\Sigma(\Gamma, I^i)$ for $k > 0$. Therefore we see that $\mathrm{Cotor}^1_\Sigma(\Gamma, \mathrm{Ker}\, d_{i-1}) = 0$, and the above sequence turns into the short exact one. q.e.d.

## §4. Computation of the differentials

The Brown-Peterson ring spectrum $BP$ at a prime $p$ gives rise to the Hopf algebroid $(BP_*, BP_*BP)$ (*cf.* [1]). In this section we consider the Hopf algebroids

$$(A, \Gamma) = (BP_*/(p, v_1, v_2, v_3), A[t_1, t_2, \cdots])$$

with coproduct $\Delta: \Gamma \to \Gamma \otimes_A \Gamma$ associated to that of the Hopf algebroid $BP_*BP$ and

$$(A, \Sigma) = (BP_*/(p, v_1, v_2, v_3), A[t_2, t_3, \cdots])$$

with coproduct $\bar{\Delta} = (\pi \otimes \pi)\Delta i: \Sigma \to \Sigma \otimes_A \Sigma$. Here the map $\pi: \Gamma \to \Sigma$ (resp. $i: \Sigma \to \Gamma$) denotes the cononical projection (resp. injection). Then $\Gamma$ is a right $\Sigma$-comodule by the structure map $\psi_\Gamma = (1_\Gamma \otimes \pi)\Delta$ and put

$$B = \Gamma \square_\Sigma A.$$

Note that the map given by the multiplication by $t_1$ from $\Gamma$ to $\Gamma$ is a $\Sigma$-comodule map. Then we have $\mathrm{Ext}^i_\Sigma(A, \Gamma) = 0$ for $i > 0$ followed from the short exact sequence $0 \to \Gamma \xrightarrow{t_1} \Gamma \to \Sigma \to 0$ and $\mathrm{Ext}^i_\Sigma(A, \Sigma) = 0$ for $i > 0$.

LEMMA 4.1. *$\mathrm{Ext}^*_\Gamma(A, B)$ is the cohomology of the resolution*

$$0 \longrightarrow B \xrightarrow{\subset} \Gamma \xrightarrow{d_0} \Gamma \otimes_A \Sigma \xrightarrow{d_1} \Gamma \otimes_A (\Sigma \otimes_A \Sigma) \longrightarrow \cdots \longrightarrow \Gamma \otimes_A (\Sigma^{\otimes n}) \xrightarrow{d_n} \cdots$$

*with differential defined by*

$$d_n x = \textstyle\sum_{i=0}^n (-1)^i \tilde{\Delta}_i x + (-1)^{n+1} x \otimes 1$$

*for $x \in \Gamma \otimes_A (\Sigma^{\otimes n})$, where*

$$\tilde{\Delta}_0 = ((1_\Gamma \otimes \pi)\Delta) \otimes 1_n \text{ and,}$$
$$\tilde{\Delta}_i = 1_\Gamma \otimes 1_{i-1} \otimes \bar{\Delta} \otimes 1_{n-i}$$

*for $i \geq 1$ and for the identity map $1_n: \Sigma^{\otimes n} \to \Sigma^{\otimes n}$.*

PROOF. Apply the functor $\Gamma \square_\Sigma$ to the cobar resolution

$$0 \longrightarrow A \overset{\eta_L}{\hookrightarrow} \Sigma \overset{\bar{d}_0}{\longrightarrow} \Sigma \otimes_A \Sigma \overset{\bar{d}_1}{\longrightarrow} \cdots,$$

and we obtain an exact sequence

$$0 \longrightarrow B \longrightarrow \Gamma \longrightarrow \Gamma \otimes_A \Sigma \longrightarrow \cdots$$

by lemma 3.2 identifying $\Gamma \square_\Sigma (\Sigma \otimes_A M) = \Gamma \otimes_A M$. A direct calculation shows that the following diagrams commute:

$$\begin{array}{ccccc} 0 \longrightarrow & \Gamma \square_\Sigma A & \overset{\eta}{\longrightarrow} & \Gamma \square_\Sigma \Sigma & \\ & & \overset{\subset}{\searrow} & \downarrow \hat{\Delta} \; \cong & \\ & & & \Gamma & \end{array}$$

for $\eta = 1_\Gamma \otimes \eta_L$ and $\hat{\Delta} = (1_\Gamma \otimes \pi)\Delta$, and

$$\begin{array}{ccc} \Gamma \square_\Sigma (\Sigma \otimes_A \Sigma^{\otimes n}) & \overset{d}{\longrightarrow} & \Delta \square_\Sigma (\Sigma \otimes_A \Sigma^{\otimes (n+1)}) \\ \hat{\Delta}(n) \uparrow \cong & & \hat{\Delta}(n+1) \uparrow \cong \\ \Gamma \otimes_A \Sigma^{\otimes n} & \overset{d_n}{\longrightarrow} & \Gamma \otimes_A \Sigma^{\otimes n} \end{array}$$

for $d = 1_\Gamma \otimes \bar{d}_n$ and $\hat{\Delta}(n) = \hat{\Delta} \otimes 1_n$. Therefore this exact sequence is the desired one, and gives a good $\Gamma$-resolution. q.e.d.

We denote this resolution by $D^*B$.

LEMMA 4.2. *There is a map $f$ of resolutions from the cobar $D_\Gamma^* B$ to $D^*B$, such that $f_{-1} = 1_B$ and*

$$f_n(\gamma \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes b) = \gamma \otimes \pi\gamma \otimes \cdots \otimes \pi\gamma_n \otimes \bar{\pi} b \in D^n B$$

*for $\gamma \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes b \in \Gamma^{\otimes n+1} \otimes_A B = D_\Gamma^n B$. Here $\pi\colon \Gamma \to \Sigma$ and $\bar{\pi}\colon B \to A$ denote the canonical projections.*

PROOF. Since $t_1$ is primitive, we compute

$$(\pi \otimes \bar{\pi})\Delta(at_1^i) = (\pi \otimes \bar{\pi})\left(a \sum_{j=0}^{i} \binom{i}{j} t_1^{i-j} \otimes t_1^j\right) = \begin{cases} a & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

for $a \in A$, which equals to $\eta_L \bar{\pi}(at_1^i)$, and we have $(\pi \otimes \bar{\pi})\Delta = \eta_L \bar{\pi}$. Then by the definition of the map $f_n$, we verify

$$f_{n+1}(1_\Gamma \otimes \tilde{d}_{n-1}) = (1_\Gamma \otimes \bar{d}_{n-1}) f_n$$

for the differentials $\tilde{d}_{n-1}$ and $\bar{d}_{n-1}$ of the cobar resolutions $D_\Gamma^* B$ and $D_\Sigma^* A$, respectively. Thus $f_{n+1}\tilde{d}_n = f_{n+1}(\Delta \otimes id - 1_\Gamma \otimes \tilde{d}_{n-1}) = (\hat{\Delta} \otimes id) f_n - (1_\Gamma \otimes \bar{d}_{n-1}) f_n$

$= d_n f_n$ as desired ($\hat{\Delta} = (1_\Gamma \otimes \pi)\Delta$ as above). q.e.d.

Noticing that $A \square_\Gamma D^* B = A \square_\Sigma D^*_\Sigma A$, Lemma 3.1 implies

PROPOSITION 4.3. *The map $f$ of Lemma 4.2 induces an isomorphism*

$$f_*: \mathrm{Ext}^*_\Gamma(A, B) \xrightarrow{\cong} \mathrm{Ext}_\Sigma(A, A).$$

We put $V = V(3)$ and $W = V(3) \wedge T(1)$. Then $BP_* V = A$ and $BP_* W = B$. Consider the Hopf algebras $\Phi = F_p[t_1, t_2, \cdots]$ and $\Psi = F_p[t_2, t_3, \cdots]$ over the prime field $F_p$ of chracteristic $p$. Then the equalities $\mathrm{Hom}^t_\Gamma(A, D^s_\Gamma A) = \mathrm{Hom}^t_\Phi(F_p, D^s_\Phi F_p)$ and $\mathrm{Hom}^t_\Gamma(A, D^s B) = \mathrm{Hom}^t_\Psi(F_p, D^s_\Psi F_p)$ for $t - s < 2p^4 - 2$ show

LEMMA 4.4. *For* $t - s < 2p^4 - 2$, $\mathrm{Ext}^{s,t}_\Gamma(A, A) = \mathrm{Ext}^{s,t}_\Phi(F_p, F_p)$ *and* $\mathrm{Ext}^{s,t}_\Gamma(A, B) = \mathrm{Ext}^{s,t}_\Psi(F_p, F_p)$.

LEMMA 4.5. $\mathrm{Ext}^{kq+1, 2p^4-2+kq}_\Psi(F_p, F_p) = 0$ *for* $k > 1$.

PROOF. We have the cocentral extensions $\Psi_i \to \Psi(i) \to \Psi(i-1)$ for $i > 0$, where $\Psi_i = F_p[t_i]$ and $\Psi(i) = F_p[t_2, t_3, \cdots, t_i]$. These lead to the Cartan-Eilenberg spectral sequences, which give the inequality

$$\mathrm{rank}\,(\otimes^i_{j=2} \mathrm{Ext}^*_{\Psi_j}(F_p, F_p))^{s,t} \geq \mathrm{rank}\,(\mathrm{Ext}^{s,t}_{\Psi(i)}(F_p, F_p)).$$

It is well known that $\mathrm{Ext}^*_{\Psi_i}(F_p, F_p) = E(h_{i,j}) \otimes F_p[b_{i,j}]$ with $|h_{i,j}| = 2p^j(p^i - 1)$ and $|b_{i,j}| = 2p^{j+1}(p^i - 1)$. Here $E$ stands for the exterior algebra and $h_{i,j}$ and $b_{i,j}$ have homology dimensions 1 and 2, respectively. We notice that $\mathrm{Ext}^{**}_\Psi(F_p, F_p) = \mathrm{Ext}^{**}_{\Psi(4)}(F_p, F_p)$ at total degree $2p^4 - 3$. Under the condition $k > 1$, we see that every element of the left hand side of the above inequality has total degree greater than $2p^4 - 3$. This implies the lemma. q.e.d.

Consider the cocentral extension $F_p[t_1] \to \Phi \xrightarrow{\pi} \Psi$, and it gives rise to the Cartan-Eilenberg spectral sequence converging to $\mathrm{Ext}_\Phi(F_p, F_p)$ with $E_2 = \mathrm{Ext}_{F_p[t_1]}(F_p, F_p) \otimes \mathrm{Ext}_\Psi(F_p, F_p)$. Here $\pi: \Phi \to \Psi$ denotes the canonical projection. By Proposition 4.3 and Lemma 4.4, we see that the edge homomorphism of this spectral sequnce is the induced map from the composition $fi$ for the inclusion $i: D^*_\Gamma A \to D^*_\Gamma B$. The generator of $\mathrm{Ext}^{2p-1, 2p^4+2p-4}_\Phi(F_p, F_p)$ is known to be the element

$$\xi = b_{20}^{p-3} h_{11} h_{20} h_{12} h_{21} h_{30}$$

of the $E_2$-term (*cf.* [5, pp. 217–218]). These show the following:

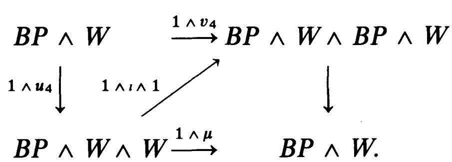
(4.6) $$\iota_* \xi = 0$$

for the map $\iota: V \to W$ induced from the unit map $\iota_1: S \to T(1)$ of the ring spectrum $T(1)$, since $\iota_*$ is the edge homomorphism of the Cartan-Eilenberg spectral sequence.

PROPOSITION 4.7. *Let $u_4$ be the generator of the $E_2$-term* $\mathrm{Ext}_\Gamma^{0,2p^4-2}(A, B)$ *of the Adams-Novikov spectral sequence for W. Then the element $u_4$ is a permanent cycle.*

PROOF. Suppose that $d_{2p-1}v_4 = k\xi$ for some $k \in \boldsymbol{Z}$ and the generator $v_4$ of the $E_2$-term of the Adams-Novikov spectral sequence for $V$. Then $u_4 = \iota_* v_4$ for the map $\iota: V \to W$, and the naturality of the differential of the spectral sequence implies $d_{2p-1}u_4 = d_{2p-1}\iota_* v_4 = \iota_* d_{2p-1} v_4 = \iota_* k\xi = k\iota_*\xi = 0$. By Lemma 4.5 we see that $\mathrm{Ext}_\Gamma^{s,t}(A, B) = 0$ for $t - s = 2p^4 - 3$ except for $(s, t) = (2p-1, 2p^4 + 2p - 4)$. Thus we have $d_r u_4 = 0$ for $r \geq 2$. q.e.d.

PROOF OF THEOREM. By Proposition 4.7, we have the map $u_4 \in \pi_* W$ which is mapped to $v_4$ of $BP_* W = B = BP_*/I_4[t_1]$ by the edge homomorphism of the Adams-Novikov spectral sequence. Since $W$ is a ring spectrum, we have the self map $\eta: W \to W$ defined by the composition $\eta = \mu(1_W \wedge u_4)$ for the multiplication $\mu$ of $W$. Then we see that $BP_*\eta = (1 \wedge \mu)_*(1 \wedge u_4)_* = v_4$. In fact, we have the commutative diagram




Therefore the cofiber of $\eta$ turns out to be the desired spectrum $W_1(4)$. q.e.d.

## References

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