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Classification of non-compact real simple generalized Jordan triple systems of the second kind

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Introduction

A non-associative algebra $\mathscr{A}$ satisfying

$$xy = yx \text{ and } x^2(xy) = x(x^2y) \qquad (x, y \in \mathscr{A})$$

is called a *Jordan algebra*. A triple product (xyz) in $\mathscr{A}$ defined by

$$(xyz) = (xy)z + (zy)x - y(xz)$$

satisfies following two identities:

(JTS 1) $$(xyz) = (zyx),$$

(JTS 2) $$(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)).$$

In general a triple system satisfying these two identities is called a *Jordan triple system*. This definition was given by Meyberg [10], though the word of Jordan triple system had already been used in limited senses [3], [13]. He extended the Koecher's construction of a Lie algebra from a given Jordan algebra to the case of Jordan triple systems. Kantor [6] extended still more this construction to the case of *generalized Jordan triple systems*, which were triple systems satisfying only the identity (JTS 2) by definition. A familiar example of generalized Jordan triple system and not Jordan triple system is the space $M_{m,n}(\boldsymbol{R})$ of $m \times n$ real matrices with the product $(XYZ) = X{}^tYZ$. In a Lie algebra with an involution σ, a subspace U satisfying $[[U, \sigma(U)], U] \subset U$ also becomes a generalized Jordan triple system by the triple product $(xyz) = [[x, \sigma(y)], z]$. Starting from a given generalized Jordan triple system, Kantor constructed a graded Lie algebra, which is called the *Kantor algebra* for the generalized Jordan triple system in this paper. A graded Lie algebra $\mathscr{G} = \sum_{i=-\infty}^{\infty} \mathscr{G}_i$ is said to be *of the n-th kind* $(n > 0)$ if $\mathscr{G}_{\pm n} \neq \{0\}$ and $\mathscr{G}_m = \{0\}$ for $|m| > n$. To a Jordan triple system, there associates a graded Lie algebra of the first kind. Since the Lie product in the Kantor algebra was not easy to explain in general style, Yamaguti [14] gave another interpretation for the Kantor algebra in case of the second kind. Moreover he defined a symmetric bilinear form on a generalized Jordan triple system of the second kind. In case of the

Jordan triple system, the form coincides with the usual trace form. If the form is positive definite, then the generalized Jordan triple system is said to be *compact*. Loos [7] classified the simple Jordan triple systems over an algebraically closed field. Also he [8] gave the classification of the real simple compact Jordan triple systems. The classification of non-compact simple Jordan triple systems was made by Neher [12]. As for generalized Jordan triple systems, Kantor [6] classified (up to weak isomorphisms) K-simple generalized Jordan triple systems of the second kind over an algebraically closed field. The classification of real simple compact generalized Jordan triple systems of the second kind was given by us [5] in case that their Kantor algebras were classical simple Lie algebras.

The aim of this paper is to classify the non-compact classical real simple generalized Jordan triple systems of the second kind. This problem is reduced to determine a representative system of a certain equivalence classes (see § 5.1) of involutive automorphisms for every compact real simple generalized Jordan triple system of the second kind. This is not so similar as in Lie algebras or Jordan triple systems, because the equivalence classes do not necessarily coincide with the conjugate classes. Our method used to determine a representative system is different from Neher's [6] for Jordan triple systems. Of course, our method is applicable to the case of Jordan triple systems.

In § 1 we summarize some known results, which are used later in this paper, about generalized Jordan triple systems. Moreover we study about modifications by involutive automorphisms. Our modifications are different from ones considered by Neher in Jordan triple systems. In § 2 we study about the structure group of a generalized Jordan triple system and the automorphism group of its Kantor algebra. In § 3 we give classification theorems (Theorem 3.5 and Theorem 3.6). In § 4 we recall from [5] the result of classification of compact simple generalized Jordan triple systems with some arrangements and improvements. In § 5 we classify the non-compact classical simple real generalized Jordan triple systems of the second kind by using classification theorems obtained in § 3.

§ 1. Basic results on generalized Jordan triple systems

1.1. Let U be a finite dimensional real vector space and let $B: U \times U \times U \to U$ be a trilinear mapping. Then the pair (U, B) is called a *triple system.* We shall often write (xyz) instead of $B(x, y, z)$. A triple system (U, B) is called a *generalized Jordan triple system* (or shortly *GJTS*) if the identity

$$(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz)) \tag{1.1}$$

is satisfied for all $u, v, x, y, z \in U$. Furthermore, if the additional condition

$$(xyz) = (zyx) \qquad \text{for all } x, y, z \in U \tag{1.2}$$

is satisfied, then (U, B) is called a *Jordan triple system* (or simply *JTS*). Starting from a given GJTS (U, B), Kantor [6] constructed a certain graded Lie algebra (or in short GLA) $\mathscr{L}(B) = \sum_{i=-\infty}^{\infty} U_i$ such that $U_{-1} = U$. We call $\mathscr{L}(B)$ the *Kantor algebra for* (U, B). We say that (U, B) is *of the n-th kind* $(n > 0)$ if $U_{\pm m} = \{0\}$ for all $m > n$ and $U_{\pm n} \neq \{0\}$. Let us put

$$\begin{aligned} &L(a, b)(x) = (abx), && S(a, b)(x) = (axb) - (bxa), \\ &R(a, b)(x) = (xab), && B_a(x, y) = (xay). \end{aligned} \tag{1.3}$$

We say that (U, B) *satisfies the condition* (A) if $B_a = 0$ implies $a = 0$.

Let (U, B) be a GJTS satisfying the condition (A). It is known that there exists a grade-reversing involutive automorphism τ_B, which is called the *grade-reversing canonical involution*, of $\mathscr{L}(B)$ ([5] Proposition 3.8). The following two lemmas are essentially due to Kantor [6].

LEMMA 1.1 ([1] Theorem 1.1). *Let (U, B) be a GJTS of the 2nd kind satisfying the condition* (A), *and let τ_B be the grade-reversing canonical involution of the Kantor algebra $\mathscr{L}(B) = \sum_{i=-2}^{2} U_i$ for (U, B). Then*

(i) *U_{-2} is the subspace of* $\mathrm{End}(U)$ *spanned by all $S(a, b)$,*
$U_{-1} = U, \qquad U_1 = \tau_B(U_{-1}), \qquad U_2 = \tau_B(U_{-2}),$
U_0 is the subspace of $\mathrm{End}(U)$ *spanned by all operators $L(a, b)$.*

(ii) *We have the following bracket relations in $\mathscr{L}(B)$:*

$$\begin{aligned} &[a, b] = S(b, a), \qquad [L(a, b), \tau_B(c)] = -\tau_B(B(b, a, c)), \\ &[L(a, b), c] = B(a, b, c), \qquad [\tau_B(a), b] = L(b, a), \\ &[\tau_B(S(a, b)), c] = \tau_B(S(a, b)c), \\ &[L(a, b), S(c, d)] = S(L(a, b)c, d) + S(c, L(a, b)d), \\ &[S(a, b), \tau_B(S(c, d))] = L(S(a, b)c, d) - L(S(a, b)d, c), \\ &[L(a, b), L(c, d)] = L(L(a, b)c, d) - L(c, L(b, a)d), \end{aligned} \tag{1.4}$$

where $a, b, c, d \in U$.

We remark that the following identity is also valid:

$$\tau_B(L(a, b)) = -L(b, a). \tag{1.5}$$

The following lemma can be proved by straightforward calculations.

LEMMA 1.2. *Let (U, B) be a GJTS satisfying the condition* (A). *Then, it is* *of the 2nd kind if and only if the following identity is valid*:

$$S(S(x, y)u, v) = S(x, y)L(u, v) + L(v, u)S(x, y) \tag{1.6}$$

for all $u, v, x, y \in U$.

1.2. Let (U, B) be a GJTS. A subspace V of U is called an *ideal* (resp. *K-ideal*) of (U, B) if the following relation is valid:

$$B(V, U, U) + B(U, V, U) + B(U, U, V) \subset V$$
$$(\text{resp. } B(V, U, U) + B(U, U, V) \subset V).$$

A GJTS (U, B) is called *simple* (resp. *K-simple*) if B is not a zero map and if (U, B) has no non-trivial ideal (resp. K-ideal). Obviously the K-simplicity implies the simplicity, but the converse is not always true. In case of compact (see 1.3) GJTS's of the 2nd kind, we proved the following

LEMMA 1.3 ([1] Theorem 3.8). *Let (U, B) be a compact GJTS of the 2nd kind. Then (U, B) is simple if and only if it is K-simple.*

1.3. Now let (U, B) be a GJTS of the 2nd kind. The symmetric bilinear form γ_B on U defined by

$$\gamma_B(x, y) = (1/2)\mathrm{Tr}\{2R(x, y) + 2R(y, x) - L(x, y) - L(y, x)\} \tag{1.7}$$

is called the *trace form* of (U, B), where $\mathrm{Tr}(f)$ means the trace of a linear endomorphism f. The form γ_B was defined by Yamaguti [14] for a wider class of triple systems. We have following relations concerning the trace form.

LEMMA 1.4. (1) *The following identity is valid* [4]:

$$\gamma_B(w, (xyz)) = \gamma_B((yxw), z) = \gamma_B((wzy), x). \tag{1.8}$$

(2) *Let φ be an isomorphism of (U, B) onto (U', B'). Then we have*

$$\gamma_{B'}(\varphi(x), \varphi(y)) = \gamma_B(x, y) \qquad \text{for } x, y \in U. \tag{1.9}$$

(3) *Let V be an ideal of (U, B) and let us denote by $\bar{B}$ the restriction of B to V. Then we have*

$$\gamma_{\bar{B}}(x, y) = \gamma_B(x, y) \qquad \text{for } x, y \in V. \tag{1.10}$$

(4) *If (U, B) satisfies the condition* (A), *then we have*

$$\gamma_B(x, y) = (-1/2)\beta(x, \tau_B(y)) = (-1/2)\beta(\tau_B(x), y) \tag{1.11}$$

for $x, y \in U$, where β is the Killing form of the Kantor algebra $\mathscr{L}(B)$.

We say that a GJTS (U, B) is *compact* (resp. *non-degenerate*) if its trace form γ_B is positive definite (resp. non-degenerate). It is known [1] that the condition (A) is always satisfied in every non-degenerate GJTS of the 2nd kind.

1.4. For a JTS (U, B), a modification by an involutive automorphism φ was defined in [11] as

$$B_\varphi(x, y, z) = B(\varphi(x), y, z) + B(x, \varphi(y), z) + B(x, y, \varphi(z)) - \varphi(B(x, y, z)).$$

But we have trouble to seek B_φ practically from this definition. So we consider a different one from this.

Let (U, B) be a GJTS. For any involutive automorphism φ of (U, B), we define a new triple product B_φ on U by

$$B_\varphi(x, y, z) = B(x, \varphi(y), z). \tag{1.12}$$

Then (U, B_φ) also becomes a GJTS. We call (U, B_φ) the *φ-modification of* (U, B). We put

$$L_\varphi(x, y)(z) = B_\varphi(x, y, z), \qquad R_\varphi(x, y)(z) = B_\varphi(z, x, y),$$
$$S_\varphi(x, y)(z) = B_\varphi(x, z, y) - B_\varphi(y, z, x).$$

Then we have obviously

$$L_\varphi(x, y) = L(x, \varphi(y)), \qquad R_\varphi(x, y) = R(\varphi(x), y), \tag{1.13}$$

$$S_\varphi(x, y) = S(x, y) \circ \varphi. \tag{1.14}$$

LEMMA 1.5. *Let (U, B) be a GJTS and φ an involutive automorphism of (U, B).*
(1) *If (U, B) satisfies the condition* (A), *then so does (U, B_φ).*
(2) *If (U, B) is of the 2nd kind, then so is (U, B_φ).*
(3) *If (U, B) is non-degenerate, then so is (U, B_φ).*

PROOF. (1) is trivial. By straightforward calculations, using (1.13), (1.14) and Lemma 1.2, we can see that (2) is true. We will prove (3). Since φ is an involutive automorphism, we get

$$\operatorname{Tr}(R(\varphi(x), y)) = \operatorname{Tr}(R(x, \varphi(y))), \qquad \operatorname{Tr}(L(\varphi(x), y)) = \operatorname{Tr}(L(x, \varphi(y))). \tag{1.15}$$

By (1.13), (1.15) and (1.7), we obtain that

$$\gamma_{B_\varphi}(x, y) = \gamma_B(x, \varphi(y)) = \gamma_B(\varphi(x), y). \tag{1.16}$$

Hence, if (U, B) is non-degenerate, then so is (U, B_φ).

PROPOSITION 1.6. *Let (U, B) be a GJTS of the 2nd kind satisfying the condition* (A) *and φ be an involutive automorphism of (U, B). Then $\mathscr{L}(B)$ is isomorphic to $\mathscr{L}(B_\varphi)$ as graded Lie algebras.*

PROOF. Let us define a linear map $f\colon \mathscr{L}(B) \to \mathscr{L}(B_\varphi)$ by

$$S(a, b) \longmapsto S_\varphi(\varphi(a), \varphi(b)), \qquad a \longmapsto \varphi(a),$$
$$L(a, b) \longmapsto L_\varphi(\varphi(a), b), \qquad \tau_B(a) \longmapsto \tau_{B_\varphi}(a),$$
$$\tau_B(S(a, b)) \longmapsto \tau_{B_\varphi}(S_\varphi(a, b)).$$

Then, using (1.4), we can check in case by case that f is an isomorphism.

PROPOSITION 1.7. *Let (U, B) be a simple GJTS and φ an involutive automorphism of (U, B). Then (U, B_φ) is a simple GJTS or the direct sum of two simple ideals, which are mutually transferred by φ.*

PROOF. Let V be a non-zero minimal ideal of (U, B_φ). Since φ is also an automorphism of (U, B_φ), $\varphi(V)$ is an ideal of (U, B_φ). Hence we see that $V + \varphi(V)$ is an ideal of (U, B). Since (U, B) is simple by the assumption, we must have $U = V + \varphi(V)$. On the other hand, from the choice of V, we have $V \cap \varphi(V) = \{0\}$ or $\varphi(V) = V$. Therefore, we see that $V + \varphi(V) = U$ (direct sum) or $V = U$. In the former case, V is obviously a simple ideal of (U, B_φ).

PROPOSITION 1.8. *Let (U, B) be a compact simple GJTS of the 2nd kind and φ be an involutive automorphism of (U, B). Then the modification (U, B_φ) is a simple GJTS of the 2nd kind.*

PROOF. By Lemma 1.3, it is known that (U, B) is K-simple. Since any K-ideal of (U, B_φ) is obviously a K-ideal of (U, B), it follows that (U, B_φ) is also K-simple, hence simple.

§ 2. Structure groups and automorphism groups

2.1. Let (U, B) be a non-degenerate GJTS and γ_B its trace form. We denote by $\tilde{\psi}$ the adjoint operator of $\psi \in \mathrm{End}(U)$ with respect to γ_B. If φ is an automorphism of (U, B), then we get $\tilde{\varphi}^{-1} = \varphi$ from (1.9). Let us put

$$\Gamma(U, B) = \{\psi \in GL(U) \mid \psi \circ L(x, y) = L(\psi(x), \tilde{\psi}^{-1}(y)) \circ \psi \quad \text{for } x, y \in U\}.$$

We remark that the identity

$$\psi \circ L(x, y) = L(\psi(x), \tilde{\psi}^{-1}(y)) \circ \psi \tag{2.1}$$

is equivalent to

$$\psi(B(x, y, z)) = B(\psi(x), \tilde{\psi}^{-1}(y), \psi(z)). \tag{2.2}$$

It is easily seen that $\Gamma(U, B)$ becomes a group with the composition of mappings and that $\Gamma(U, B)$ contains the automorphism group $\mathrm{Aut}(U, B)$ of (U, B). The following lemma is well known in case that (U, B) is a JTS.

LEMMA 2.1. *For any non-degenerate GJTS (U, B), the group $\Gamma(U, B)$ is self-adjoint, that is, $\tilde{\psi} \in \Gamma(U, B)$ for any $\psi \in \Gamma(U, B)$.*

PROOF. For any $\psi \in \Gamma(U, B)$, using (1.8), we have

$$\begin{aligned}\gamma_B(\tilde{\psi}(xyz), u) &= \gamma_B((xyz), \psi(u)) = \gamma_B(z, B(y, x, \psi(u))) \\ &= \gamma_B(\tilde{\psi}(z), B(\psi^{-1}(y), \tilde{\psi}(x), u)) = \gamma_B(B(\tilde{\psi}(x), \psi^{-1}(y), \tilde{\psi}(z)), u).\end{aligned}$$

Since γ_B is non-degenerate, we get

$$\tilde{\psi}(B(x, y, z)) = B(\tilde{\psi}(x), \psi^{-1}(y), \tilde{\psi}(z)).$$

This means $\tilde{\psi} \in \Gamma(U, B)$. Hence $\Gamma(U, B)$ is a self-adjoint group.

The group $\Gamma(U, B)$ is called the *structure group* of (U, B).

EXAMPLE. Let (U, B) be a GJTS as follows (see §4):

$$U = M_{p,q}(C); \qquad B(X, Y, Z) = XY^*Z + ZY^*X - ZX^*Y.$$

For any pair $(P, Q) \in GL(p, C) \times GL(q, C)$, we define a linear endomorphism $[P, Q]$ on U by $[P, Q](X) = PXQ^{-1}$. Using the relation ([1])

$$\gamma_B(X, Y) = 2(p + 2q)\,\mathrm{Re}(\mathrm{Tr}(XY^*)),$$

where $\mathrm{Re}(\alpha)$ means the real part of a complex number α, we obtain $[P, Q]^\sim = [P^*, Q^*]$. If Q is a unitary matrix, then $[P, Q] \in \Gamma(U, B)$. Furthermore, if P is also a unitary matrix, then $[P, Q] \in \mathrm{Aut}(U, B)$.

PROPOSITION 2.2. *Let φ and ψ be involutive automorphisms of a non-degenerate GJTS (U, B). Then two modifications (U, B_φ) and (U, B_ψ) are isomorphic with each other if and only if there exists an element $\omega \in \Gamma(U, B)$ satisfying $\tilde{\omega} \circ \psi \circ \omega = \varphi$.*

PROOF. The "if"-part is easily proved. Now we assume that ω is an isomorphism of (U, B_φ) onto (U, B_ψ). Then, from (1.16) and (1.9), we have

$$\gamma_B(\omega(x), \psi \circ \omega(y)) = \gamma_{B_\psi}(\omega(x), \omega(y)) = \gamma_{B_\varphi}(x, y) = \gamma_B(x, \varphi(y)).$$

Since γ_B is non-degenerate, we get $\tilde{\omega} \circ \psi \circ \omega = \varphi$. On the other hand, since ω is an isomorphism, we have

$$\begin{aligned}\omega(B(x, \varphi(y), z)) &= \omega(B_\varphi(x, y, z)) = B_\psi(\omega(x), \omega(y), \omega(z)) \\ &= B(\omega(x), \psi \circ \omega(y), \omega(z)).\end{aligned}$$

Substituting y by $\varphi^{-1}(y)$ in this identity, we get

$$\omega(B(x, y, z)) = B(\omega(x), \psi \circ \omega \circ \varphi^{-1}(y), \omega(z)) = B(\omega(x), \tilde{\omega}^{-1}(y), \omega(z)).$$

This implies $\omega \in \Gamma(U, B)$, which completes the proof of the "only if"-part.

2.2. The following lemma is analogous to our previous one ([5] Lemma 3.11).

LEMMA 2.3. *Let (U, B) be a non-degenerate GJTS of the 2nd kind and $\Gamma(U, B)$ be its structure group. Then any element $\psi \in \Gamma(U, B)$ induces a grade-preserving automorphism $\mathscr{L}(\psi)$ of $\mathscr{L}(B)$. Furthermore it satisfies*

$$\mathscr{L}(\psi) \circ \tau_B = \tau_B \circ \mathscr{L}(\tilde{\psi}^{-1}). \tag{2.3}$$

PROOF. Using Lemma 1.1, we define a linear transformation $\mathscr{L}(\psi)$ on $\mathscr{L}(B)$ as

$$\begin{aligned} &S(a, b) \longmapsto S(\psi(a), \psi(b)), && x \longmapsto \psi(x), \\ &L(a, b) \longmapsto L(\psi(a), \tilde{\psi}^{-1}(b)), && \tau_B(y) \longmapsto \tau_B(\tilde{\psi}^{-1}(y)), \\ &\tau_B(S(a, b)) \longmapsto \tau_B(S(\tilde{\psi}^{-1}(a), \tilde{\psi}^{-1}(b))) && \end{aligned} \tag{2.4}$$

for any $a, b, x, y \in U$. Then it is easy to check that $\mathscr{L}(\psi)$ is a grade-preserving automorphism of $\mathscr{L}(B)$. Moreover, from the definition of $\mathscr{L}(\psi)$, we have

$$(\tau_B \circ \mathscr{L}(\tilde{\psi}^{-1}))(x) = \tau_B(\tilde{\psi}^{-1}(x)) = \mathscr{L}(\psi)(\tau_B(x)) = (\mathscr{L}(\psi) \circ \tau_B)(x), \tag{2.5}$$

$$(\tau_B \circ \mathscr{L}(\tilde{\psi}^{-1}))(\tau_B(y)) = \tau_B(\tau_B(\psi(y))) = \psi(y) = (\mathscr{L}(\psi) \circ \tau_B)(\tau_B(y)). \tag{2.6}$$

Hence we see that (2.3) is valid on $U_{-1} + U_1$. Since $\mathscr{L}(B)$ is generated by $U_{-1} + U_1$, it is also valid on $\mathscr{L}(B)$.

Next we put

$$\begin{aligned} &\mathrm{Aut}_+ \mathscr{L}(B) = \{\sigma \in \mathrm{Aut}(\mathscr{L}(B)) \mid \sigma \text{ is grade-preserving}\}, \\ &\mathrm{Aut}_- \mathscr{L}(B) = \{\sigma \in \mathrm{Aut}(\mathscr{L}(B)) \mid \sigma \text{ is grade-reversing}\}, \\ &\mathrm{Aut}_\pm (\mathscr{L}(B), \tau_B) = \{\sigma \in \mathrm{Aut}_\pm \mathscr{L}(B) \mid \sigma \circ \tau_B = \tau_B \circ \sigma\}. \end{aligned}$$

Let us consider a map $\mathscr{L}: \psi \mapsto \mathscr{L}(\psi)$ of $\Gamma(U, B)$ into $\mathrm{Aut}_+ \mathscr{L}(B)$. The map $\mathscr{L}$ is a monomorphism. In fact, the identity

$$\mathscr{L}(\psi \circ \varphi) = \mathscr{L}(\psi) \circ \mathscr{L}(\varphi) \tag{2.7}$$

is obviously valid on $U = U_{-1}$. Furthermore, by use of (2.4), it is valid on U_1. Hence it is also valid on $\mathscr{L}(B)$. It is clear that $\mathscr{L}$ is injective.

For any involutive automorphism τ of $\mathscr{L}(B)$, we define a symmetric bilinear form β_τ on $\mathscr{L}(B)$ by

$$\beta_\tau(X, Y) = \beta(X, \tau(Y)),$$

where β is the Killing form of $\mathscr{L}(B)$.

LEMMA 2.4. *Let (U, B) be a non-degenerate GJTS of the 2nd kind. Then we have*

$$\tau_B \circ \sigma \circ \tau_B = (\sigma^*)^{-1} \qquad \text{for } \sigma \in \operatorname{Aut}(\mathscr{L}(B)), \tag{2.8}$$

$$\mathscr{L}(\tilde{\psi}) = \mathscr{L}(\psi)^* \qquad \text{for } \psi \in \Gamma(U, B), \tag{2.9}$$

where σ^* denotes the adjoint operator of σ with respect to β_{τ_B}.

PROOF. Let $\sigma \in \operatorname{Aut}(\mathscr{L}(B))$. Then we see that

$$\begin{aligned}\beta(X, Y) &= \beta(\sigma(X), \sigma(Y)) = \beta_{\tau_B}(\sigma(X), \tau_B \circ \sigma(Y)) \\ &= \beta_{\tau_B}(X, \sigma^* \circ \tau_B \circ \sigma(Y)) = \beta(X, \tau_B \circ \sigma^* \circ \tau_B \circ \sigma(Y)).\end{aligned}$$

Since β is non-degenerate, we get $\tau_B \circ \sigma^* \circ \tau_B \circ \sigma = \mathbf{1}$, where $\mathbf{1}$ denotes the identity map on $\mathscr{L}(B)$. This implies (2.8). Next let $\psi \in \Gamma(U, B)$. By use of (2.3) and (2.7), we have

$$\begin{aligned}\beta(\mathscr{L}(\tilde{\psi})(X), Y) &= \beta(X, \mathscr{L}(\tilde{\psi})^{-1}(Y)) = \beta(X, \mathscr{L}(\tilde{\psi}^{-1})(Y)) \\ &= \beta(X, \tau_B \circ \mathscr{L}(\psi) \circ \tau_B(Y)) = \beta_{\tau_B}(X, \mathscr{L}(\psi) \circ \tau_B(Y)) \\ &= \beta_{\tau_B}(\mathscr{L}(\psi)^*(X), \tau_B(Y)) = \beta(\mathscr{L}(\psi)^*(X), Y).\end{aligned}$$

Hence we get $\mathscr{L}(\tilde{\psi}) = \mathscr{L}(\psi)^*$.

THEOREM 2.5. *Let (U, B) be a non-degenerate GJTS of the second kind and $\mathscr{L}(B)$ be its Kantor algebra. Then the following relations are valid:*

$$\mathscr{L}(\Gamma(U, B)) = \operatorname{Aut}_+ \mathscr{L}(B), \tag{2.10}$$

$$\mathscr{L}(\operatorname{Aut}(U, B)) = \operatorname{Aut}_+(\mathscr{L}(B), \tau_B). \tag{2.11}$$

PROOF. We have already proved that $\mathscr{L}(\Gamma(U, B)) \subset \operatorname{Aut}_+ \mathscr{L}(B)$. For any $\sigma \in \operatorname{Aut}_+ \mathscr{L}(B)$, we denote by ψ the restriction of σ on U. Then we have

$$\sigma^*(y) = \tilde{\psi}(y) \qquad \text{for } y \in U. \tag{2.12}$$

In fact, using (1.11), we have

$$\begin{aligned}\gamma_B(\tilde{\psi}(y), z) &= \gamma_B(y, \psi(z)) = (-1/2)\beta(\tau_B(y), \psi(z)) \\ &= (-1/2)\beta_{\tau_B}(y, \sigma(z)) = (-1/2)\beta_{\tau_B}(\sigma^*(y), z) \\ &= (-1/2)\beta(\sigma^*(y), \tau_B(z)) = \gamma_B(\sigma^*(y), z).\end{aligned}$$

Using (2.8) and (2.12), we have

$$\begin{aligned}\psi(B(x, y, z)) &= \sigma([[\tau_B(y), x], z]) = [[\sigma \circ \tau_B(y), \sigma(x)], \sigma(z)] \\ &= [[\tau_B \circ (\sigma^*)^{-1}(y), \sigma(x)], \sigma(z)]\end{aligned}$$

$$= B(\sigma(x), (\sigma^*)^{-1}(y), \sigma(z)) = B(\psi(x), \tilde{\psi}^{-1}(y), \psi(z)).$$

This means that $\psi \in \Gamma(U, B)$. Moreover, using again (2.4), (2.8) and (2.12), we have

$$\mathscr{L}(\psi)(x + \tau_B(y)) = \psi(x) + \tau_B(\tilde{\psi}^{-1}(y)) = \sigma(x) + \tau_B((\sigma^*)^{-1}(y))$$
$$= \sigma(x) + \sigma(\tau_B(y)) = \sigma(x + \tau_B(y)).$$

This means that $\mathscr{L}(\psi) = \sigma$ on $U_{-1} + U_1$, hence on $\mathscr{L}(B)$. Therefore we see that $\mathscr{L}$ maps $\Gamma(U, B)$ onto $\mathrm{Aut}_+ \mathscr{L}(B)$. Next we get from (2.3) that $\mathscr{L}(\mathrm{Aut}(U, B)) \subset \mathrm{Aut}_+(\mathscr{L}(B), \tau_B)$. Conversely, for any $\sigma \in \mathrm{Aut}_+(\mathscr{L}(B), \tau_B)$, there exists $\psi \in \Gamma(U, B)$ such that $\mathscr{L}(\psi) = \sigma$. Hence we have $\mathscr{L}(\psi) \circ \tau_B = \tau_B \circ \mathscr{L}(\psi)$. From this and Lemma 2.4, we get $\mathscr{L}(\tilde{\psi}^{-1}) = \mathscr{L}(\psi)$, hence $\tilde{\psi}^{-1} = \psi$. This means $\psi \in \mathrm{Aut}(U, B)$. Therefore we have proved that $\mathscr{L}(\mathrm{Aut}(U, B)) = \mathrm{Aut}_+(\mathscr{L}(B), \tau_B)$.

COROLLARY. *Let (U, B) be a non-degenerate GJTS of the 2nd kind. Then we have*

$$\Gamma(U, B) = \{\sigma|_U \mid \sigma \in \mathrm{Aut}_+ \mathscr{L}(B)\},$$
$$\mathrm{Aut}(U, B) = \{\sigma|_U \mid \sigma \in \mathrm{Aut}_+(\mathscr{L}(B), \tau_B)\},$$

where $\sigma|_U$ denotes the restriction of σ on U.

§3. Classification Theorems

3.1. At first, we show the following

THEOREM 3.1. *Let (U, B) be a non-degenerate real GJTS of the 2nd kind and τ_B the grade-reversing canonical involution on $\mathscr{L}(B)$. Then there exists a grade-reversing Cartan involution σ commuting with τ_B.*

PROOF. Since $\mathscr{L}(B)$ is semisimple ([1] Proposition 2.4), there exists a grade-reversing Cartan involution ν of $\mathscr{L}(B)$. Then the form β_ν is negative definite by definition. Now we put $\rho = \tau_B \circ \nu$. Then the automorphism ρ of $\mathscr{L}(B)$ is self-adjoint with respect to the inner product, which is defined by

$$(X, Y) = -\beta_\nu(X, Y). \tag{3.1}$$

In fact, we have

$$(\rho(X), Y) = -\beta(\tau_B \circ \nu(X), \nu(Y)) = -\beta(\nu(X), \tau_B \circ \nu(Y)) = (X, \rho(Y)).$$

It follows that ρ^2 is a positive definite symmetric operator. Therefore, there exists a derivation D on $\mathscr{L}(B)$ satisfying

$$\exp D = \rho^2. \tag{3.2}$$

Now let us put

$$\sigma = \exp(1/4)D \circ \nu \circ \exp(-1/4)D. \tag{3.3}$$

Then we see that σ is a Cartan involution commuting with τ_B (see [9] p. 153). Moreover, since ρ is grade-preserving, the automorphism $\exp(1/4)D$ is also grade-preserving. It follows from (3.3) that σ is grade-reversing.

3.2. Let (U, B) be a GJTS satisfying the condition (A). An involutive automorphism φ of (U, B) is called a *Cartan involution* if the φ-modification (U, B_φ) is compact.

THEOREM 3.2. *For every non-degenerate GJTS (U, B) of the 2nd kind, there exists a Cartan involution on it.*

PROOF. By Theorem 3.1, there exists a grade-reversing Cartan involution σ of $\mathscr{L}(B)$ commuting with τ_B. Put

$$\varphi = \sigma \circ \tau_B|_U. \tag{3.4}$$

Then φ is an automorphism of (U, B) by Corollary of Theorem 2.5. It is trivial that φ is involutive. Using (1.11), (1.16) and (3.4), we have the relation

$$\begin{aligned}\gamma_{B_\varphi}(x, x) &= \gamma_B(x, \varphi(x)) = (-1/2)\beta(x, \tau_B \circ \varphi(x)) \\ &= (-1/2)\beta(x, \sigma(x)) = (-1/2)\beta_\sigma(x, x).\end{aligned}$$

Hence the form γ_{B_φ} is positive definite, hereby φ is a Cartan involution.

Let (U, B) be a non-degenerate GJTS of the 2nd kind and φ be a Cartan involution of it. Then φ is also an automorphism of the compact GJTS (U, B_φ). Obviously (U, B) is the φ-modification of (U, B_φ). Hence we have the following

THEOREM 3.3. *Any non-degenerate GJTS of the 2nd kind is obtained by a modification of a compact GJTS.*

3.3. For a given GJTS (U, B), let us consider the direct sum $(U + U, B + B)$ as triple system. This direct sum also becomes a GJTS. A linear map $\varphi: U + U \to U + U$ defined by $\varphi((x, y)) = (y, x)$ is an involutive automorphism of $(U + U, B + B)$. Hereafter we shall denote by $(\tilde{U}, \tilde{B})$ the φ-modification of $(U + U, B + B)$, that is,

$$\tilde{B}((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (B(x_1, y_2, z_1), B(x_2, y_1, z_2)). \tag{3.5}$$

We call $(\tilde{U}, \tilde{B})$ the *special direct sum* of (U, B).

PROPOSITION 3.4. *If (U, B) is a compact simple GJTS of the 2nd kind, then $(\tilde{U}, \tilde{B})$ is a simple GJTS of the 2nd kind.*

PROOF. Let W be a non-zero ideal of $(\tilde{U}, \tilde{B})$. Let us put

$$V = \{x \in U \mid (x, 0) \in W\}.$$

We will prove that V is a non-zero K-ideal of (U, B). Let us assume that $V = \{0\}$. From the assumption of W being nonzero, there exists a non-zero element (x, y) in W. For any $u, v \in U$, we have

$$(B_y(u, v), 0) = (B(u, y, v), 0) = \tilde{B}((u, 0), (x, y), (v, 0)) \in W.$$

Since $V = \{0\}$, we get $B_y(u, v) = 0$, hence $B_y = 0$. Since a simple GJTS satisfies the condition (A) ([1] Proposition 2.5), it follows that $y = 0$, that is, $(x, 0) \in W$. Using again the assumption $V = \{0\}$, we obtain $x = 0$, which contradicts to the choice of (x, y). Therefore V is the non-zero subset of U. For $x, y \in U$ and $z \in V$, we have

$$(B(z, x, y), 0) = \tilde{B}((z, 0), (0, x), (y, 0)) \in W,$$
$$(B(x, y, z), 0) = \tilde{B}((x, 0), (0, y), (z, 0)) \in W.$$

Hence we see that $B(z, x, y)$ and $B(x, y, z) \in V$. Therefore V is a K-ideal of (U, B). On the other hand, since (U, B) is also K-simple by Lemma 1.3, we must have $V = U$. It follows that $U + \{0\} \subset W$. Using this relation, we get

$$(0, B(x, y, z)) = \tilde{B}((0, x), (y, 0), (0, z)) \in W$$

for any $x, y, z \in U$. Since $B(U, U, U) = U$, we get $\{0\} + U \subset W$. Therefore we have $W = U + U = \tilde{U}$.

THEOREM 3.5. *Any non-compact simple GJTS of the 2nd kind is obtained as*
(1) *a modification of a compact simple GJTS of the 2nd kind by an involutive automorphism, or*
(2) *the special direct sum of a compact simple GJTS of the 2nd kind.*

PROOF. Let (U, B) be a non-compact simple GJTS of the 2nd kind. Since it is non-degenerate ([1] Theorem 2.8), by Theorem 3.3, there exist a compact GJTS (U, B') and an involutive automorphism φ of (U, B') such that $B = B'_\varphi$. We remark that we also have $B_\varphi = B'$. Now we assume that (U, B') is not simple. Then, by Proposition 1.7, we see that $U = V + \varphi(V)$ (direct sum) and V is a simple ideal of (U, B'). We denote the restriction $B'|_{V \times V \times V}$ by the same symbol B'. From (1.10), we see that an ideal of a compact GJTS of the 2nd kind is also a compact GJTS. Hence (V, B') is a compact simple GJTS of the 2nd kind. Identifying $\varphi(V)$ and V, we see that (U, B) is isomorphic to

$(\tilde{V}, \tilde{B}')$. In fact, we define a linear map $f: U \to \tilde{V}$ by $f(x+\varphi(y)) = (x, y)$ $(x, y \in V)$. Then we have

$$\begin{aligned} f(B(u+\varphi(v), w+\varphi(x), y+\varphi(z))) &= f(B'(u+\varphi(v), x+\varphi(w), y+\varphi(z))) \\ &= f(B'(u, x, y)+\varphi(B'(v, w, z))) = (B'(u, x, y), B'(v, w, z)) \\ &= \tilde{B}'((u, v), (w, x), (y, z)) = \tilde{B}'(f(u+\varphi(v)), f(w+\varphi(x)), f(y+\varphi(z))). \end{aligned}$$

Hence f is a homomorphism of (U, B) to $(\tilde{V}, \tilde{B}')$. Since f is obviously bijective, we have proved that (U, B) is isomorphic with $(\tilde{V}, \tilde{B}')$. Therefore (U, B) is isomorphic to the special direct sum of the compact simple GJTS (V, B').

THEOREM 3.6. *Let (U, B) be a non-degenerate GJTS of the 2nd kind, and let φ and ψ be involutive automorphisms of (U, B). Then the modification (U, B_φ) is isomorphic with (U, B_ψ) if and only if there exists a grade-preserving automorphism σ of $\mathscr{L}(B)$ such that*

$$\sigma^{-1} \circ \tau_B \circ \mathscr{L}(\varphi) \circ \sigma = \tau_B \circ \mathscr{L}(\psi). \tag{3.6}$$

PROOF. Assume that (U, B_φ) and (U, B_ψ) are isomorphic with each other. From Proposition 2.2, there exists an element $\omega \in \Gamma(U, B)$ such that $\tilde{\omega} \circ \varphi \circ \omega = \psi$. Hence we get $\mathscr{L}(\tilde{\omega}) \circ \mathscr{L}(\varphi) \circ \mathscr{L}(\omega) = \mathscr{L}(\psi)$. It follows from (2.3) that

$$\mathscr{L}(\omega)^{-1} \circ \tau_B \circ \mathscr{L}(\varphi) \circ \mathscr{L}(\omega) = \tau_B \circ \mathscr{L}(\psi).$$

Conversely, let σ be a grade-preserving automorphism of $\mathscr{L}(B)$ satisfying (3.6). Then, by Theorem 2.5, there exists $\mu \in \Gamma(U, B)$ such that $\mathscr{L}(\mu) = \sigma$. Using (2.8), we get $\mathscr{L}(\mu)^* \circ \mathscr{L}(\varphi) \circ \mathscr{L}(\mu) = \mathscr{L}(\psi)$. Hence, using again (2.9), we get $\tilde{\mu} \circ \varphi \circ \mu = \psi$. It follows from Proposition 2.2 that (U, B_φ) and (U, B_ψ) are isomorphic with each other.

§4. Compact classical simple GJTS's of the 2nd kind

In [5], we gave the classification of compact classical simple GJTS's of the 2nd kind. Here again we will write it down by another representation with some improvements. Throughout this section, we will use following notations.

$\boldsymbol{R}$ (resp. $\boldsymbol{C}$, $\boldsymbol{H}$): the set of all real (resp. complex, quaternion) numbers,
${}^tX$: the transposed matrix of a matrix X,
X^*: the transposed conjugate matrix of X,
I_n : the identity matrix of degree n,
$J_n = (a_{ij})$: the matrix of degree n such that $a_{ij} = \delta_{i,n+1-j}$, where $\delta_{i,j}$ denotes the Kronecker's delta,

$$\tilde{J}_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes J_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \qquad \hat{J}_n = I_n \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$K_n = J_n \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, where i denotes the imaginary unit,

$$A_{n,i} = \begin{pmatrix} & & J_i \\ & I_{n-2i} & \\ J_i & & \end{pmatrix}, \qquad A'_{n,i} = A_{n,i} \otimes I_2,$$

$M_{p,q}(K)$: the vector space of all $p \times q$ matrices with entries in K, where $K = \boldsymbol{R}, \boldsymbol{C}$,

$M_{p,q}(\boldsymbol{H}) = \{X \in M_{2p,2q}(\boldsymbol{C}) | \bar{X}\hat{J}_q = \hat{J}_p X\}$,

$M_n(K) = M_{n,n}(K)$,

$Alt'_n(K) = \{X \in M_n(K) | {}^tXJ_n + J_nX = 0\}$.

Theorem 4.1 ([5]). *Compact classical real simple GJTS's (U, B) of the 2nd kind are classified (up to isomorphisms) as follows:*

1. $U = M_{p,q}(K) \times M_{q,r}(K)$, $K = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}$, $p \le r$;

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1 Y_1^* Z_1 + Z_1 Y_1^* X_1 - Z_1 X_2 Y_2^* \\ X_2 Y_2^* Z_2 + Z_2 Y_2^* X_2 - Y_1^* X_1 Z_2 \end{pmatrix}.$$

2. $U = M_{p,q}(\boldsymbol{R})$, $2 \le p$;

$$B_r(X, Y, Z) = X\,{}^tYZ + Z\,{}^tYX - ZA_{q,r}\,{}^tXYA_{q,r},$$

$0 \le r < q/2$	if q is odd or $(p, q) = (2, 2)$,
$0 \le r \le q/2$	otherwise.

3. $U = M_{p,q}(\boldsymbol{C})$;

$$B_r(X, Y, Z) = XY^*Z + ZY^*X - ZA_{q,r}X^*YA_{q,r},$$

$0 \le r < q/2$	if q is odd or $(p, q) = (1, 2), (1, 4), (2, 2)$,
$0 \le r \le q/2$	otherwise.

4. $U = M_{p,2q}(K)$, where $K = \boldsymbol{R}, \boldsymbol{C}$, $(p, q) \ne (1, 1)$;

$$B(X, Y, Z) = XY^*Z + ZY^*X + Z\tilde{J}_q\,{}^tX\bar{Y}\tilde{J}_q.$$

5. $U = M_{p,q}(\boldsymbol{C})$, $2 \le p$, $(p, q) \ne (2, 2)$;

$$B(X, Y, Z) = XY^*Z + ZY^*X - Z\,{}^tX\bar{Y}.$$

6. $U = M_{p,q}(\boldsymbol{H})$;

$$B_r(X, Y, Z) = XY^*Z + ZY^*X - ZA'_{q,r}X^*YA'_{q,r}, \qquad 0 \le r \le q/2.$$

7. $U = M_{p,q}(\boldsymbol{H}), \quad (p, q) \neq (1, 1), (1, 2);$

$B(X, Y, Z) = XY^*Z + ZY^*X + ZK_qX^*YK_q.$

8. $U = M_{1,n}(K) \times Alt'_n(K)$, where $K = \boldsymbol{R}, \boldsymbol{C}, \quad 4 \leq n;$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right)$$
$$= \begin{pmatrix} X_1Y_1^*Z_1 + Z_1Y_1^*X_1 - Z_1X_2Y_2^* \\ X_2Y_2^*Z_2 + Z_2Y_2^*X_2 - Y_1^*X_1Z_2 - Z_2J_n{}^tX_1\bar{Y}_1J_n \end{pmatrix}.$$

§5. Clssification of non-compact classical simple GJTS's

Throughout this section, we keep the notations in the previous section, We will also use following notations for matrices:

$$S(p, i) = \begin{pmatrix} -I_i & 0 \\ 0 & I_{p-i} \end{pmatrix} \quad (0 \leq i \leq p), \quad \begin{matrix} S(p, i) = I_p & (i < 0), \\ S(p, i) = -I_p & (i > p), \end{matrix}$$

$$H(q, j) = \begin{pmatrix} -I_j & 0 & 0 \\ 0 & I_{q-2j} & 0 \\ 0 & 0 & -I_j \end{pmatrix} \quad (0 \leq j \leq q/2),$$

$$K(q; j, k) = \begin{pmatrix} 0 & 0 & J_j \\ 0 & H(q-2j, k) & 0 \\ J_j & 0 & 0 \end{pmatrix} \quad (0 \leq j \leq q/2, 0 \leq k + j \leq q/2),$$

$L(q, r; j, k) =$

$$\frac{1}{2}\begin{pmatrix} S(r, k) + S(r, j) & 0 & (S(r, k) - S(r, j))J_r \\ 0 & 2J_{q-2r}S(q-2r, k-r)J_{q-2r} & 0 \\ J_r(S(r, k) - S(r, j)) & 0 & J_r(S(r, k) + S(r, j))J_r \end{pmatrix}$$
$$(0 \leq j \leq r, 0 \leq k \leq q-r),$$

$$\hat{S}(q, j) = S(q, j) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$S'(p, i) = S(p, i) \otimes I_2 = S(2p, 2i),$$

$$L'(q, r; j, k) = L(q, r; j, k) \otimes I_2.$$

Moreover we will use following notations for mappings:

$$[P, Q, R](X, Y) = (PXQ^{-1}, QYR^{-1}),$$
$$\omega(X, Y) = (\bar{X}, \bar{Y}), \quad \tau(X, Y) = (-{}^tY, -{}^tX),$$

for $(X, Y) \in M_{p,q}(K) \times M_{q,r}(K)$;

$$[P, Q](X) = PXQ^{-1}, \quad \omega(X) = \bar{X}$$

for $X \in M_{p,q}(K)$.

5.1. Let (U, B) be a compact simple GJTS. We put

$\mathrm{Inv}(U, B) =$ the set of all involutive automorphisms of (U, B),
$\mathrm{Inv}(\mathscr{L}(B)) =$ the set of all involutive automorphisms of $\mathscr{L}(B)$,
$\mathrm{Inv}_-(\mathscr{L}(B), \tau_B) = \mathrm{Inv}(\mathscr{L}(B)) \cap \mathrm{Aut}_-(\mathscr{L}(B), \tau_B)$.

For $\sigma, \rho \in \mathrm{Inv}_-(\mathscr{L}(B), \tau_B)$, we say that σ is *equivalent* to ρ (denoted by $\sigma \sim \rho$) if $\xi^{-1} \circ \sigma \circ \xi = \rho$ for some $\xi \in \mathrm{Aut}_+(\mathscr{L}(B))$. We denote by $\mathscr{R}$ a representative system of equivalence classes of $\mathrm{Inv}_-(\mathscr{L}(B), \tau_B)$ under this equivalence relation. Moreover, we put

$$\mathscr{S} = \{\tau_B \circ \sigma|_U \,|\, \sigma \in \mathscr{R}\}.$$

For $\varphi, \psi \in \mathrm{Aut}(U, B)$, we say that φ is *equivalent* to ψ under $\Gamma(U, B)$ if there exists an element $\omega \in \Gamma(U, B)$ such that $\tilde{\omega} \circ \varphi \circ \omega = \psi$. Then we see that the set $\mathscr{S}$ is a representative system of equivalence classes under $\Gamma(U, B)$ of $\mathrm{Inv}(U, B)$.

For classification of non-compact simple GJTS's, by Theorem 3.5 and Theorem 3.6, it is sufficient to decide the set $\mathscr{S}$, equivalently the set $\mathscr{R}$, for every compact simple GJTS.

5.2. Let (U, B) be a compact classical real simple GJTS of the 2nd kind. It is known ([1]) that its Kantor algebra $\mathscr{L}(B)$ is a non-compact simple Lie algebra and the grade-reversing canonical involution τ_B is a Cartan involution of $\mathscr{L}(B)$. For brevity we put $\mathscr{G} = \mathscr{L}(B)$. Let $\mathscr{G} = \mathscr{K} + \mathscr{P}$ be the Cartan decomposition of $\mathscr{G}$ by τ_B, that is,

$$\mathscr{K} = \{X \in \mathscr{G} \,|\, \tau_B(X) = X\}, \qquad \mathscr{P} = \{X \in \mathscr{G} \,|\, \tau_B(X) = -X\}.$$

Let G be a connected classical simple Lie group, whose Lie algebra is isomorphic to $\mathscr{G}$, and let K be the analytic subgroup of G corresponding to $\mathscr{K}$. Let us denote by $\mathrm{Inn}(\mathscr{G})$ (resp. $\mathrm{Inn}(\mathscr{G}, \tau_B)$) the set of all inner automorphisms (resp. inner automorphisms commuting with τ_B) of $\mathscr{G}$. Then it is known that $\mathrm{Inn}(\mathscr{G}) = \mathrm{Ad}_{\mathscr{G}}(G)$, $\mathrm{Inn}(\mathscr{G}, \tau_B) = \mathrm{Ad}_{\mathscr{G}}(K)$ and $\mathrm{Inn}(\mathscr{G}, \tau_B)$ is a maximal compact subgroup of $\mathrm{Inn}(\mathscr{G})$. Furthermore, since G is the self-adjoint subgroup of $GL(n, \boldsymbol{C})$, we can obtain the group K as $K = G \cap U(n)$, where $U(n)$ denotes the group of unitary matrices.

The following proposition is used in process of deciding representative systems $\mathscr{S}$ in §5.5.

Proposition 5.1 ([2]). *Let G be a compact but not necessarily connected Lie group and S be a Cartan subgroup of G generated by z. And let G_0 (resp. S_0) be the connected component of the identity of G (resp. S). Then any $g \in G_0 z$ is conjugate to an element of $S_0 z$ via conjugation by an element of G_0.*

5.3. For example, we will firstly consider about the following GJTS (U, B):

$$U = M_{p,q}(\boldsymbol{R}) \times M_{q,r}(\boldsymbol{R}), \qquad p \leq r;$$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1{}^tY_1Z_1 + Z_1{}^tY_1X_1 - Z_1X_2{}^tY_2 \\ X_2{}^tY_2Z_2 + Z_2{}^tY_2X_2 - {}^tY_1X_1Z_2 \end{pmatrix}.$$

In this case, we have following results:

Kantor algebra: $\mathscr{G} = \mathscr{L}(B) = \mathfrak{sl}(n, \boldsymbol{R})$, where $n = p + q + r$,

Cartan involution: $\tau_B(X) = -{}^tX$,

Characteristic element:

$$E = \begin{pmatrix} (\alpha - 1)I_p & 0 & 0 \\ 0 & \alpha I_q & 0 \\ 0 & 0 & (\alpha - 1)I_r \end{pmatrix}, \text{ where } \alpha = \frac{p - r}{n},$$

$\mathrm{Inn}(\mathscr{G}) = \mathrm{Ad}_{\mathscr{G}}(SL(n, \boldsymbol{R}))$,

$\mathrm{Aut}(\mathscr{G}) = \mathrm{Ad}_{\mathscr{G}}(SL'(n, \boldsymbol{R})) \cup \tau_B \circ \mathrm{Ad}_{\mathscr{G}}(SL'(n, \boldsymbol{R}))$, where

$$SL'(n, \boldsymbol{R}) = \{P \in GL(n, \boldsymbol{R}) \mid \det P = \pm 1\}.$$

Now we assume that $p < r$. Then we have

$$\mathrm{Aut}_+ \mathscr{G} = \{\mathrm{Ad}(P) \mid \mathrm{Ad}(P)E = E\}$$

$$= \left\{ \mathrm{Ad}(P) \;\middle|\; P = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix} \in SL'(n, R),\ \begin{matrix} P_1 \in M_p(\boldsymbol{R}), \\ P_2 \in M_q(\boldsymbol{R}), \\ P_3 \in M_r(\boldsymbol{R}) \end{matrix} \right\},$$

$\mathrm{Inn}_- \mathscr{G} = \phi$,

$$\mathrm{Aut}_-(\mathscr{G}, \tau_B) = \left\{\tau_B \circ \mathrm{Ad}(P) \mid P = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix} \in O(n)\right\},$$

If $\tau_B \circ \mathrm{Ad}(P) \in \mathrm{Inv}_-(\mathscr{G}, \tau_B)$, then we have $P^2 = I$ or $P^2 = -I$, where the latter occurs only when n is even. If $P^2 = I$, then P is a symmetric orthogonal matrix. Hence we get

$$\mathrm{Ad}(P) \sim \mathrm{Ad}\begin{pmatrix} S(p,i) & & \\ & S(q,j) & \\ & & S(r,k) \end{pmatrix}$$

$$\left(0 \le i \le p,\ 0 \le j \le \frac{q}{2},\ 0 \le k \le r\right)$$

under $\mathrm{Aut}_+(\mathscr{G}, \tau_B)$. Therefore we see

$$\tau_B \circ \mathrm{Ad}(P) \sim \tau_B \circ \mathrm{Ad}\begin{pmatrix} S(p,i) & & \\ & S(q,j) & \\ & & S(r,k) \end{pmatrix} \tag{5.1}$$

$$\left(0 \le i \le p,\ 0 \le j \le \frac{q}{2},\ 0 \le k \le r\right).$$

If $P^2 = -I$, then $P_i (i = 1, 2, 3)$ is a skew-symmetric orthogonal matrix. Hence, we have that p, q and r are all even and that

$$\mathrm{Ad}(P) \sim \mathrm{Ad}(\hat{J}_{n/2})$$

under $\mathrm{Aut}_+(\mathscr{G}, \tau_B)$. Hence we also get

$$\tau_B \circ \mathrm{Ad}(P) \sim \tau_B \circ \mathrm{Ad}(\hat{J}_{n/2}). \tag{5.2}$$

From (5.1) and (5.2) we see that

$$\mathscr{R} = \{\tau_B \circ \mathrm{Ad}\begin{pmatrix} S(p,i) & & \\ & S(q,j) & \\ & & S(r,k) \end{pmatrix} \left(0 \le i \le p, 0 \le j \le \frac{q}{2}, 0 \le k \le r\right),$$

$$\tau_B \circ \mathrm{Ad}(\hat{J}_{n/2}) \quad \text{if } p, q \text{ and } r \text{ are even}\}.$$

Since any element $X = (X_1, X_2) \in U$ is imbedded in $\mathscr{L}(B)$ as

$$X = \begin{pmatrix} 0 & X_1 & 0 \\ 0 & 0 & X_2 \\ 0 & 0 & 0 \end{pmatrix},$$

we have

$$\mathrm{Ad}(P)X = PXP^{-1} = \begin{pmatrix} 0 & P_1X_1P_2{}^{-1} & 0 \\ 0 & 0 & P_2X_2P_3{}^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore we get

$$\mathscr{S} = \{[S(p, i), S(q, j), S(r, k)] \ (0 \le i \le p, 0 \le j \le q/2, 0 \le k \le r), \\ [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{r/2}] \quad \text{if } p, q \text{ and } r \text{ are even}\}.$$

Next we assume that $p = r$. Then we have

$$\mathrm{Aut}_-(\mathscr{G}, \tau_B) = \{\tau_B \circ \mathrm{Ad}(P) \mid P = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix} \in O(n)\}$$

$$\cup \{\mathrm{Ad}(Q) \mid Q = \begin{pmatrix} & & Q_1 \\ & Q_2 & \\ Q_3 & & \end{pmatrix} \in O(n)\}.$$

Let $\mathrm{Ad}(Q) \in \mathrm{Inv}_-(\mathscr{G}, \tau_B)$. Then Q is a symmetric or skew-symmetric (the latter occurs only when q is even) orthogonal matrix. If Q is symmetric (resp. skew-symmetric), then there exist matrices T_1, T_2 and T_3 satisfying

$$T_1 Q_1 T_3^{-1} = I_p, \qquad T_1, T_3 \in SL'(p, \boldsymbol{R}),$$
$$T_2 Q_2 T_2^{-1} = S(q, j) \quad \text{for some } j \quad (\text{resp. } T_2 Q_2 T_2^{-1} = \hat{J}_{q/2}).$$

Therefore, putting

$$T = \begin{pmatrix} T_1 & & \\ & T_2 & \\ & & T_3 \end{pmatrix} \in SL'(n, \mathbf{R}),$$

we have

$$TQT^{-1} = \begin{pmatrix} & & I_p \\ & S(q, j) & \\ I_p & & \end{pmatrix} \qquad (\text{resp. } TQT^{-1} = \begin{pmatrix} & & I_p \\ & \hat{J}_{q/2} & \\ -I_p & & \end{pmatrix}).$$

Hence we get

$$(5.3) \quad \mathrm{Ad}(Q) \sim \mathrm{Ad}\begin{pmatrix} & & I_p \\ & S(q, j) & \\ I_p & & \end{pmatrix} \text{ or } \mathrm{Ad}(Q) \sim \mathrm{Ad}\begin{pmatrix} & & I_p \\ & \hat{J}_{q/2} & \\ -I_p & & \end{pmatrix}$$

under $\mathrm{Aut}_+ \mathscr{G}$. Therefore we see that

$$\mathscr{R} = \{\tau_B \circ \mathrm{Ad}\begin{pmatrix} S(p, i) & & \\ & S(q, j) & \\ & & S(r, k) \end{pmatrix} \quad \left(0 \le i \le p, 0 \le j \le \frac{q}{2}, 0 \le k \le r\right),$$

$$\mathrm{Ad}\begin{pmatrix} & & I_p \\ & S(q, j) & \\ I_p & & \end{pmatrix} \qquad (0 \leq j \leq q),$$

$$\mathrm{Ad}\begin{pmatrix} & & I_p \\ & \hat{J}_{q/2} & \\ -I_p & & \end{pmatrix} \qquad \text{if } q \text{ is even},$$

$$\tau_B \circ \mathrm{Ad}(\hat{J}_{n/2}) \qquad \text{if } p, q \text{ and } r \text{ are even}\}.$$

From the following identity

$$(\tau_B \circ \mathrm{Ad}(Q))X = \begin{pmatrix} 0 & -{}^t(Q_2X_2Q_1{}^{-1}) & 0 \\ 0 & 0 & -{}^t(Q_3X_1Q_2{}^{-1}) \\ 0 & 0 & 0 \end{pmatrix},$$

we see

$$\begin{aligned} \mathscr{S} = \{ & [S(p, i), S(q, j), S(r, k)] && (0 \leq i \leq p, 0 \leq j \leq q/2, 0 \leq k \leq r), \\ & \tau \circ [I_p, S(q, j), I_p] && (0 \leq j \leq q/2), \\ & \tau \circ [I_p, \hat{J}_{q/2}, -I_p] && \text{if } q \text{ is even}, \\ & [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{r/2}] && \text{if } p, q \text{ and } r \text{ are even}\}. \end{aligned}$$

5.4. For another example, we will consider about the following GJTS

$$U = M_{p,q}(\boldsymbol{R}), \qquad 2 \leq p;$$

$$B_r(X, Y, Z) = X{}^tYZ + Z{}^tYX - ZA_{q,r}{}^tXYA_{q,r}, \qquad 0 \leq r < q/2.$$

For brevity, we put $m = p + r$, $n = p + q - r$. In this case, we have following facts:

Kantor algebra: $\mathscr{G} = \mathscr{L}(B) = \mathfrak{so}'(m, n)$

$$= \{X \in \mathfrak{sl}(m+n, \boldsymbol{R}) | {}^tXA_{m+n,m} + A_{m+n,m}X = 0\},$$

Cartan involution: $\tau_B(X) = -{}^tX = \mathrm{Ad}(A_{m+n,m})X$,

Characteristic element: $E = \begin{pmatrix} -I_p & & \\ & 0_q & \\ & & I_p \end{pmatrix}$,

$\mathrm{Inn}(\mathscr{G}) = \mathrm{Ad}_{\mathscr{G}}(SO'(m, n)^\circ)$, where

$$SO'(m, n) = \{P \in SL(m+n, \boldsymbol{R}) | {}^tPA_{m+n,m}P = A_{m+n,m}\},$$

$SO'(m, n)^\circ =$ the identity connected component of $SO'(m, n)$,

$\mathrm{Aut}(\mathscr{G}) = \mathrm{Ad}_{\mathscr{G}}(O'(m, n))$, where

$$O'(m, n) = \{P \in GL(m+n, \boldsymbol{R}) \mid {}^t P A_{m+n,m} P = A_{m+n,m}\},$$

$$\mathrm{Aut}_+ \mathscr{G} = \left\{\mathrm{Ad}(P) \mid P = \begin{pmatrix} P_1 & & \\ & P_2 & \\ & & P_3 \end{pmatrix} \in O'(m, n)\right\},$$

$$\mathrm{Aut}_- \mathscr{G} = \left\{\mathrm{Ad}(Q) \mid Q = \begin{pmatrix} & & Q_1 \\ & Q_2 & \\ Q_3 & & \end{pmatrix} \in O'(m, n)\right\},$$

$$\mathrm{Aut}_-(\mathscr{G}, \tau_B) = \left\{\mathrm{Ad}(Q) \mid Q = \begin{pmatrix} & & Q_1 \\ & Q_2 & \\ Q_3 & & \end{pmatrix} \in O'(m, n) \cap O(m+n)\right\}.$$

Let $\mathrm{Ad}(Q) \in \mathrm{Inv}_-(\mathscr{G}, \tau_B)$. Then we have

$$Q^2 \in \text{the center of } SO'(m, n) = \begin{cases} \pm I_{m+n} & \text{if } m \text{ and } n \text{ are even,} \\ I_{m+n} & \text{otherwise.} \end{cases}$$

Let us assume that $Q^2 = I$. Then we have

$$\begin{aligned} &Q_1 \in O(p), \qquad Q_2 \in O'(r, q-r) \cap O(q), \qquad Q_3 = J_p Q_1 J_p, \\ &Q_2{}^2 = I, \qquad Q_1 Q_3 = I, \qquad \det Q_2 = (-1)^p. \end{aligned}$$

Since $Q_3 = Q_1{}^{-1}$, we have $(Q_1 J_p)^2 = I$. Furthermore, since $Q_1 J_p$ is an orthogonal matrix, it is symmetric. Hence there exists

$$T_1 \in O(p) \qquad \text{such that} \qquad T_1 (Q_1 J_p) T_1{}^{-1} = S(p, i) \qquad (0 \le i \le p).$$

As for Q_2, the problem is troublesome. In case of $Q_2 \in SO'(r, q-r)^\circ$, there exists a matrix $T_2 \in SO'(r, q-r)^\circ \cap O(q)$ such that

$$T_2 Q_2 T_2{}^{-1} = L(q, r; 2j, 2k) A_{q,r} \qquad (0 \le 2j \le r,\ 0 \le 2k \le q-r).$$

If Q_2 is not contained in $SO'(r, q-r)^\circ$, then we examine about the conjugacy of Q_2 in case by case. For example, let us assume that $Q_2 \in SO'(r, q-r) - SO'(r, q-r)^\circ$. Now we put

$$C = \begin{pmatrix} S(r, 1) & 0 \\ 0 & J_q S(q-r, 1) J_q \end{pmatrix}.$$

Since $Q_2 \in C \circ SO'(r, q-r)^\circ$, by Proposition 5.1, there exists a matrix $T_2 \in SO'(r, q-r)^\circ \cap O(q)$ satisfying $T_2 Q_2 T_2{}^{-1} = C \circ L(q, r; 2j, 2k) A_{q,r}$. On the other hand, it is easily seen that $C \circ L(q, r; 2j, 2k) A_{q,r}$ is conjugate to $L(q, r;$

$2j-1, 2k-1)A_{q,r}$. Hence we may consider that $T_2Q_2T_2{}^{-1} = L(q, r; 2j-1, 2k-1)A_{q,r}$. In case of $Q_2 \notin SO'(r, q-r)$, similarly we see that Q_2 is conjugate to $L(q, r; 2j-1, 2k)A_{q,r}$ or $L(q, r; 2j, 2k-1)A_{q,r}$. Hence we get together that Q_2 is conjugate to $L(q, r; j, k)A_{q,r}$ by an element of $SO'(r, q-r)^\circ \cap O(q)$. Let us put

$$T = \begin{pmatrix} T_1 & & \\ & T_2 & \\ & & J_pT_1J_p \end{pmatrix}.$$

Then $T \in SO'(m, n)^\circ \cap O(m+n)$ and

$$TQT^{-1} = \begin{pmatrix} & & S(p, i)J_p \\ & L(q, r; j, k)A_{q,r} & \\ J_pS(p, i) & & \end{pmatrix}$$
$$(0 \leq i \leq p, 0 \leq j \leq r, 0 \leq k \leq q-r).$$

Therefore we get

$$\mathrm{Ad}(Q) \sim \mathrm{Ad}\begin{pmatrix} & & S(p, i)J_p \\ & L(q, r; j, k)A_{q,r} & \\ J_pS(p, i) & & \end{pmatrix}.$$

Since $\mathrm{Ad}(T) \in \mathrm{Inn}_+(\mathscr{G}, \tau_B)$, we have

$$\tau_B \circ \mathrm{Ad}(Q) \sim \tau_B \circ \mathrm{Ad}\begin{pmatrix} & & S(p, i)J_p \\ & L(q, r; j, k)A_{q,r} & \\ J_pS(p, i) & & \end{pmatrix}. \tag{5.4}$$

Since any element $X \in U$ is imbedded in $\mathscr{L}(B)$ as

$$X = \begin{pmatrix} 0 & X & 0 \\ 0 & 0 & X' \\ 0 & 0 & 0 \end{pmatrix}, \qquad X' = -A_{q,r}{}^tXJ_p,$$

we have

$$(\tau_B \circ \mathrm{Ad}(Q))X = \begin{pmatrix} 0 & Q_1J_pXA_{q,r}Q_2^{-1} & 0 \\ 0 & 0 & -Q_2{}^tXQ_3{}^{-1} \\ 0 & 0 & 0 \end{pmatrix}$$

for every matrix

$$Q = \begin{pmatrix} & & Q_1 \\ & Q_2 & \\ Q_3 & & \end{pmatrix}.$$

Hence, considering the restriction of the map (5.4) on U, we obtain an element $[S(p, i), L(q, r; j, k)]$ of $\mathscr{S}$. Since $[P, Q] = [-P, -Q]$, we may assume that i runs from 0 to $p/2$.

Next let us assume that $Q^2 = -I$. Then $Q_1 J_p$ and Q_2 are alternative matrices and satisfy $(Q_1 J_p)^2 = -I$, $Q_2{}^2 = -I$. Hence p, q and r must be even. There exist $T_1 \in O(p)$ and $T_2 \in SO'(r, q-r)^\circ \cap O(q)$ such that

$$T_1(Q_1 J_p) T_1{}^{-1} = \hat{J}_{p/2}, \qquad T_2 Q_2 T_2{}^{-1} = \hat{S}(q/2, r/2) A_{q,r},$$

Putting T as above mentioned, we get

$$TQT^{-1} = \begin{pmatrix} 0 & 0 & \hat{J}_{p/2} J_p \\ 0 & \hat{S}(q/2, r/2) A_{q,r} & 0 \\ J_p \hat{J}_{p/2} & 0 & 0 \end{pmatrix}. \tag{5.5}$$

Hence we also see $[\hat{J}_{p/2}, \hat{S}(q/2, r/2)] \in \mathscr{S}$. Therefore we get

$$\begin{aligned} \mathscr{S} = \{ & [S(p, i), L(q, r; j, k)] && (0 \le i \le p/2,\ 0 \le j \le r,\ 0 \le k \le q-r), \\ & [\hat{J}_{p/2}, \hat{S}(q/2, r/2)] && \text{if } p, q \text{ and } r \text{ are even}\}. \end{aligned}$$

5.5. In this last paragraph, we list up the representative systems $\mathscr{S}$ of equivalence classes of involutive automorphisms for all compact classical simple GJTS's of the 2nd kind.

1. $U = M_{p,q}(\boldsymbol{R}) \times M_{q,r}(\boldsymbol{R})$, $p \le r$;

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1{}^tY_1 Z_1 + Z_1{}^tY_1 X_1 - Z_1 X_2{}^tY_2 \\ X_2{}^tY_2 Z_2 + Z_2{}^tY_2 X_2 - {}^tY_1 X_1 Z_2 \end{pmatrix}.$$

(1) In case of $p < r$:

$$\begin{aligned} & [S(p, i), S(q, j), S(r, k)] && (0 \le i \le p,\ 0 \le j \le q/2,\ 0 \le k \le r), \\ & [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{r/2}] && \text{if } p, q \text{ and } r \text{ are even.} \end{aligned}$$

(2) In case of $p = r$:

$$\begin{aligned} & [S(p, i), S(q, j), S(p, k)] && (0 \le i \le k \le p,\ 0 \le j \le q/2), \\ & [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{p/2}] && \text{if } p \text{ and } q \text{ are even,} \\ & \tau \circ [I_p, S(q, j), I_p] && (0 \le j \le q/2), \\ & \tau \circ [-I_p, \hat{J}_{q/2}, I_p] && \text{if } q \text{ is even.} \end{aligned}$$

2. $U = M_{p,q}(\boldsymbol{C}) \times M_{q,r}(\boldsymbol{C}), \quad p \leq r;$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1Y_1^*Z_1 + Z_1Y_1^*X_1 - Z_1X_2Y_2^* \\ X_2Y_2^*Z_2 + Z_2Y_2^*X_2 - Y_1^*X_1Z_2 \end{pmatrix}.$$

(1) In case of $p < r$:

$[S(p, i), S(q, j), S(r, k)] \qquad (0 \leq i \leq p, 0 \leq j \leq q/2, 0 \leq k \leq r),$

$\omega,$

$\omega \circ [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{r/2}] \qquad$ if p, q and r are even.

(2) In case of $p = r$:

$[S(p, i), S(q, j), S(p, k)] \qquad (0 \leq i \leq k \leq p, 0 \leq j \leq q/2),$

$\omega,$

$\tau,$

$\omega \circ \tau \circ [I_p, S(q, j), I_p] \qquad (0 \leq j \leq q/2),$

$\tau \circ [-I_p, \hat{J}_{q/2}, I_p] \qquad$ if q is even,)

$\omega \circ [\hat{J}_{p/2}, \hat{J}_{q/2}, \hat{J}_{p/2}] \qquad$ if p and q are even.

3. $U = M_{p,q}(\boldsymbol{H}) \times M_{q,r}(\boldsymbol{H}), \quad p \leq r;$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1Y_1^*Z_1 + Z_1Y_1^*X_1 - Z_1X_2Y_2^* \\ X_2Y_2^*Z_2 + Z_2Y_2^*X_2 - Y_1^*X_1Z_2 \end{pmatrix}.$$

(1) In case of $p < r$:

$[S'(p, i), S'(q, j), S'(r, k)] \qquad (0 \leq i \leq p, 0 \leq j \leq q/2, 0 \leq k \leq r),$

$\omega.$

(2) In case of $p = r$:

$[S'(p, i), S'(q, j), S'(p, k)] \qquad (0 \leq i \leq k \leq p, 0 \leq j \leq q/2),$

$\omega,$

$\tau,$

$\omega \circ \tau.$

4. $U = M_{p,q}(\boldsymbol{R}), \quad 2 \leq p;$

$B_r(X, Y, Z) = X^tYZ + Z^tYX - ZA_{q,r}{}^tXYA_{q,r}, \qquad 0 \leq r < q/2.$

$[S(p, i), L(q, r; j, k)] \qquad (0 \leq i \leq p/2, 0 \leq j \leq r, 0 \leq k \leq q - r),$

$[\hat{J}_{p/2}, \hat{S}(q/2, r/2)] \qquad$ if p, q and r are even.

5. $U = M_{p,2q}(\boldsymbol{R})$, $2 \le p$, $(p, q) \ne (2, 1)$;

$B(X, Y, Z) = X^tYZ + Z^tYX - ZJ_{2q}{}^tXYJ_{2q}$.

$[S(p, i), K(2q; j, k)]$ $(0 \le i \le p/2, 0 \le j \le q, 0 \le k \le q - j)$,

$[S(p, i), S(2q, q)]$ $(0 \le i \le p/2)$,

$[\tilde{J}_{p/2}, \tilde{J}_q]$ if p is even.

6. $U = M_{p,2q}(\boldsymbol{R})$, $(p, q) \ne (1, 1)$;

$B(X, Y, Z) = X^tYZ + Z^tYX + Z\tilde{J}_q{}^tXY\tilde{J}_q$.

$[S(p, i), H(2q, j)]$ $(0 \le i \le p/2, 0 \le j \le q)$,

$[S(p, i), S(2q, q)]$ $(0 \le i \le p/2)$,

$[\tilde{J}_{p/2}, \tilde{J}_qH(2q, j)]$ $(0 \le j \le q)$ if p is even.

7. $U = M_{p,q}(\boldsymbol{C})$;

$B_r(X, Y, Z) = XY^*Z + ZY^*X - ZA_{q,r}X^*YA_{q,r}$, $0 \le r \le q/2$.

$[S(p, i), L(q, r; j, k)]$ $(0 \le i \le p/2, 0 \le j \le r, 0 \le k \le q - r)$,

ω,

$\omega \circ [\hat{J}_{p/2}, \hat{S}(q/2, r/2)]$ if p, q and r are even.

8. $U = M_{p,2q}(\boldsymbol{C})$, $4 \le p + q$;

$B(X, Y, Z) = XY^*Z + ZY^*X - ZJ_{2q}X^*YJ_{2q}$.

$[S(p, i), K(2q; j, k)]$ $(0 \le i \le p/2, 0 \le j \le q, 0 \le k \le q - j)$,

$[S(p, i), S(2q, q)]$ $(0 \le i \le p/2)$,

ω,

$\omega \circ [I_p, S(2q, q)]$,

$\omega \circ [\tilde{J}_{p/2}, \tilde{J}_q]$ if p is even.

9. $U = M_{p,2q}(\boldsymbol{C})$, $(p, q) \ne (1, 1)$;

$B(X, Y, Z) = XY^*Z + ZY^*X + Z\tilde{J}_q{}^tX\overline{Y}\tilde{J}_q$.

$[S(p, i), H(2q, j)]$ $(0 \le i \le p/2, 0 \le j \le q)$,

$[S(p, i), S(2q, q)]$ $(0 \le i \le p/2)$,

ω,

$\omega \circ [\tilde{J}_{p/2}, \tilde{J}_qH(2q, j)]$ $(0 \le j \le q)$ if p is even.

10. $U = M_{p,q}(\boldsymbol{C}), \quad 2 \le p, \quad (p, q) \ne (2, 2);$

$B(X, Y, Z) = XY^*Z + ZY^*X - Z^tX\bar{Y}.$

$[S(p, i), S(q, j)] \qquad (0 \le i \le p/2, 0 \le j \le q),$

$\omega \circ [I_p, S(q, j)] \qquad (0 \le j \le q/2),$

$\omega \circ [\tilde{J}_{p/2}, \tilde{J}_{q/2}] \qquad \text{if } p \text{ and } q \text{ are even.}$

11. $U = M_{p,q}(\boldsymbol{H});$

$B_r(X, Y, Z) = XY^*Z + ZY^*X - ZA'_{q,r}X^*YA'_{q,r}, \qquad 0 \le r \le q/2.$

$[S'(p, i), L'(q, r; j, k)] \qquad (0 \le i \le p/2, 0 \le j \le r, 0 \le k \le q - r),$

$\omega.$

12. $U = M_{p,2q}(\boldsymbol{H});$

$B_q(X, Y, Z) = XY^*Z + ZY^*X - ZJ'_{2q}X^*YJ'_{2q}.$

$[S'(p, i), K'(2q; j, k)] \qquad (0 \le i \le p/2, 0 \le j \le q, 0 \le k \le q - j),$

$[S'(p, i), S'(2q, q)] \qquad (0 \le i \le p/2),$

$\omega,$

$\omega \circ [I_{2p}, S'(2q, q)].$

13. $U = M_{p,q}(\boldsymbol{H}), \quad (p, q) \ne (1, 1), (1, 2);$

$B(X, Y, Z) = XY^*Z + ZY^*X + ZK_qX^*YK_q.$

$[S'(p, i), K'(q; j, k)] \qquad (0 \le i \le p/2, 0 \le j \le q/2, 0 \le k \le q/2 - j),$

$\omega,$

$[S'(p, i), S'(q, q/2)] \qquad (0 \le i \le p/2) \qquad \text{if } q \text{ is even,}$

14. $U = M_{1,n}(\boldsymbol{R}) \times Alt'_n(\boldsymbol{R}), \quad 4 \le n;$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right) = \begin{pmatrix} X_1{}^tY_1Z_1 + Z_1{}^tY_1X_1 - Z_1X_2{}^tY_2 \\ X_2{}^tY_2Z_2 + Z_2{}^tY_2X_2 - {}^tY_1X_1Z_2 - Z_2J_n{}^tX_1Y_1J_n \end{pmatrix}.$$

$[I_1, S(n, i), J_nS(n, i)J_n] \qquad (0 \le i \le n).$

15. $U = M_{1,n}(\boldsymbol{C}) \times Alt'_n(\boldsymbol{C}), \quad 4 \le n;$

$$B\left(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}\right)$$

$$=\begin{pmatrix} X_1Y_1^*Z_1+Z_1Y_1^*X_1-Z_1X_2Y_2^* \\ X_2Y_2^*Z_2+Z_2Y_2^*X_2-Y_1^*X_1Z_2-Z_2J_n{}^tX_1\bar{Y}_1J_n \end{pmatrix}.$$

$[I_1, S(n, i), J_nS(n, i)J_n] \qquad (0 \leq i \leq n),$

$\omega.$

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