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Ascendant subalgebras of hyperfinite Lie algebras

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Introduction

A Lie algebra does not necessarily have the lattice of ascendant subalgebras or the lattice of subideals. So it is an interesting problem to present sufficient conditions for a Lie algebra to have these lattices, and furthermore, to have them as complete lattices. Recent works on this problem include Aldosray [1, § 1], the author [5, 6, 7, 9], and Kawamoto and Nomura [13]. The purpose of this paper is to present further results concerning the family of ascendant subalgebras of a hyperfinite Lie algebra. We shall present some equivalent conditions to be ascendant for a subalgebra of a hyperfinite or hyperfinite-and-abelian Lie algebra, and conclude that every hyperfinite-and-abelian Lie algebra over a field of characteristic zero has the complete lattice of ascendant subalgebras.

In Section 2 we shall first prove that if H is a subalgebra of a hyperfinite Lie algebra L over a field of characteristic zero, then the condition $H \operatorname{asc} L$ is equivalent to each of the following: (a) $H \operatorname{asc} \langle H, x \rangle$ for every $x \in L$; (b) $H \operatorname{ser} \langle H, x \rangle$ for every $x \in L$ (Proposition 1). We shall secondly prove that for a subalgebra H of a hyperfinite Lie algebra L, $H \operatorname{asc} L$ (resp. $H \operatorname{wasc} L$) if and only if $H \operatorname{ser} L$ (resp. $H \operatorname{wser} L$)(Theorem 1), and conclude that in every hyperfinite Lie algebra, the intersection of any family of ascendant (resp. weakly ascendant) subalgebras is always ascendant (resp. weakly ascendant)(Corollary 1).

In Section 3 we shall first prove that if H is a subalgebra of a hyperfinite-and-abelian Lie algebra L, then the condition $H \operatorname{asc} L$ is equivalent to each of the following: (a) $H \operatorname{asc} \langle H, x \rangle$ for every $x \in L$; (b) $H \operatorname{wasc} L$; (c) $H \operatorname{wasc} \langle H, x \rangle$ for every $x \in L$; (d) $H \operatorname{ser} L$; (e) $H \operatorname{ser} \langle H, x \rangle$ for every $x \in L$; (f) $H \operatorname{wser} L$; (g) $H \operatorname{wser} \langle H, x \rangle$ for every $x \in L$ (Proposition 2). Secondly we shall prove that in every hyperfinite-and-abelian Lie algebra over a field of characteristic zero, both the join and the intersection of any family of ascendant subalgebras are always ascendant (Theorem 2).

1. Preliminaries

Throughout this paper we are always concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified. Notation and terminology are mainly based on [3]. We explain some terms which we use here.

Let L be a Lie algebra over a field $\mathfrak{k}$. $H \leq L$ (resp. $H \lhd L$, $H \operatorname{si} L$) denotes that H is a subalgebra (resp. an ideal, a subideal) of L. Angular brackets $\langle\ \rangle$ denote the subalgebra generated by their contents. Let H be a subalgebra of L and ρ be an ordinal. H is a ρ-step ascendant subalgebra (resp. a ρ-step weakly ascendant subalgebra) of L, denoted by $H \lhd^{\rho} L$ (resp. $H \leq^{\rho} L$), if there exists a family $\{H_\alpha : \alpha \leqq \rho\}$ of subalgebras (resp. subspaces) of L such that

(a) $H_0 = H$ and $H_\rho = L$,

(b) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for any ordinal $\alpha < \rho$,

(c) $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leqq \rho$.

Then the family $\{H_\alpha : \alpha \leqq \rho\}$ is said to be an ascending series (resp. a weakly ascending series) from H to L. H is an ascendant subalgebra (resp. a weakly ascendant subalgebra) of L, denoted by $H \operatorname{asc} L$ (resp. $H \operatorname{wasc} L$), if $H \lhd^{\rho} L$ (resp. $H \leq^{\rho} L$) for some ordinal ρ. H is a serial subalgebra (resp. a weakly serial subalgebra) of L, denoted by $H \operatorname{ser} L$ (resp. $H \operatorname{wser} L$), if there exist a totally ordered set Σ and a family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(a) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,

(b) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,

(c) $L \setminus H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$,

(d) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

Then the family $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ is said to be a series (resp. a weak series) from H to L.

A class $\mathfrak{X}$ is a collection of Lie algebras together with their isomorphic copies and 0-dimensional Lie algebras. Lie algebras in a class $\mathfrak{X}$ is called $\mathfrak{X}$-algebras. $\mathfrak{A}$ (resp. $\mathfrak{E}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{F}\mathrm{t}$, $\mathfrak{N}$, $\mathrm{R}\mathfrak{N}$, $\mathfrak{Z}$) is the class of Lie algebras which are abelian (resp. Engel, soluble, finite-dimensional, Fitting, nilpotent, residually nilpotent, hypercentral). Let $\mathfrak{X}$ be a class of Lie algebras. H is said to be an $\mathfrak{X}$-subalgebra (resp. an $\mathfrak{X}$-ideal) of L if $H \leq L$ (resp. $H \lhd L$) and $H \in \mathfrak{X}$. $\mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L}(\lhd)\mathfrak{X}$) is the class of Lie algebras in which every finitely generated subalgebra is contained in some $\mathfrak{X}$-subalgebra (resp. $\mathfrak{X}$-ideal). In particular, Lie algebras in the class $\mathrm{L}\mathfrak{F}$ (resp. $\mathrm{L}(\lhd)\mathfrak{F}$) are called locally (resp. ideally) finite Lie algebras. The class $\acute{\mathrm{E}}(\lhd)\mathfrak{X}$ of Lie algebras is defined as follows:

$L \in \acute{\mathrm{E}}(\lhd)\mathfrak{X}$ if there exist an ordinal ρ and a family $\{L_\alpha : \alpha \leqq \rho\}$ of ideals of L such that

(a) $L_0 = \{0\}$ and $L_\rho = L$,

(b) $L_\alpha \leq L_{\alpha+1}$ and $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$ for any ordinal $\alpha < \rho$,

(c) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for any limit ordinal $\lambda \leqq \rho$.

In this case we say that L has an ascending $\mathfrak{X}$-series $\{L_\alpha : \alpha \leqq \rho\}$ of ideals. In particular, Lie algebras in the class $\acute{\mathrm{E}}(\lhd)\mathfrak{F}$ (resp. $\acute{\mathrm{E}}(\lhd)\mathfrak{A}$, $\acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A})$) are called hyperfinite (resp. hyperabelian, hyperfinite-and-abelian) Lie algebras. Finally let Δ be either the relation asc or wasc. Then we need the classes $\mathfrak{L}^{\infty}(\Delta)$ and $\mathfrak{L}_{\infty}(\Delta)$ defined in [7] as follows:

$L\in\mathfrak{L}^{\infty}(\Delta)$ if $\langle H_\lambda\colon \lambda\in\Lambda\rangle \Delta L$ whenever $H_\lambda \Delta L$ $(\lambda\in\Lambda)$;

$L\in\mathfrak{L}_{\infty}(\Delta)$ if $\bigcap_{\lambda\in\Lambda}H_\lambda \Delta L$ whenever $H_\lambda \Delta L$ $(\lambda\in\Lambda)$.

2. Hyperfinite Lie algebras

In this section we shall present some equivalent conditions for a subalgebra to be ascendant in a hyperfinite Lie algebra. In particular, we shall prove that in a hyperfinite Lie algebra, every serial subalgebra is ascendant, and so the intersection of any family of ascendant subalgebras is ascendant.

We need the following two lemmas about the class $\acute{\mathrm{E}}(\lhd)\mathfrak{F}$ of hyperfinite Lie algebras.

Lemma 1 ([12, Corollary 3.3] and [16, Lemma 4.1]). *Over any field* $\mathfrak{k}$, $\mathrm{L}(\lhd)\mathfrak{F}\leq\acute{\mathrm{E}}(\lhd)\mathfrak{F}\leq\mathrm{L}\mathfrak{F}$.

Lemma 2 ([11, Proposition 6]). *Over any field* $\mathfrak{k}$, $\mathfrak{C}\cap\acute{\mathrm{E}}(\lhd)\mathfrak{F}=\mathfrak{Z}$.

Remark 1. In Lemma 1 both inclusions hold for any field $\mathfrak{k}$. In fact, let X be an abelian Lie algebra over $\mathfrak{k}$ with basis $\{x_0, x_1,\cdots\}$ and σ be the derivation of X defined by $x_0\sigma=0$ and $x_{i+1}\sigma=x_i$ $(i\geqq 0)$. Form the split extension $L=X\dot{+}\langle\sigma\rangle$. Then it is well known (cf. [3, p.119]) that $L\in\mathfrak{Z}\leq\acute{\mathrm{E}}(\lhd)(\mathfrak{F}\cap\mathfrak{A})$. However, since $\langle\sigma^L\rangle=L\notin\mathfrak{F}$, we have $L\notin\mathrm{L}(\lhd)\mathfrak{F}$. Therefore $\mathrm{L}(\lhd)\mathfrak{F}<\acute{\mathrm{E}}(\lhd)\mathfrak{F}$. In order to show the second inclusion, we consider the McLain Lie algebra $M=\mathscr{L}_{\mathfrak{k}}(\boldsymbol{Z})$ over $\mathfrak{k}$ (cf.[3, p.111]), where $\boldsymbol{Z}$ is the set of integers with natural ordering. Then it is well known (cf. [3, p.119]) that $M\in\mathfrak{F}\mathfrak{t}\leq\mathrm{L}\mathfrak{N}\leq\mathrm{L}(\mathfrak{F}\cap\mathrm{E}\mathfrak{A})$ and $M\notin\mathfrak{Z}$. Therefore by Lemma 2 we have $M\notin\acute{\mathrm{E}}(\lhd)\mathfrak{F}$. Thus $\acute{\mathrm{E}}(\lhd)\mathfrak{F}<\mathrm{L}\mathfrak{F}$.

Remark 2. We can regard the class $\mathfrak{C}$ as the class of Lie algebras in which every 1-dimensional subalgebra is weakly ascendant. So we define a new class $\mathfrak{C}^\wedge$, generalizing the class $\mathfrak{C}$, as follows: for a Lie algebra L,

$$L\in\mathfrak{C}^\wedge \text{ if } \langle x\rangle \text{ wser } L \text{ for every } x\in L.$$

Then it is verified that $\mathfrak{C}<\mathfrak{C}^\wedge$. For example, let X be an abelian Lie algebra over a field $\mathfrak{k}$ with basis $\{x_0, x_1,\cdots\}$ and τ be the derivation of X defined by $x_i\tau=x_{i+1}$ $(i\geqq 0)$. Form the split extension $L=X\dot{+}\langle\tau\rangle$. Then since $L\in\mathrm{R}\mathfrak{N}$, by [8, Corollary 2.7(2) and Theorem 2.9(1)] we have $L\in\mathfrak{C}^\wedge$. But we clearly have $L\notin\mathfrak{C}$. We should note that Lemma 2 holds even if we replace the class $\mathfrak{C}$ by the class $\mathfrak{C}^\wedge$. In fact, it is not hard to see that $\mathfrak{C}^\wedge\cap\mathrm{L}\mathfrak{F}=\mathrm{L}\mathfrak{N}$. Thus by making use of Lemmas 1 and 2 we have $\mathfrak{C}^\wedge\cap\acute{\mathrm{E}}(\lhd)\mathfrak{F}=\mathfrak{Z}$.

Let H be a subalgebra of an ideally finite Lie algebra L over a field $\mathfrak{k}$ of characteristic zero. In [2, Theorem 5.3.7] Aldosray has proved that $H \operatorname{asc} L$ if and only if $H \operatorname{asc} \langle H, x \rangle$ for every $x \in L$. We can generalize this result in the following

PROPOSITION 1. *Let L be a hyperfinite Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Then the following conditions are equivalent*:

(1) H asc L.

(2) H asc $\langle H, x \rangle$ *for every* $x \in L$.

(3) H ser $\langle H, x \rangle$ *for every* $x \in L$.

PROOF. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume that $H \operatorname{ser} \langle H, x \rangle$ for every $x \in L$. Since by Lemma 1 L is locally finite, owing to [14, Theorem 4] we have $H \operatorname{ser} L$. Let K denote the intersection of the ideals I of H such that $H/I \in \mathrm{L}\mathfrak{N}$. Then by [14, Theorem 5 and Corollary 6] we have $K \lhd L$ and $H/K \leq \rho(L/K)$, where $\rho(L/K)$ is the Hirsch-Plotkin radical of L/K. By using Lemma 2 we have $\rho(L/K) \in \mathrm{L}\mathfrak{N} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathfrak{Z}$. Hence $H/K \operatorname{asc} \rho(L/K) \lhd L/K$ and therefore $H \operatorname{asc} L$.

Before showing the main theorem of this section we need a certain result about vector spaces.

LEMMA 3. *Let H be a subalgebra of a Lie algebra L such that the quotient space L/H is finite-dimensional as a vector space. Then*:

(1) H ser L *if and only if* H si L.

(2) H wser L *if and only if* H wsi L.

Now we have the main theorem of this section.

THEOREM 1. *Let L be a hyperfinite Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L. Then*:

(1) H asc L *if and only if* H ser L.

(2) H wasc L *if and only if* H wser L.

PROOF. Assume that $H \operatorname{ser} L$ (resp. $H \operatorname{wser} L$). Since L is hyperfinite, L has an ascending $\mathfrak{F}$-series $\{L_\alpha : \alpha \leqq \rho\}$ of ideals. Let $\alpha < \rho$ and let θ denote the natural homomorphism $L \to L/L_\alpha$. Since L is locally finite, owing to [3, Proposition 13.2.4] (resp. [4, Proposition 2.5]) we have $\theta(H) \operatorname{ser} \theta(L)$ (resp. $\theta(H)$ wser $\theta(L)$). It follows that $\theta(H)$ ser $\theta(H + L_{\alpha+1})$ (resp. $\theta(H)$ wser $\theta(H + L_{\alpha+1})$). Since $\theta(L_{\alpha+1}) \in \mathfrak{F}$, the quotient space $\theta(H + L_{\alpha+1})/\theta(H)$ is finite-dimensional as a vector space. Hence by Lemma 3 $\theta(H)$ si $\theta(H + L_{\alpha+1})$ (resp. $\theta(H)$ wsi $\theta(H + L_{\alpha+1})$) and therefore $H + L_\alpha$ si $H + L_{\alpha+1}$ (resp. $H + L_\alpha$ wsi $H + L_{\alpha+1}$). Thus we have $H \operatorname{asc} L$ (resp. $H \operatorname{wasc} L$).

As in [9] we denote by $\mathfrak{D}(\text{ser}, \text{asc})$ (resp. $\mathfrak{D}(\text{wser}, \text{wasc})$) the class of Lie algebras in which every serial (resp. weakly serial) subalgebra is ascendant (resp. weakly ascendant). Then by using [9, Theorem 2.2], we can easily see that $\mathfrak{D}(\text{ser}, \text{asc}) \leq \mathfrak{L}_\infty(\text{asc})$ and $\mathfrak{D}(\text{wser}, \text{wasc}) \leq \mathfrak{L}_\infty(\text{wasc})$. The following corollary is immediately deduced from this result and Theorem 1.

COROLLARY 1. *Over any field* $\mathfrak{k}$, $\acute{\mathrm{E}}(\lhd)\mathfrak{F} \leq \mathfrak{L}_\infty(\text{asc}) \cap \mathfrak{L}_\infty(\text{wasc})$.

REMARK 3. It is well known (cf. [3, Lemma 3.1.1]) that over a field $\mathfrak{k}$ of characteristic $p > 0$, there exists a finite-dimensional, soluble Lie algebra in which the join of a certain pair of 1-dimensional subideals is not ascendant. This implies that over a field $\mathfrak{k}$ of characteristic $p > 0$, a hyperfinite Lie algebra does not necessarily have the lattice of ascendant subalgebras or the lattice of weakly ascendant subalgebras. On the other hand, in [4, Example 5.1] we have constructed a finite-dimensional Lie algebra over a field $\mathfrak{k}$ of characteristic zero, in which the join of a certain pair of 1-dimensional weak subideals is not weakly ascendant. This implies that over a field $\mathfrak{k}$ of characteristic zero, a hyperfinite Lie algebra does not necessarily have the lattice of weakly ascendant subalgebras. However, it is not known whether a hyperfinite Lie algebra over a field $\mathfrak{k}$ of characteristic zero always has the lattice of ascendant subalgebras.

3. Hyperfinite-and-abelian Lie algebras

In this section we shall first present several equivalent conditions for a subalgebra of a hyperfinite-and-abelian Lie algebra to be ascendant, and shall secondly prove that over a field $\mathfrak{k}$ of characteristic zero, every hyperfinite-and-abelian Lie algebra has the complete lattice of ascendant subalgebras.

Concerning the class $\acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A})$ of hyperfinite-and-abelian Lie algebras, we need

LEMMA 4 ([12, Corollary 3.3] and [16, Lemma 4.2]). *Over any field* $\mathfrak{k}$, $\mathrm{L}(\lhd)(\mathfrak{F} \cap \mathrm{E}\mathfrak{A}) \leq \acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A}) \leq \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$.

REMARK 4. Consider the examples in Remark 1, we can easily see that in Lemma 4 both inclusions hold for any field $\mathfrak{k}$.

Let H be a subalgebra of a Lie algebra L over a field $\mathfrak{k}$. It has been proved in [16, Theorems 3.1 and 3.2] that if $L \in \mathrm{L}(\lhd)(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$ then the following conditions (1)–(4) are equivalent:

(1) H asc L.

(2) H asc $\langle H, x \rangle$ for every $x \in L$.

(3) H wasc L.

(4) H wasc $\langle H, x\rangle$ for every $x \in L$.

On the other hand, in [14] Stewart has indicated that if $L \in \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$ then $H \operatorname{ser} L$ if and only if $H \operatorname{ser} \langle H, x\rangle$ for every $x \in L$. From this result and [4, Theorem 2.7] it is immediately deduced that if $L \in \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$ then the following conditions (5)–(8) are equivalent:

(5) H ser L.

(6) H ser $\langle H, x\rangle$ for every $x \in L$.

(7) H wser L.

(8) H wser $\langle H, x\rangle$ for every $x \in L$.

Moreover, by [4, Theorem 3.1], if $L \in \mathrm{L}(\lhd)\mathfrak{F}$ then the conditions (3) and (7) are equivalent. Thus we conclude that if $L \in \mathrm{L}(\lhd)(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$, then all the conditions from (1) to (8) are equivalent. Furthermore, by using Theorem 1 we can generalize this result in the following

PROPOSITION 2. *Let L be a hyperfinite-and-abelian Lie algebra over a field $\mathfrak{k}$ and H be a subalgebra of L. Then the preceding conditions from* (1) *to* (8) *are equivalent.*

PROOF. It suffices to show that (8) implies (1). By Lemma 4 we have $L \in \mathrm{L}(\mathfrak{F} \cap \mathrm{E}\mathfrak{A})$. Therefore the conditions from (5) to (8) are equivalent as before. Furthermore, by Theorem 1(1) the conditions (1) and (5) are equivalent. It follows that (8) implies (1).

Let Δ be any of the relations $\leq$, asc, wasc, ser, wser. For a Lie algebra L, as in [5, 7] we use $\mathscr{S}_L(\Delta)$ to denote the family of subalgebras H of L such that $H \Delta L$. In particular, $\mathscr{S}_L(\leq)$ is the lattice of subalgebras. Let L be a hyperfinite-and-abelian Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Then by Proposition 2 we have

$$\mathscr{S}_L(\mathrm{asc}) = \mathscr{S}_L(\mathrm{wasc}) = \mathscr{S}_L(\mathrm{ser}) = \mathscr{S}_L(\mathrm{wser}).$$

Moreover, as a direct consequence of [7, Theorem 3.10(2)] and [10, Corollary 3.7], we can see that $\langle H, K\rangle \in \mathscr{S}_L(\mathrm{asc})$ whenever $H, K \in \mathscr{S}_L(\mathrm{asc})$. This means that $\mathscr{S}_L(\mathrm{asc})$ is a sublattice of $\mathscr{S}_L(\leq)$. In the rest of this paper we shall prove that $\mathscr{S}_L(\mathrm{asc})$ is furthermore a complete lattice. The proof depends on the technique of 'formal power series algebras' (cf. [3, §4.1]). We briefly explain this technique.

Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Let $\mathfrak{k}_0 = \mathfrak{k}\langle t\rangle$ be the field of formal power series in the indeterminate t, and $L^\dagger$ be the set of all formal power series

$$x = \sum_{r \geq n} x_r t^r, \ n = n(x) \in \boldsymbol{Z}, \ x_r \in L.$$

Let $y = \sum y_r t^r \in L^\dagger$ and $\alpha = \sum \alpha_r t^r \in \mathfrak{k}_0$, and define $x + y$, $[x, y]$ and αx according to the rules:

$x+y=\sum(x_r+y_r)t^r$;
$[x, y]=\sum z_r t^r$, where $z_r=\sum_{i+j=r}[x_i, y_j]$;
$\alpha x=\sum w_r t^r$, where $w_r=\sum_{i+j=r}\alpha_i x_j$.

These rules make $L^{\uparrow}$ into a Lie algebra over $\mathfrak{k}_0$. Let $A \subseteq L$ and let $A^{\uparrow}$ denote the set of elements $x=\sum x_r t^r \in L^{\uparrow}$ with $x_r \in A$. Let $\{A_i : i \in I\}$ be the family of finite-dimensional subspaces of L, and define $L^{\wedge}=\bigcup_{i\in I}A^{\uparrow}$. Then it is shown that $L^{\wedge}$ is a subalgebra of $L^{\uparrow}$. Let $A^{\wedge}$ denote the set of elements $x=\sum x_r t^r \in L^{\wedge}$ with $x_r \in A$.

Let d be a derivation of L, and define a mapping $\exp(t\,d)$ of $L^{\uparrow}$ as follows: for each $x=\sum x_r t^r \in L^{\uparrow}$,

$$x^{\exp(td)}=\sum u_r t^r, \quad \text{where } u_r=\sum_{i+j=r}x_i d^j/j!.$$

Then it is shown that $\exp(t\,d)$ is a Lie automorphism of $L^{\uparrow}$.

Let $x=\sum_{r\geqq n}x_r t^r \in L^{\uparrow}$ with $x_n \neq 0$. Then x_n is called the first coefficient of x. Let $M \subseteq L^{\uparrow}$ and let $M^{\downarrow}$ denote the set of elements $x \in L$ such that $x=0$ or x is the first coefficient of some element of M. In particular, if $M \subseteq L^{\wedge}$ then $M^{\downarrow}$ is denoted by $M^{\vee}$. Then by using [3, Lemmas 4.1.1 and 4.1.2], we can easily show the following

LEMMA 5. *Let σ be an ordinal. Then*:
(1) *If $H \lhd^{\sigma} K \leq L$, then $H^{\wedge} \lhd^{\sigma} K^{\wedge} \leq L^{\wedge}$.*
(2) *If $M \lhd^{\sigma} N \leq L^{\wedge}$, then $M^{\vee} \lhd^{\sigma} N^{\vee} \leq L$.*
(3) *If $H \leq^{\sigma} K \leq L$, then $H^{\wedge} \leq^{\sigma} K^{\wedge} \leq L^{\wedge}$.*
(4) *If $M \leq^{\sigma} N \leq L^{\wedge}$, then $M^{\vee} \leq^{\sigma} N^{\vee} \leq L$.*

Aldosray [1, Lemma 1.1] has proved that for subalgebras H, K of L, if K is maximal with respect to $K \operatorname{si} L$ and $K \leq H$, then $K \lhd H$. Concerning ascendant subalgebras of locally finite, hyperabelian Lie algebras, we can prove the analogous result in the following lemma, which is a key lemma to prove the main theorem of this section

LEMMA 6. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic zero and H, K be subalgebras of L. Assume that $L \in \mathrm{L}\mathfrak{F} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. If K is maximal with respect to $K \operatorname{asc} L$ and $K \leq H$, then $K \lhd H$.*

PROOF. Suppose that K is not an ideal of H. Since $K \operatorname{asc} H$, there is an ascending series $\{K_\alpha : \alpha \leqq \rho\}$ from K to H. Then we can find the first ordinal μ such that K is not an ideal of K_μ. Clearly μ is not zero or a limit ordinal, and so $K \lhd K_{\mu-1}$. There exist $a \in K$ and $x \in K_\mu$ such that $[a, x] \notin K$. Put $e=\exp(t\,\mathrm{ad}_L x)$. Since $L \in \mathrm{L}\mathfrak{F}$, $\mathrm{ad}_L x$ is a locally finite derivation of L in the sense of [3, p.85]. It follows from [3, Lemma 4.2.2] that $H^{\wedge e}=H^{\wedge}$ and $L^{\wedge e}$

$= L^\wedge$. Since $K(\mathrm{ad}_L x)^n \subseteq K_{\mu-1} (n \geqq 0)$, we have $K^{\wedge e} \subseteq K^{\uparrow e} \subseteq K^{\uparrow}_{\mu-1}$. Therefore $K^{\wedge e} \subseteq L^\wedge \cap K^{\uparrow}_{\mu-1} = K^{\wedge}_{\mu-1}$. Since $K^\wedge \lhd K^{\wedge}_{\mu-1}$, we have $[K^\wedge, K^{\wedge e}] \subseteq K^\wedge$. By Lemma 5(1) $K^\wedge \mathrm{asc} L^\wedge$ and hence $K^{\wedge e} \mathrm{asc} L^\wedge$. Using [7, Theorem 2.5] we have $K^\wedge + K^{\wedge e} \mathrm{wasc} L^\wedge$. It follows from Lemma 5(4) that $(K^\wedge + K^{\wedge e})^\vee \mathrm{wasc} L$. Put $J = (K^\wedge + K^{\wedge e})^\vee$. By [15, Theorem 1] we have $J \mathrm{asc} L$. Since $K \leq J \leq H$, by the maximality of K we have $K = J$. Since $a^e - a \in K^\wedge + K^{\wedge e}$, we have

$$[a, x] \in \{a^e - a\}^\downarrow = \{a^e - a\}^\vee \subseteq J = K.$$

This is a contradiction. Therefore K must be an ideal of H.

LEMMA 7. *Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic zero, and assume that $L \in \mathrm{L}\mathfrak{F} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Then $L \in \mathfrak{L}^\infty(\mathrm{asc})$ if and only if $\mathscr{S}_L(\mathrm{asc})$ is closed under the formation of unions of totally ordered chains.*

PROOF. The implication in one direction is clear. Assume that $\mathscr{S}_L(\mathrm{asc})$ is closed under the formation of unions of totally ordered chains. Let $\{H_\lambda : \lambda \in \Lambda\}$ be a subset of $\mathscr{S}_L(\mathrm{asc})$ and put $J = \langle H_\lambda : \lambda \in \Lambda \rangle$. By our assumption and Zorn's lemma, L has a subalgebra K maximal with respect to $K \mathrm{asc} L$ and $K \leq J$. Then by Lemma 6 K must be an ideal of J. By making use of [7, Theorem 2.5] and [15, Theorem 1], for every $\lambda \in \Lambda$ we have $K + H_\lambda \mathrm{asc} L$. Then the maximality of K implies that $K + H_\lambda = K$. It follows that $J = K \mathrm{asc} L$. Thus we obtain $L \in \mathfrak{L}^\infty(\mathrm{asc})$.

Now we can prove the main theorem of this section.

THEOREM 2. *Over a field $\mathfrak{k}$ of characteristic zero,*

$$\acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A}) \leq \mathfrak{L}^\infty(\mathrm{asc}) \cap \mathfrak{L}_\infty(\mathrm{asc}).$$

PROOF. By Corollary 1 we have $\acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A}) \leq \mathfrak{L}_\infty(\mathrm{asc})$. Let L be a hyperfinite-and-abelian Lie algebra over $\mathfrak{k}$ and $\{H_\lambda : \lambda \in \Lambda\}$ be a totally ordered subset of $\mathscr{S}_L(\mathrm{asc})$. Put $H = \bigcup_{\lambda \in \Lambda} H_\lambda$. Let F be a finite-dimensional subalgebra of L. Then $H_\lambda \cap F \mathrm{si} F$ for all $\lambda \in \Lambda$. It is well known that F has the complete lattice of subideals. Hence $H \cap F = \langle H_\lambda \cap F : \lambda \in \Lambda \rangle \mathrm{si} F$. Since L is locally finite, owing to [3, Proposition 13.2.4] we have $H \mathrm{ser} L$. It follows from Theorem 1(1) that $H \mathrm{asc} L$. Thus by Lemma 7 we have $L \in \mathfrak{L}^\infty(\mathrm{asc})$.

REMARK 5. The Hartley example (cf. [3. Lemma 3.1.1]) shows that if the ground field $\mathfrak{k}$ is of characteristic $p > 0$, then the statement of Theorem 2 becomes a failure. On the other hand, we should note that if we replace the class $\acute{\mathrm{E}}(\lhd)(\mathfrak{F} \cap \mathfrak{A})$ by the class $\mathrm{L}\mathfrak{F} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$ in the statement of Theorem 2, then it also becomes a failure. In fact, let L denote the McLain Lie algebra $\mathscr{L}_\mathfrak{k}(N)$ over a field $\mathfrak{k}$ (cf. [3, p. 111]), where N is the set of positive integers with natural ordering. Then L has a basis $\{a_{ij} : i, j \in N, i < j\}$ with multiplications $[a_{ij}, a_{kl}]$

$= \delta_{jk}a_{il} - \delta_{li}a_{kj}$. (Here δ_{**} is the Kronecker delta.) It is well known that $L \in \mathfrak{F}\mathfrak{t}$, whence $L \in \mathrm{L}\mathfrak{F} \cap \acute{\mathrm{E}}(\lhd)\mathfrak{A}$. Put $A = \langle a_{1j} : j \geqq 2 \rangle$ and $H = \langle a_{ij} : j > i \geqq 2 \rangle$. Then $L = A + H$ and $A \cap H = \{0\}$. It is not hard to show that $I_L(H) \cap A = \{0\}$, where $I_L(H)$ is the idealiser of H in L. By the modular law we have $I_L(H) = I_L(H) \cap (A + H) = H$. This implies that H is not an ascendant subalgebra of L. However, since $L \in \mathfrak{F}\mathfrak{t}$, $\langle h \rangle \operatorname{si} L$ for every $h \in H$. Therefore we have $L \notin \mathfrak{L}^{\infty}(\mathrm{asc})$.

From Proposition 2 and Theorem 2, we can immediately deduce the following

COROLLARY 2. *Let L be a hyperfinite-and-abelian Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Then the following families coincide with each other and are complete sublattices of the lattice $\mathscr{S}_L(\leq)$:*

$$\mathscr{S}_L(\mathrm{asc}),\ \mathscr{S}_L(\mathrm{wasc}),\ \mathscr{S}_L(\mathrm{ser}),\ \mathscr{S}_L(\mathrm{wser}).$$

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